Smoothness of Nonlinear Median-Interpolation Subdivision

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Abstract

We give a refined analysis of the Hölder regularity for the limit functions arising from a nonlinear pyramid algorithm for robust removal of non-Gaussian noise proposed by Donoho and Yu [6, 7, 17]. The synthesis part of this algorithm can be interpreted as a nonlinear triadic subdivision scheme where new points are inserted based on local quadratic polynomial median interpolation and imputation. We introduce the analogon of the Donoho-Yu scheme for dyadic refinement, and show that its limit functions are in C^α for $\alpha > \log_4(128/31) = 1.0229....$ In the triadic case, we improve the lower bound of $\alpha > \log(135/121) = 0.0997...$ previously obtained in [6] to $\alpha > \log_3(135/53) = 0.8510....$ These lower bounds are relatively close to the anticipated upper bounds of $\log_2(16/7) = 1.1982...$ in the dyadic resp. 1 in the triadic cases, and have been obtained by deriving recursive inequalities for the ℓ_∞ norm of second rather than first order differences of the sequences arising in the subdivision process.

1 Introduction

This paper is concerned with the smoothness analysis of a univariate nonlinear stationary subdivision scheme introduced in [6, 7, 17] in connection with an application to heavy-tail noise removal. The nonlinear subdivision operator $S_{\text{med}} : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ associated with the Donoho-Yu scheme has dilation factor $r = 3$ and is defined by a local interpolation/imputation procedure using median values of polynomials of fixed even degree. More generally, similar operators (both linear and nonlinear ones) can be introduced for arbitrary $r \geq 2$, by using a suitable functional

$$\text{int} \;:\quad (p, I) \quad \longmapsto \quad \text{int}(p; I) \in \mathbb{R}\,, \tag{1}$$

that is well-defined for polynomials p of degree $\leq k$ and closed bounded intervals $I = [\alpha, \beta]$. We give the definition of S_{int} for $k = 2$, i.e., in the case of quadratic polynomials which we will deal with in most parts of this paper. Let $m := \{m_i \,:\, i \in \mathbb{Z}\} \in \ell_\infty(\mathbb{Z})$ be given. For any three consecutive entries m_{i-1}, m_i, m_{i+1}, $i \in \mathbb{Z}$, in the *interpolation step* we define a polynomial p_{int} of degree ≤ 2 from the conditions

$$\text{int}(p_{\text{int}}; I^\nu) = m_{i+\nu}, \quad I^\nu := [2\nu - 1, 2\nu + 1], \quad \nu = -1, 0, 1. \tag{2}$$

The choice of I^ν as reference intervals is one of convenience. We assume unique solvability of (2), independently of the data m. In the *imputation step*, we determine $\tilde{m} := S_{\mathrm{int}} m$ by setting

$$\widetilde{m}_{ir+\mu} = \mathrm{int}(p_{\mathrm{int}}; \tilde{I}^\mu), \quad \tilde{I}^\mu := [-1 + \frac{2\mu}{r}, -1 + \frac{2(\mu+1)}{r}], \quad \mu = 0, \ldots, r-1 , \tag{3}$$

for the r subintervals of the center reference interval $I^0 = [-1, 1]$. This definition automatically ensures shift-invariance of the scheme. Obviously, extending (2) to more intervals I^ν, $\nu = -l, \ldots, l$, leads to definitions of p_{int} and S_{int} for arbitrary even $k = 2l$.

The interpolation functionals that will be used below include the *median* $\mathrm{med}(p; I)$ which coincides with the unique $\lambda \in \mathbb{R}$ such that

$$\mathrm{meas}(\{x \in I \,:\, p(x) \geq \lambda\}) = \mathrm{meas}(\{x \in I \,:\, p(x) \leq \lambda\}) .$$

Equivalently, $\mathrm{med}(p; I)$ represents the best constant approximation to p in $L_1(I)$. For theoretical properties, especially the unique solvability of (2), of median interpolation with polynomials of arbitrary degree, see [9]. Explicit formulas for $k = 2, r = 3$ can be found in [6]. We also consider examples based on *midpoint values*

$$\mathrm{mid}(p; [\alpha, \beta]) = p((\alpha + \beta)/2,$$

and *average values*

$$\mathrm{ave}(p; I) = |I|^{-1} \int_I p(x)\, dx.$$

These lead to the linear subdivision operators S_{mid}, S_{ave}, respectively. Further examples will be mentioned in the remarks in Section 3.

A stationary subdivision scheme with subdivision operator S creates a sequence $\{m^j \in \ell_\infty(\mathbb{Z}) : j \geq 0\}$ by repeatedly applying S to an initially given $m \in \ell_\infty(\mathbb{Z})$, i.e., $m^0 = m$ and $m^{j+1} = Sm^j = S^{j+1}m$, $j \geq 0$. We say that the subdivision scheme with initial sequence m converges to a function f if the associated sequence

$$h^j(x) = m_i^j , \qquad x \in [r^{-j}i, r^{-j}(i+1)) , \quad i \in \mathbb{Z} , \quad j \geq 0 , \tag{4}$$

of step functions satisfies

$$h^j(x) \quad \longrightarrow \quad f(x) \equiv f(x; m) , \quad x \in \mathbb{R} , \qquad j \to \infty , \tag{5}$$

in an appropriate sense. Throughout this paper, we will only consider convergence in $L_\infty(\mathbb{R})$, and the main objective is to find bounds for the Hölder exponent of the limit function $f(\cdot; m)$ which hold for arbitrary $m \in \ell_\infty(\mathbb{Z})$.

As is well-known, the convergence and smoothness properties of linear subdivision schemes can be studied by using the theory of refinable functions, see, e.g., [13, 2, 8, 4, 15, 16]. Thus, the challenge are nonlinear schemes such as S_{med}. For $r = 3$, it was shown in [6, 17] that the subdivision scheme S_{med} always converges to a limit function with Hölder exponent $\alpha > \log_3(135/121) = 0.0997....$ In [6] it was conjectured that the limit functions f satisfy a much stronger estimate

$$|f(x+h) - f(x)| \leq Ch|\log h| , \qquad 0 < h < 1 , \tag{6}$$

and that this cannot be improved. In other words, limit functions of the triadic subdivision scheme S_{med} are conjectured to be in C^{α} for any $\alpha < 1$ but not necessarily in C^1.

The fact that there is a significant gap between the conjecture (6) and the established lower bound for the exact Höelder exponent has triggered our interest in this subject. The conjecture (6) is based on the observation that S_{med} is closely related to S_{mid}. The intuition behind this is simple: we have $\text{med}(f;I) = \text{mid}(f;I)$ for each monotone and continuous function f on I, and polynomials of low degree consist of a few monotone pieces. More precisely, for polynomials of exact degree $k = 2$ inequality $\text{med}(p;I) \neq \text{mid}(p;I)$ happens if and only if the unique extremal point c of p belongs to the middle portion $((3\alpha+\beta)/4, (\alpha+3\beta)/4)$ of $I = [\alpha, \beta]$, and the deviations between median and midpoint values (and thus between $S_{\text{med}}m$ and $S_{\text{mid}}m$) remain relatively small. Thus, it can be expected that S_{med} inherits many of its properties from the linear scheme S_{mid}. For the latter, limit functions satisfy (6) which, together with numerical evidence, leads to the conjecture (6), see [17, 6] for complete details.

Our aim is to provide better lower bounds for the exact Hölder exponents which come closer to the projected sharp result. We demonstrate our approach in all detail for the dyadic case $r = 2$. One shortcoming of the analysis in [17, 6] is that it is based on considering only the behavior of first order differences expressed by the quantities $\Delta_1(m) := \sup_i |m_{i+1} - m_i|$. The lower bound for the Hölder smoothness of the triadic S_{med} from [6] follows from the inequality

$$\Delta_1(\widetilde{m}) \leq \frac{121}{135}\Delta_1(m) . \tag{7}$$

Since the constant in (7) satisfies $121/135 < 1$ this implies sufficient geometric decay of the sequence $\{\Delta_1(m^j) \,:\, j \geq 0\}$ which is crucial for establishing convergence as well as C^{α}-regularity for some small $\alpha > 0$. This approach is obviously restricted to $\alpha \leq 1$, and usually does not lead to very impressive lower bounds. E.g., in order to establish (6) one would need a constant $1/3$ in (7) which is impossible since the constant in (7) is optimal. For the analogous inequality in the case $r = 2$, the best constant is 1, and insufficient to yield any statement on Hölder smoothness. One possible improvement is to directly estimate the constant in $\Delta_1(m^{j+s}) \leq \gamma_s \Delta_1(m^j)$ for some $s > 1$ which usually improves significantly upon the result for $s = 1$ since often $\gamma_s << \gamma_1^s$ for large enough s. However, this is technically hard to achieve since it means to analyze a more involved scheme (the composition of s subdivision steps of the original method).

The alternative we pursue here is to take into account higher order difference behavior which is the standard approach for linear subdivision schemes [8, 4, 15]. It turns out that results improve substantially if we include quantities such as $\Delta_2(m) := \sup_i |m_{i-1} - 2m_i + m_{i+1}|$. We will replace (7) by proving

$$\Delta_2(\widetilde{m}) \leq \gamma(r)\Delta_2(m) \tag{8}$$

for $r = 2, 3$. For $r = 2$, we obtain $\gamma(2) = 1/2$ which incidently yields (6) for the limit functions obtained by the dyadic scheme S_{med}. With considerable extra effort, the C^{α} property can be established for some $\alpha > 1$, too. For $r = 3$, we establish (8) with $\gamma(3) = 53/135$.

It is worthwhile to comment on some related research we became aware of while working on the final draft of this paper. Nonlinear subdivision schemes and their smoothness properties have first be considered in the context of shape-preserving interpolation of univariate data (see [11, 12] for examples and further references). A general class of *quasi-linear* or *data-dependent subdivision* *schemes*, and a theory on how to treat their stability and smoothness properties, was introduced in [3]. Roughly speaking, the proposal in [3] amounts to identifying an appropriate family of linear subdivision operators $\mathcal{S} := \{S_w \,:\, w \in \ell_\infty(\mathbb{Z})\}$ such that

$$Sm = S_m m \qquad \forall\, m \in \ell_\infty(\mathbb{Z}) \,. \tag{9}$$

The basic assumptions on $\mathcal{S}$ that enable a systematic treatment of the Hölder smoothness issue are *uniform locality* and *polynomial reproduction* of a certain order for all $S_w \in \mathcal{S}$. For exact definitions, we refer to [3]. Hölder exponents can then be estimated by using factorization techniques and the joint spectral radius $\rho_\infty(\mathcal{S}_n)$ of an associated family $\mathcal{S}_n$ of derived subdivision schemes for differences. To apply this theory, there is one little obstacle: For given S, the requirement (9) does not define $\mathcal{S}$ in a unique way, and [3] does not provide a canonical construction either. Although for $S_{\rm med}$ and $r = 2$, it is possible to construct $\mathcal{S}$ such that all S_w are uniformly local and reproduce linear polynomials, this leads to a continuum of different S_w in $\mathcal{S}$, and does not seem to simplify the technicalities. We also claim that we can get stronger estimates in some cases. Thus, as appealing the theory in [3] looks, we decided to stay with our straightforward approach which can be viewed as a particular shortcut for implementing the general theory, and subsumes the factorization and joint spectral radius estimation steps of [3] into the derivation of inequalities such as (8).

Another recent paper we wish to mention is [5], where some results on the smoothness of the limit functions arising from nonlinear perturbations of a linear subdivision scheme are discussed. One of the basic results (see [5, Theorems 3.3 resp. 4.4]) states that the smoothness of the limit functions of a nonlinear subdivision process $\tilde{S}$ can be estimated from the smoothness exponent μ of a "nearby" linear subdivision operator $\hat{S}$ (with the same dilation factor r) and the best constant ν for which one has

$$\|m^{j+1} - \hat{S}m^j\|_{\ell_\infty} = \|(\tilde{S} - \hat{S})m^j\|_{\ell_\infty} = \mathrm{O}(r^{-j\nu}) \,, \quad m^{j+1} = \tilde{S}m^j \,, \quad j \geq 0 \,. \tag{10}$$

Although this does not remove the need in facilitating some kind of recursive analysis for $\tilde{S}$ (since the arguments m^j of the inequality are defined by $\tilde{S}$, and not $\hat{S}$), concentrating on estimates for for the perturbations $\tilde{S}m - Sm$ will help in technical terms. This is especially advantageous since we already have a good candidate $\hat{S} = S_{\rm mid}$ for our nonlinear $\tilde{S} = S_{\rm med}$.

The paper is organized in the following way. In Section 2 we consider the scheme $S_{\rm med}$ in detail. We give all the formulas and estimates necessary for establishing the local estimates for second order differences formulated in Theorem 1 for $r = 2$. To improve the readability, some of the elementary but tedious technical arguments of the proof have been postponed into the appendix 4. The modifications for $r = 3$ are listed in Subsection 2.2, since they are straightforward, less detail is given there.

With the recursive inequalities for second order differences at hand, the statements on lower bounds for the Hölder exponent follow by standard arguments. This is done in Section 3 for $r = 2$ and $r = 3$. For comparison, we also give the exact Hölder exponents for the linear schemes associated with midpoint and average interpolation. E.g., for $r = 2$ the midpoint interpolation scheme $S_{\rm mid}$ has an exact Hölder exponent of $\log_2(7/16) = 1.1982\ldots$ Numerical evidence indicates that this is also the exact Hölder exponent for $S_{\rm med}$ which is in line with the above-mentioned conjecture for $r = 3$ from [6]. We conclude with a brief discussion of some extensions and open problems.

2 Recursive inequalities for S_{med}

2.1 The case $r = 2$

We will start with some explicit formulas for S_{med} in the case $r = 2$, and derive some preliminary results. We will preserve the notation from the introduction, and analyze most of the time a single subdivision step $m \to \widetilde{m} = S_{\mathrm{med}} m$. One of the technical differences with [6], where similar formulas can be found for $r = 3$, is that we consistently represent quadratic polynomials in the form $p(x) = a(x-c)^2 + b$, and parametrize all formulas w.r.t. the value of the center c. Using this form, we have

$$\mathrm{med}(p; I) = \begin{cases} p((\alpha+\beta)/2) = a((\alpha+\beta)/2 - c)^2 + b\,, & |c - (\alpha+\beta)/2| \geq |I|/4\,, \\ p(c \pm |I|/4) = a|I|^2/16 + b\,, & |c - (\alpha+\beta)/2| < |I|/4\,, \end{cases} \tag{11}$$

for any interval $I = [\alpha, \beta]$ ($|I| = \beta - \alpha$), see [6, Proposition 2.1]. This nonlinear relation enters both interpolation and imputation steps, and is responsible for the nonlinearity of S_{med}.

The necessary formulas for S_{med} will be represented as perturbations with respect to the corresponding linear expressions for S_{mid}. Because of shift-invariance, it is enough to consider m_{-1}, m_0, m_1 as the three consecutive data for the interpolation/imputation steps (2), (3). Let $p_{\mathrm{med}}(x)$, $p_{\mathrm{mid}}(x)$ be the quadratic interpolants satisfying (2) for the median and midpoint value functionals, respectively. Obviously,

$$p_{\mathrm{mid}}(x) = \frac{x(x-2)}{8} m_{-1} + \frac{4 - x^2}{4} m_0 + \frac{x(x+2)}{8} m_1\ . \tag{12}$$

In the *exceptional case* $\delta_2 := m_{-1} - 2m_0 + m_1 = 0$, these polynomials become linear and coincide:

$$p_{\mathrm{med}}(x) = p_{\mathrm{mid}}(x) = m_0 + \frac{m_1 - m_{-1}}{4} x \qquad (\delta_2 = 0)\ . \tag{13}$$

If $\delta_2 \neq 0$ then $p_{\mathrm{med}}(x) = a(x-c)^2 + b$ with $a \neq 0$. For this *generic case*, we have

Lemma 1 *If $\delta_2 = m_{-1} - 2m_0 + m_1 \neq 0$ then the quadratic interpolants $p_{\mathrm{med}}(x) = a(x-c)^2 + b$ and $p_{\mathrm{mid}}(x)$ to the data (m_{-1}, m_0, m_1) satisfy*

$$p_{\mathrm{med}}(x) = p_{\mathrm{mid}}(x) - \frac{2\epsilon_0(4 - x^2) + \epsilon_{-2} x(x-2) + \epsilon_2 x(x+2)}{8(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \delta_2\ , \tag{14}$$

where the quantities $\epsilon_n := (1 - 4(c-n)^2)_+ \in [0, 1]$ depend on c and vanish for $c \notin (n - 1/2, n + 1/2)$, $n = -2, 0, 2$. Moreover, we have

$$a = \frac{4\delta_2}{32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2}\ . \tag{15}$$

The elementary proof can be found in the Appendix. For the imputation step and $r = 2$, (11) immediately implies

Lemma 2 *If $\delta_2 = m_{-1} - 2m_0 + m_1 \neq 0$ then the quadratic interpolant $p_{\text{med}}(x) = a(x-c)^2 + b$ to the data (m_{-1}, m_0, m_1) satisfies*

$$\begin{aligned} \operatorname{med}(p_{\text{med}}; [-1,0]) &= p_{\text{med}}(-\frac{1}{2}) + \frac{a}{16}\hat{\epsilon}_{-1/2} , \qquad \hat{\epsilon}_{-1/2} := (1 - 4(1+2c)^2)_+ , \\ \operatorname{med}(p_{\text{med}}; [0,1]) &= p_{\text{med}}(\frac{1}{2}) + \frac{a}{16}\hat{\epsilon}_{1/2} , \qquad \hat{\epsilon}_{1/2} := (1 - 4(1-2c)^2)_+ , \end{aligned} \tag{16}$$

where a is defined by (15).

Putting things together, we have the following representation formula (it also holds for the exceptional case $\delta_2 = 0$ if we formally set $c = \pm\infty$).

Proposition 1 *Let $\widetilde{m} = S_{\text{med}} m$ and $\hat{m} = S_{\text{mid}} m$ for $r = 2$. Then, using the notation introduced in Lemma 1 and 2, we have*

$$\begin{aligned} \widetilde{m}_0 &= \hat{m}_0 - \delta_2 \frac{30\epsilon_0 + 5\epsilon_{-2} - 3\epsilon_2 - 8\hat{\epsilon}_{-1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \equiv \hat{m}_0 - \delta_2 \alpha_0 , \\ \widetilde{m}_1 &= \hat{m}_1 - \delta_2 \frac{30\epsilon_0 - 3\epsilon_{-2} + 5\epsilon_2 - 8\hat{\epsilon}_{1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \equiv \hat{m}_1 - \delta_2 \alpha_1 , \end{aligned} \tag{17}$$

whereas

$$\hat{m}_0 = \frac{5}{32} m_{-1} + \frac{15}{16} m_0 - \frac{3}{32} m_1 , \quad \hat{m}_1 = -\frac{3}{32} m_{-1} + \frac{15}{16} m_0 + \frac{5}{32} m_1 . \tag{18}$$

Using the shift-invariance properties of S_{med} resp. S_{mid}, similar formulas hold for all $\widetilde{m}_{2i}$, $\widetilde{m}_{2i+1}$ resp. $\hat{m}_{2i}$, $\hat{m}_{2i+1}$, $i \in \mathbb{Z}$.

Since $\widetilde{m}_0 = \operatorname{med}(p_{\text{med}}; [-1,0])$ and $\hat{m}_0 = p_{\text{mid}}(-1/2)$ (similarly for $\widetilde{m}_1, \hat{m}_1$), (17) simply follows from substituting (14) and (15) into (16). The local rule (18) for the linear subdivision operator S_{mid} follows by evaluating (12) for $x = \pm 1/2$.

We are now in position to state the main technical result of this paper.

Theorem 1 *Let $r = 2$, $m \in \ell_\infty(\mathbb{Z})$, and denote by $\widetilde{m} = S_{\text{med}} m$ resp. $\widetilde{\widetilde{m}} = S_{\text{med}} \widetilde{m}$ the result of one resp. two subdivision steps with S_{med}. Then we have*

$$\Delta_2(\widetilde{m}) \leq \frac{1}{2} \Delta_2(m) , \tag{19}$$

and

$$\Delta_2(\widetilde{\widetilde{m}}) \leq \frac{31}{128} \Delta_2(m) , \tag{20}$$

The constant in (19) is sharp (equality is achieved for $m = (\ldots, 0, 1, 0, 1, \ldots)$) while the constant in (20) is not final.

Proof. We establish (19) while postponing the proof of the slightly better estimate (20) into the Appendix. By Proposition 1, we can express the differences $\tilde{\delta}_{2,i'} := \widetilde{m}_{i'-1} - 2\widetilde{m}_{i'} + \widetilde{m}_{i'+1}$ of $\widetilde{m}$ by the corresponding second differences $\delta_{2,i} := m_{i-1} - 2m_i + m_{i+1}$ of m. By shift-invariance, it suffices to consider $i' = 0, -1$. Obviously, $\tilde{\delta}_{2,0}$, $\tilde{\delta}_{2,-1}$ depend on the result of two of the above described interpolation/imputation steps involving quadratic polynomials, one with respect to the data (m_{-1}, m_0, m_1) (see above, the formulas involve a parameter c and give $\widetilde{m}_0$, $\widetilde{m}_1$), the other with respect to the data (m_{-2}, m_{-1}, m_0) computes the values $\widetilde{m}_{-2}$, $\widetilde{m}_{-1}$ and involves another parameter c^*. Quantities which depend on c^* will be indicated by an asterisque. Thus, from (17), (18) we have

$$\begin{aligned} \widetilde{m}_{-2} &= \frac{5}{32}m_{-2} + \frac{15}{16}m_{-1} - \frac{3}{32}m_0 - \delta_{2,-1}\alpha_0^* \,, \\ \widetilde{m}_{-1} &= -\frac{3}{32}m_{-2} + \frac{15}{16}m_{-1} + \frac{5}{32}m_0 - \delta_{2,-1}\alpha_1^* \,, \end{aligned}$$

while

$$\begin{aligned} \widetilde{m}_{0} &= \frac{5}{32}m_{-1} + \frac{15}{16}m_{0} - \frac{3}{32}m_1 - \delta_{2,0}\alpha_0 \,, \\ \widetilde{m}_{1} &= -\frac{3}{32}m_{-1} + \frac{15}{16}m_{0} + \frac{5}{32}m_1 - \delta_{2,0}\alpha_1 \,. \end{aligned}$$

A straightforward calculation shows that

$$\tilde{\delta}_{2,0} = -(\frac{3}{32} + \alpha_1^*)\delta_{2,-1} + (\frac{11}{32} + \beta_0)\delta_{2,0} \,, \quad \beta_0 := 2\alpha_0 - \alpha_1 \,. \tag{21}$$

Similarly, we obtain

$$\tilde{\delta}_{2,-1} = (\frac{11}{32} + \beta_1^*)\delta_{2,-1} - (\frac{3}{32} + \alpha_0)\delta_{2,0} \,, \quad \beta_1^* := 2\alpha_1^* - \alpha_0^* \,. \tag{22}$$

From (17) we see that

$$\alpha_0,\ \alpha_1 \in [-1/128, 1/32] \,, \qquad c \in \mathbb{R} \,, \tag{23}$$

where the interval boundaries are sharp: we have $\alpha_0 = \alpha_1 = 1/32$ iff $c = 0$, and $\alpha_0 = -1/128$ resp. $\alpha_1 = -1/128$ iff $c = -1/2$ resp. $c = 1/2$. Similarly, since

$$\begin{aligned} \beta_0 = 2\alpha_0 - \alpha_1 &= \frac{30\epsilon_0 + 13\epsilon_{-2} - 11\epsilon_2 + 8\hat{\epsilon}_{1/2} - 16\hat{\epsilon}_{-1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \,, \\ \beta_1 = 2\alpha_1 - \alpha_0 &= \frac{30\epsilon_0 - 11\epsilon_{-2} + 13\epsilon_2 + 8\hat{\epsilon}_{-1/2} - 16\hat{\epsilon}_{1/2}}{32(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \,, \end{aligned}$$

we obtain

$$\beta_0,\ \beta_1 \in [-1/64, 1/32] \,, \qquad c \in \mathbb{R} \,, \tag{24}$$

again with sharp interval boundaries. Note that for the exceptional case $\delta_2 = 0$ all these expressions vanish. Both (23) and (24) follow by elementary calculus, some details are given below. Substituting into (21), (22), we arrive at

$$\begin{aligned} |\tilde{\delta}_{2,0}| &\le |\frac{3}{32} + \alpha_1^*||\delta_{2,-1}| + |\frac{11}{32} + \beta_0||\delta_{2,0}| \le \frac{1}{2}\max(|\delta_{2,-1}|, |\delta_{2,0}|) \,, \\ |\tilde{\delta}_{2,-1}| &\le |\frac{11}{32} + \beta_1^*||\delta_{2,-1}| + |\frac{3}{32} + \alpha_0||\delta_{2,0}| \le \frac{1}{2}\max(|\delta_{2,-1}|, |\delta_{2,0}|) \end{aligned}$$

Because of shift-invariance, this gives (19). If we take $(m_{-2}, m_{-1}, m_0, m_1) = (0,1,0,1)$ then the above local estimates turn into equality. More generally, this happens if and only if $c = c^* = 0$ and $\delta_{2,-1} = -\delta_{2,0} \neq 0$.

Remark 1 *We note that there is no useful analog of (7) for $r = 2$. All what can be proved is*

$$\Delta_1(\widetilde{m}) \leq \Delta_1(m) \ , \qquad r = 2 \ . \tag{25}$$

We leave this as an exercise. The constant 1 *cannot be improved since equality in (25) is attained for the same alternating sequence $m = (\ldots, 0, 1, 0, 1, \ldots)$ that gave equality in (19).*

2.2 The case $r = 3$

We outline the changes. Lemma 1 can be used as is while the formulas of Lemma 2 need to be replaced by

$$\begin{aligned}
\mathrm{med}(p_{\mathrm{med}}; [-1, -\tfrac{1}{3}]) &= p_{\mathrm{med}}(-\tfrac{2}{3}) + \frac{a}{36}\tilde{\epsilon}_{-2/3} \ , \qquad & \tilde{\epsilon}_{-2/3} &:= (1 - 4(1+3c)^2)_+ \ , \\
\mathrm{med}(p_{\mathrm{med}}; [-\tfrac{1}{3}, \tfrac{1}{3}]) &= p_{\mathrm{med}}(0) + \frac{a}{36}\tilde{\epsilon}_0 \ , & \tilde{\epsilon}_0 &:= (1 - 36c^2)_+ \ , \\
\mathrm{med}(p_{\mathrm{med}}; [\tfrac{1}{3}, 1]) &= p_{\mathrm{med}}(\tfrac{2}{3}) + \frac{a}{36}\tilde{\epsilon}_{2/3} \ , & \tilde{\epsilon}_{2/3} &:= (1 - 4(1-3c)^2)_+ \ ,
\end{aligned} \tag{26}$$

where a is defined by (15). This gives

Proposition 2 *Let $\tilde{m} = S_{\mathrm{med}}m$ and $\hat{m} = S_{\mathrm{mid}}m$ for $r = 3$. Using the same notation as in Subsection 2.1, we obtain*

$$\begin{aligned}
\widetilde{m}_0 &= \hat{m}_0 - \delta_2 \frac{8\epsilon_0 + 2\epsilon_{-2} - \epsilon_2 - \tilde{\epsilon}_{-2/3}}{9(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \equiv \hat{m}_0 - \delta_2 \hat{\alpha}_0 \ , \\
\widetilde{m}_1 &= \hat{m}_1 - \delta_2 \frac{9\epsilon_0 - \tilde{\epsilon}_0}{9(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \equiv \hat{m}_2 - \delta_2 \hat{\alpha}_1 \ , \\
\widetilde{m}_2 &= \hat{m}_2 - \delta_2 \frac{8\epsilon_0 - \epsilon_{-2} + 2\epsilon_2 - \tilde{\epsilon}_{2/3}}{9(32 - 2\epsilon_0 + \epsilon_{-2} + \epsilon_2)} \equiv \hat{m}_2 - \delta_2 \hat{\alpha}_2 \ ,
\end{aligned} \tag{27}$$

whereas

$$\hat{m}_0 = \frac{2}{9}m_{-1} + \frac{8}{9}m_0 - \frac{1}{9}m_1 \ , \quad \hat{m}_1 = m_0 \ , \quad \hat{m}_2 = -\frac{1}{9}m_{-1} + \frac{8}{9}m_0 + \frac{2}{9}m_1 \ . \tag{28}$$

Using the shift-invariance properties of S_{med} resp. S_{mid}, similar formulas hold for all $\widetilde{m}_{3i}$, $\widetilde{m}_{3i+1}$, $\widetilde{m}_{3i+2}$ resp. $\hat{m}_{3i}$, $\hat{m}_{3i+1}$, $\hat{m}_{3i+2}$, $i \in \mathbb{Z}$.

From Proposition 2, the following recursive formulas for second order differences can be derived:

$$\begin{aligned}
\tilde{\delta}_{2,-1} &= (2/9 + \hat{\beta}_2^*)\delta_{2,-1} - (1/9 + \hat{\alpha}_0)\delta_{2,0} \ , \quad \hat{\beta}_2^* := 2\hat{\alpha}_2^* - \hat{\alpha}_1^* \ , \\
\tilde{\delta}_{2,0} &= -(1/9 + \hat{\alpha}_2^*)\delta_{2,-1} + (2/9 + \hat{\beta}_0)\delta_{2,0} \ , \quad \hat{\beta}_0 := 2\hat{\alpha}_0 - \hat{\alpha}_1 \ , \\
\tilde{\delta}_{2,1} &= (1/9 + \hat{\beta}_1)\delta_{2,0} \ , \quad \hat{\beta}_1 := 2\hat{\alpha}_1 - \hat{\alpha}_0 - \hat{\alpha}_2 \ .
\end{aligned} \tag{29}$$

The asterique notation has the same meaning as in Subsection 2.1. By shift-invariance, similar equalities hold for all $\tilde{\delta}_{2,i}$.

Since we are looking for an analog of the simpler inequality (19), straightforward estimates for the quantities $\hat{\alpha}_i, \hat{\beta}_i$ will suffice. From the formulas given in (27) and (29), we find

$$\hat{\phi}(c) := \max_{i=1,\ldots,3} (|\hat{\alpha}_i|, |\hat{\beta}_i|) \leq \frac{4}{135} , \tag{30}$$

with equality attained if and only if $c = 0$ in which case $\hat{\alpha}_0 = \hat{\alpha}_2 = \hat{\beta}_0 = \hat{\beta}_2 = 4/135$. This can again easily be verified by elementary calculus. Using (30) in conjunction with (29), we arrive at

$$|\tilde{\delta}_{2,0}| \leq (\frac{1}{9} + |\hat{\alpha}_2^*| + \frac{2}{9} + |\hat{\beta}_0|) \max(|\delta_{2,-1}|, |\delta_{2,0}|) \leq \frac{53}{135} \Delta_2(m) ,$$

similarly for the other $\tilde{\delta}_{2,i}$. Thus, we have

Theorem 2 *Let $r = 3$, $m \in \ell_\infty(\mathbb{Z})$, and $\widetilde{m} = S_{\mathrm{med}} m$. Then we have*

$$\Delta_2(\widetilde{m}) \leq \frac{53}{153} \Delta_2(m) , \tag{31}$$

The constant in (46) is sharp (equality is achieved for $m = (\ldots, 0, 1, 0, 1, \ldots)$).

3 Hölder regularity results

The Hölder class $C^\alpha = C^\alpha(\mathbb{R})$ is conventionally defined for $l \leq \alpha < l+1$ and integer $l \geq 0$ as the class of l times continuously differentiable functions $f(x)$ where (for $0 < \beta := \alpha - l < 1$) the seminorm

$$|f|_{C^\alpha} := \sup_{x \in \mathbf{R},\, h>0} h^{-\beta} |f^{(l)}(x+h) - f^{(l)}(x)|$$

is finite. Alternatively, one can define the same spaces by using higher order differences of the functions $f(x)$ themselves. There is some ambiguity about the definition for integer $\alpha = l$ (in our definition C^l coincides with the space of l times continuously differentiable functions) but we will not go into these issues. A norm in C^α can be defined by setting

$$\|f\|_{C^\alpha} := \|f\|_C + |f|_{C^\alpha} ,$$

where $\|f\|_C := \sup_{x \in \mathbf{R}} |f(x)|$ and $|f|_{C^l} = \|f^{(l)}\|_C$ for integer l.

We now derive the smoothness results for S_{med} . Let

$$s_\infty(S) = \inf_{m \in \ell_\infty(\mathbf{Z})} \sup\{\alpha \geq 0 \, : \, f(\cdot; m) \in C^\alpha(\mathbb{R})\} ,$$

denote the *exact Hölder exponent* associated with the subdivision scheme S. Here, $f(\cdot; m)$ is the limit function obtained from the subdivison scheme with initial m, as defined in (5). For the S considered in this paper, the existence and continuity of $f(\cdot; m)$ is guaranteed, and $s_\infty(S)$ well-defined.

Theorem 3 *The Hölder exponent for the quadratic median-interpolation scheme S_{med} satisfies*

$$s_\infty(S_{\mathrm{med}}) > \log_4(128/31) = 1.0229... , \quad r = 2 ,$$

resp.

$$s_\infty(S_{\mathrm{med}}) \geq \log_3(135/53) = 0.8510... , \quad r = 3 .$$

For comparison, the linear subdivision schemes based on midpoint resp. average interpolation satisfy

$$s_\infty(S_{\mathrm{mid}}) = \log_2(16/7) = 1.1926... , \qquad s_\infty(S_{\mathrm{ave}}) = 1 , \quad r = 2 ,$$

resp.

$$s_\infty(S_{\mathrm{mid}}) = 1 , \qquad s_\infty(S_{\mathrm{ave}}) = 0.8173... , \quad r = 3 ,$$

Proof. We will use the following corollary to [5, Theorem 3.3] (the proof given for $r = 2$ in [5] easily extends to arbitrary dilation factors). Fix $r \geq 2$, and let the linear subdivision operator $\hat{S}$ be defined by

$$(\hat{S}m)_i = \sum_{l \in \mathbf{Z}} \hat{s}_{i-rl} m_l , \quad k \in \mathbb{Z} ,$$

with finitely supported mask $\{\hat{s}_i : i \in \mathbb{Z}\}$. Assume that $\hat{S}$ has order $K \geq 1$ of polynomial reproduction, for precise definitions, see [5, 3]. In the examples considered in this section, we always have $K = 3$. Define $\Delta_k(m) := \|\delta_k(m)\|_{\ell_\infty(\mathbf{Z})}$ as the ℓ_∞-norm of the k-th order difference sequence

$$\delta_k(m) := \delta(\delta_{k-1}(m)) , \quad k \geq 1 \quad (\delta_0(m) = m)$$

of the sequence $m \in \ell_\infty(\mathbb{Z})$ which is consistent with our earlier definitions for $k = 1, 2$. The first order difference operator $\delta \equiv \delta_1$ is defined by $\delta(m)_i = m_{i+1} - m_i$, $i \in \mathbb{Z}$.

Proposition 3 *Let $r \geq 2$, let $\hat{S}$ be as described. Suppose that for some $k = 1, \ldots, K$, there are constants $C_\mu, \mu > 0$ such that*

$$\Delta_k(\hat{S}^j m) \leq C_\mu \|m\|_{\ell_\infty(\mathbf{Z})} r^{-j\mu} , \qquad j \geq 0 . \tag{32}$$

If $\tilde{S}$ is some well-defined but otherwise arbitrary (linear or nonlinear) subdivision operator such that

$$\|\tilde{S}^{j+1} m - \hat{S}\tilde{S}^j m\|_{\ell_\infty(\mathbf{Z})} \leq C_\nu \|m\|_{\ell_\infty(\mathbf{Z})} r^{-j\nu} , \qquad j \geq 0 , \tag{33}$$

for some constants $C_\nu, \nu > 0$, then

$$s_\infty(\tilde{S}) \geq \min(\mu, \nu) , \qquad s_\infty(\hat{S}) \geq \mu . \tag{34}$$

We have everything ready in order to apply this result. Set $\tilde{S} = S_{\mathrm{med}}$, $\hat{S} = S_{\mathrm{mid}}$, $K = 3$, and $k = 2$. Denote $m^j := \tilde{S}^j m$. By Proposition 1 ($r = 2$) resp. 2 ($r = 3$) we have

$$\|\tilde{S}^{j+1} m - \hat{S}\tilde{S}^j m\|_{\ell_\infty(\mathbf{Z})} = \|\tilde{S}m^j - \hat{S}m^j\|_{\ell_\infty(\mathbf{Z})} \leq C\Delta_2(m^j) .$$

For $r=2$, the recursive application of (20) implies

$$\Delta_2(m^j) \le \frac{31}{128}\Delta_2(m^{j-2}) \ \le C\left(\frac{31}{128}\right)^{[j/2]} \max(\Delta_2(m^0),\Delta_2(m^1)) \le C\|m\|_{\ell_\infty(\mathbf{Z})}2^{-j\nu}$$

for all $j \ge 0$, where $\nu = \log_4(128/31)$. We can even take a slightly larger ν as follows from the proof of (20). If $r=3$ then from (31) we conclude that

$$\Delta_2(m^j) \le \left(\frac{53}{153}\right)^j \Delta_2(m) \ \le 4\|m\|_{\ell_\infty(\mathbf{Z})}3^{-j\nu}\ , \quad j \ge 0\ ,$$

where $\nu = \log_3(135/53)$. This gives (33), with values ν depending on r.

The other assumption (32) follows directly from (21), (22) (for $r=2$) resp. (29) if we drop the extra terms which describe the "difference" between $S_{\rm med}m$ and $S_{\rm mid}m$. This gives

$$\Delta_2(\hat{S}m) \le \frac{3+11}{32}\Delta_2(m) = \frac{7}{16}\Delta_2(m)\,, \quad r=2\ ,$$

resp.

$$\Delta_2(\hat{S}m) \le \frac{1+2}{9}\Delta_2(m) = \frac{1}{3}\Delta_2(m)\,, \quad r=3\ ,$$

and yields (32) with $\mu = \log_2(16/7)$ resp. $\mu = \log_3(3) = 1$.

Now, Proposition 3 implies almost all the statements in Theorem 3. The exact Hölder exponents for the linear schemes $S_{\rm mid}$ and $S_{\rm ave}$ can be obtained by standard methods from [8, 4, 15, 16]. As an example, let us consider the linear scheme $S_{\rm mid}$ for $r=2$ whose explicit formulas are given by (18). Note that in contrast to the case $r=3$ (compare the formulas in (28)), this is *not* an interpolatory scheme. If we associate the support of its mask with $\{-2,\ldots,3\}$, its symbol $P(z)$ ($z=e^{-i\omega}$) takes the form

$$P(z) = \frac{1}{64}(-3(z^{-2}+z^3)+5(z^{-1}+z^2)+30(1+z^2)) = \frac{(1+z)^3}{64z}(14-6\cos\omega)\ .$$

Thus, $P(z)$ satisfies sum rules of order 3 which corresponds to quadratic polynomial reproduction. We state without proof that the stability condition is satisfied and that $s_2(S_{\rm mid}) = 1.6469...$, where

$$s_2(S) = \inf_{m\neq 0}\sup\{s\ge 0\ :\ f(\cdot;m)\in H^s(\mathbb{R})\}\ ,$$

denotes the exact Sobolev smoothness exponent of the limit functions of the subdivision scheme S. For the numerical computation of smoothness exponents, we have used the Matlab routines developed for [10], see also [1].

To compute the Hölder smoothness exponent, we will consider the factorized symbol

$$P_1(z) = \frac{1}{8}\cos^2\frac{\omega}{2}(14-6\cos\omega)\ ,$$

which is the symbol of the derived subdivision scheme $S_{\rm mid,1}$ associated with the first order divided differences, and make use of the fact that $P_1(z)\ge 0$ for all $z=e^{-i\omega}$. Thus, according to a result for

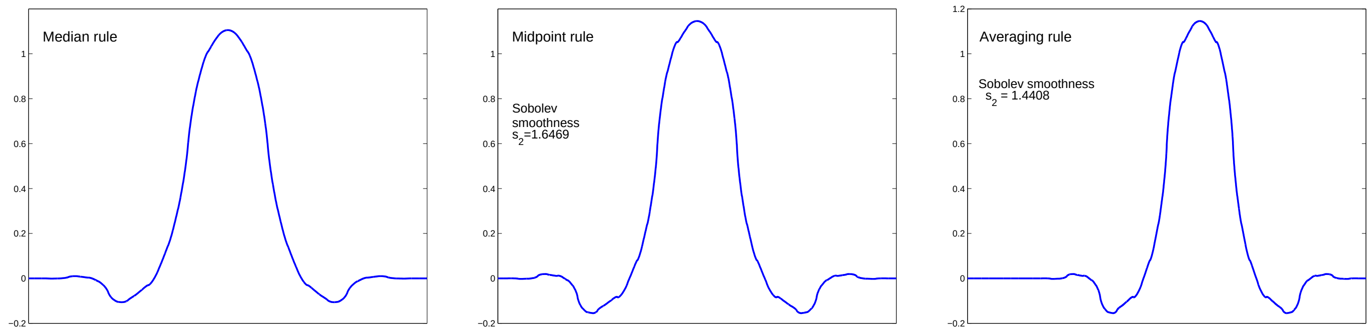

Figure 1: Limit functions for the Dirac sequence (dyadic case)

scalar subdivision schemes with non-negative symbols (see, e.g., [16, Proposition 4.5]), $s_\infty(S_{\mathrm{mid},1})$ can be computed from the eigenvalues of a finite matrix which yields

$$s_\infty(S_{\mathrm{mid}}) = 1 + s_\infty(S_{\mathrm{mid},1}) = 1 + \log_2(8/7) = \log_2(16/7) \ .$$

Details (as well as the consideration of $r = 3$ and of S_{ave})) are left to the reader.

Remark 2 *Figure 1 shows the graphs of the limit functions by applying the three schemes for $r = 2$ to a Dirac sequence $m = (\ldots, 0, 0, 1, 0, 0, \ldots)$. They reveal that S_{med} indeed seems to be a close relative of S_{mid}. For the linear schemes, the numerically computed Sobolev exponents are also shown.*

Remark 3 *The following table records the numbers $\rho_l^j(m) = \Delta_l(m^j)/\Delta_l(m^{j-1})$, $j = 1, \ldots, 10$, $l = 1, 2$, for the Dirac sequence $(\ldots, 0, 1, 0, \ldots)$ and for the alternating sequence $(\ldots, 0, 1, 0, 1, 0, \ldots)$ as initial sequence for the subdivision, see columns 1 and 2. Recall that the alternating sequence turned out to be extremal for the estimates (19) and (25) which is reflected in the corresponding values in the first row. For both sequences ρ_2^j very quickly approaches the value $7/16 = 0.4375$, while ρ_1^j tends to $1/2$. The same pattern is observed for randomly generated initial sequences (columns 3-5, the sequence for column 3 was monotone). Since the best constant in*

$$\Delta_2(m^{j+s}) \le \gamma_{2,s}\Delta_2(m^j) \ , \tag{35}$$

is given by

$$\gamma_{2,s} = \sup_{m\in\ell_\infty(\mathbf{Z})} \prod_{i=1}^{s} \rho_2^i(m) \ ,$$

and delivers a lower bound

$$s_\infty(S_{\mathrm{med}}) \ge \liminf_{s\to\infty} \log_2(\gamma_{2,s}^{-1/s}) = \liminf_{s\to\infty} (\min_m \frac{1}{s}\sum_{i=1}^{s} \log_2(1/\rho_2^i(m))) \ , \tag{36}$$

(this can be shown in the same way as was done in Theorem 3), we see strong numerical support for the conjecture that

$$s_\infty(S_{\mathrm{med}}) = s_\infty(S_{\mathrm{mid}}) \ . \tag{37}$$

However, this as well as the corresponding conjecture from [6] remain open.

	Dirac		Alternating		Random					
j	ρ_1^j	ρ_2^j	ρ_1^j	ρ_2^j	ρ_1^j	ρ_2^j	ρ_1^j	ρ_2^j	ρ_1^j	ρ_2^j
1	0.8438	0.4219	1.0000	0.5000	0.5672	0.3437	0.7389	0.4373	0.7952	0.4812
2	0.6578	0.4078	0.6875	0.4375	0.5518	0.3599	0.6415	0.4280	0.6476	0.4337
3	0.6038	0.4123	0.6164	0.4268	0.5411	0.4082	0.5965	0.4336	0.5997	0.4222
4	0.5752	0.4276	0.5826	0.4375	0.5332	0.4174	0.5708	0.4353	0.5823	0.4362
5	0.5572	0.4317	0.5620	0.4375	0.5273	0.4339	0.5542	0.4362	0.5651	0.4368
6	0.5449	0.4341	0.5483	0.4375	0.5226	0.4259	0.5428	0.4368	0.5504	0.4371
7	0.5361	0.4356	0.5385	0.4375	0.5189	0.4365	0.5345	0.4371	0.5400	0.4373
8	0.5294	0.4364	0.5313	0.4375	0.5160	0.4369	0.5282	0.4373	0.5324	0.4374
9	0.5243	0.4369	0.5258	0.4375	0.5135	0.4372	0.5234	0.4374	0.5267	0.4374
10	0.5203	0.4371	0.5214	0.4375	0.5115	0.4373	0.5196	0.4374	0.5221	0.4375

Table 1: Numerical behavior of the ratios ρ_r^j

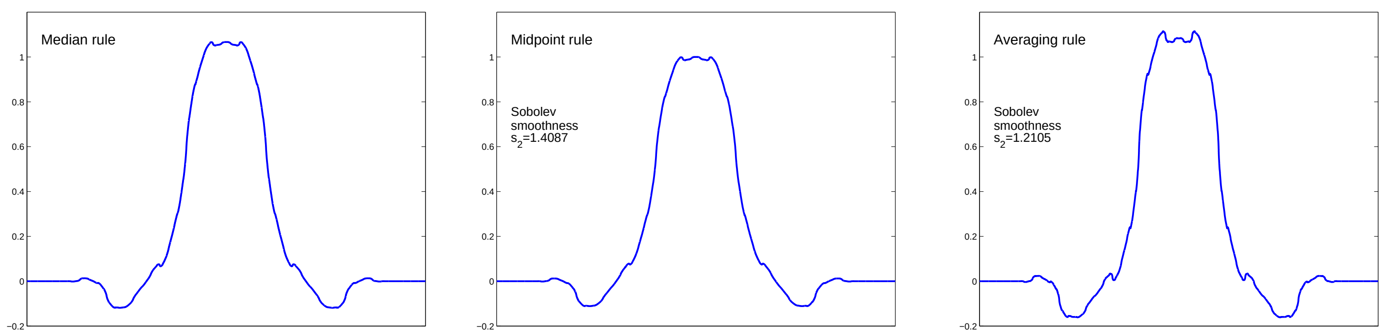

Figure 2: Limit functions for the Dirac sequence (triadic case)

Remark 4 *In Figure 2, we show the graphs of the limit functions (after* 10 *subdivision steps) corresponding to the Dirac sequence* $m = (\ldots, 0, 0, 1, 0, 0, \ldots)$ *for the triadic schemes. Again the similarity w.r.t. smoothness between* S_{med} *and* S_{mid} *is evident.*

We conclude with a few remarks about extensions and open problems directly related to the above case study. It is our conviction that the subject will attract further attention and could be a source of some "cross-fertilization" with iterated mapping theory and nonlinear analysis.

- As was pointed out in [9] the median $\text{med}(f; I)$ of a continuous function can equivalently be defined as the L_1-best constant approximation of f on the interval I. Obviously, the average value $\text{ave}(f; I)$ corresponds to the case of L_2-best constant approximation. What about using the L_p-best constant approximation instead? This would lead to a family of nonlinear schemes S_{L_p} which depends on the parameter $1 \le p \le \infty$ and contains $S_{\text{med}} = S_{L_1}$ and $S_{\text{ave}} = S_{L_2}$ as special cases. In particular, $p = \infty$ leads to another nonlinear subdivision scheme which can be treated in a fashion similar to the case $p = 1$ of median interpolation (see the forthcoming paper [14]). The associated linear scheme should be called *endpoint-average interpolation scheme* since for monotone f the L_∞-best constant approximation on

$I=[\alpha,\beta]$ is given by $(f(\alpha)+f(\beta))/2$. For $p\neq 1,2,\infty$, it is not clear at all how to define such an associated linear scheme. Judging from the particular cases treated so far, it is natural to conjecture that the smoothness exponents associated with S_{L_p} decrease with increasing p.

- The case $k=2$ (quadratic polynomial interpolation/imputation) has been emphasized since it is the only case where the median interpolation problem can be found by explicit formulas. This is a prerequisite for the above approach to smoothness analysis. Since [7, 6] only defines schemes for even k, their next complicated case would involve 4-th degree polynomials, and hardly seems accessible to an elementary analysis.

  The natural generalization of the Donoho-Yu schemes for odd k and $r=2$ is as follows. Let $k=2k'-1$, $k'\geq 1$, be given. In the interpolation step, we find a unique polynomial p_{med} of degree $\leq k$ such that

$$\mathrm{med}(p_{\mathrm{med}};[2\nu,2(\nu+1)])=m_{i+\nu},\quad \nu=-k',\dots,k'-1,$$

  after which we find $\widetilde{m}_{2i-1},\widetilde{m}_{2i}$ by median imputation on the intervals $[-1,0]$, $[0,1]$, respectively. Repeating this local procedure for all $i\in\mathbb{Z}$, we define $\tilde{m}=S_{\mathrm{med}}m$. For other interpolation functionals, we proceed in a similar fashion.

  For $k=1$, i.e., when linear polynomials are used for the interpolation/imputation steps, the above considered schemes are the same, i.e., $S_{\mathrm{med}}=S_{\mathrm{mid}}=S_{L_p}$ for all $1\leq p\leq\infty$, and incidentally coincide with the quadratic B-spline subdivision scheme,

$$\widetilde{m}_{2i-1}=\frac{3}{4}m_{i-1}+\frac{1}{4}m_i,\quad \widetilde{m}_{2i}=\frac{1}{4}m_{i-1}+\frac{3}{4}m_i,$$

  which possesses relatively high smoothness exponents, i.e.,

$$s_\infty(S_{\mathrm{med}})=2\ ,\qquad s_2(S_{\mathrm{med}})=5/2\qquad(\ k=1\ )\ .$$

  The scheme has an order of polynomial reproduction $K=3$, although it is defined only with linear polynomials. We do not know whether this classical linear scheme behaves in a satisfactory way w.r.t. the heavy tail noise removal problems discussed in [7, 6].

  For $k=3$ (cubic polynomials) we have preliminarily investigated the linear midpoint scheme S_{mid}. Its explicit formula is

$$\begin{aligned}\tilde{m}_{2i-1}&=-\frac{7}{128}m_{i-2}+\frac{105}{128}m_{i-1}+\frac{35}{128}m_i-\frac{5}{128}m_{i+1}\ ,\\ \tilde{m}_{2i}&=-\frac{5}{128}m_{i-2}+\frac{35}{128}m_{i-1}+\frac{105}{128}m_i-\frac{7}{128}m_{i+1}\ ,\end{aligned}$$

  $i\in\mathbb{Z}$. The symbol is $P(z)=(1+z)^5(18-5(z+z^{-1}))/256$, the forefactor reveals $K=5$, the smoothness computations show

$$s_\infty(S_{\mathrm{mid}})=2.8310...\ ,\qquad s_2(S_{\mathrm{mid}})=3.2596...\qquad(\ k=3\ )\ .$$

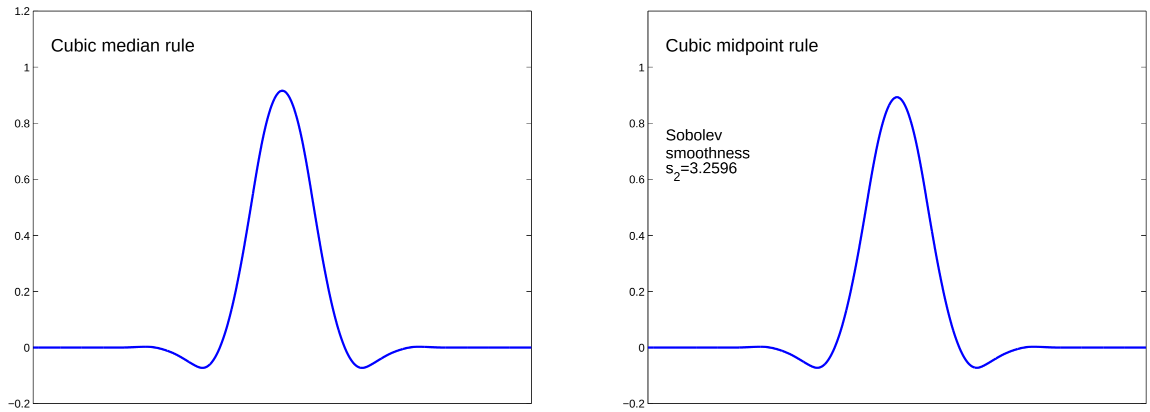

Figure 3: Limit functions for the Dirac sequence (cubic case $k=3$, $r=2$)

Thus, the associated nonlinear cubic median-interpolation scheme $S_{\rm med}$ might be an interesting scheme to look at. It can be shown that $\mathrm{med}(p;I)$ (and consequently the imputation step) can be computed for any cubic polynomial p and interval I by explicit formulas. However, for cubic median interpolation we still have to resort to a numerical scheme (see, e.g., [7]). Numerical tests indicate that limit functions for $S_{\rm med}$ are certainly in $C^2(\mathbb{R})$, and that $s_\infty(S_{\rm med}) = s_\infty(S_{\rm mid})$ seems probable again. We have currently no means to verify either of these conjectures. Figure 3 shows the graphs of the limit functions for the Dirac sequence for both $S_{\rm med}$ and $S_{\rm mid}$.

- Let us return to the connection of our case study with [3]. For simplicity, consider the case $k=r=2$ of $S_{\rm med}$, see Subsection 2.1. From (21), (22) we can identify a family $\mathcal{S}_2 = \{S_{2,w} : w \in \ell_\infty(\mathbb{Z})\}$ of derived subdivision schemes for second order differences associated with $S_{\rm med}$. More precisely, these formulas imply that

$$\delta_2(S_{\rm med} m) = S_{2,m}\delta_2(m) \;, \tag{38}$$

where $S_{2,m}$ is a linear operator represented by a bi-infinite block-matrix with coefficients depending on m and acting on $\ell_\infty(\mathbb{Z})$ according to

$$\begin{pmatrix} \tilde\delta_{2i-1} \\ \tilde\delta_{2i} \end{pmatrix} = \begin{pmatrix} 11/32+\beta_1^{i-1} & -(3/32+\alpha_0^i) \\ -(3/32+\alpha_1^{i-1}) & 11/32+\beta_0^i \end{pmatrix} \begin{pmatrix} \delta_{i-1} \\ \delta_i \end{pmatrix} \;, \quad i \in \mathbb{Z} \;. \tag{39}$$

Here we have denoted $\delta_2 := \delta_2(m)$, $\tilde\delta_2 := \delta_2(S_{\rm med} m)$. The expressions and estimates for $\alpha_0^i, \alpha_1^i, \beta_0^i, \beta_1^i$ as functions of the parameter $c = c^i$ associated with the interpolation/imputation step for the data (m_{i-1}, m_i, m_{i+1}) can be found in Proposition 1 and Subsection 2.1.

In [3, Proposition 1], the existence of such a family $\mathcal{S}_2$ is shown to be the consequence of uniform locality and polynomial reproduction of order ≥ 2. As suggested in [3], the smoothness analysis in Hölder-Besov spaces of the original scheme $S_{\rm med}$ can be facilitated by using the ℓ_p joint spectral radius of $\mathcal{S}_2$. As this example shows, the definition of ℓ_p joint spectral radii, given in [3] for arbitrary families $\mathcal{S} = \{S_w : w \in \ell_p(\mathbb{Z})\}$ of linear maps acting on $\ell_p(\mathbb{Z})$, $1 \le p \le \infty$, by

$$\rho_p(\mathcal{S}) = \limsup_{j\to\infty} \sup_{w^0,\ldots,w^{j-1}} \|S_{w^{j-1}} \ldots S_{w^0}\|_{\ell_p}^{1/j}, \tag{40}$$

will sometimes lead to overly pessimistic results. Indeed, from [3] we deduce that $s_\infty(S_{\mathrm{med}}) \ge \log_2(1/\rho_\infty(\mathcal{S}_2))$, where $\mathcal{S}_2$ is given by (39). However, since $\rho_\infty(\mathcal{S}_2) = 1/2$, this does not yield the best possible result. The upper bound $\rho_\infty(\mathcal{S}_2) \le 1/2$ follows from (19). The constant $1/2$ cannot be improved since by taking $w^s = w := (\ldots, 0, 1, 0, 1, \ldots)$ for all s we obtain

$$\rho_\infty(\mathcal{S}_2) \ge \limsup_{j\to\infty} \|S_{2,w}^j\|_{\ell_\infty}^{1/j} \ge \rho_\infty(S_{2,w}) = 1/2 \; . \tag{41}$$

The last equality is obvious since for the alternating sequence $w = (\ldots, 0, 1, 0, 1, \ldots)$ we have $c^i = 0$ and $\alpha_0^i = \alpha_1^i = \beta_0^i = \beta_1^i = 1/32$ which means that all 2×2 submatrices occuring in the matrix representation of $S_{2,w}$ (see the definition (39)) are of the form

$$A = \begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix} \; .$$

With this, the spectral radius value $\rho_\infty(S_{2,w}) = 1/2$ is obtained.

It follows from (41) that the theorems in [3], if used as is, fail to establish any C^α, $\alpha \ge 1$, for S_{med} and $r = 2$. This can easily be repaired by replacing the definition (40) by its following nonlinear version:

$$\rho_p(\mathcal{S}, \tilde{S}) = \limsup_{j\to\infty} \sup_{w} \|S_{\tilde{S}^{j-1}w} \ldots S_{\tilde{S}w} S_w\|_{\ell_p}^{1/j} \; . \tag{42}$$

Our estimate (20) provides an upper bound for $\rho_\infty(\mathcal{S}_2, S_{\mathrm{med}})$ by considering $j = 2$, and this upper bound leads to a lower bound for $s_\infty(S_{\mathrm{med}})$. For further details, see also [14].

4 Appendix

Proof of Lemma 1. This is a consequence of (11). Indeed, if $c \ne (-5/2, -3/2) \cup (-1/2, 1/2) \cup (3/2, 5/2)$ then on all three intervals I^i, $i = -1, 0, 1$, the first case in (11) applies, and we have $p_{\mathrm{med}}(x) = p_{\mathrm{mid}}(x)$ and

$$\delta_2 = p_{\mathrm{med}}(-2) - 2p_{\mathrm{med}}(0) + p_{\mathrm{med}}(2) = 8a \; .$$

This gives (14) and (15) for this range of c.

Let us next consider the case $c \in I_A := (-1/2, 1/2)$. Then by (11) we have

$$m_{-1} = p_{\mathrm{med}}(-2) = a(c+2)^2 + b \; , \quad m_1 = p_{\mathrm{med}}(2) = a(c-2)^2 + b \; ,$$

which yields

$$a = -\frac{m_1 - m_{-1}}{8c} \; , \quad b = \frac{m_1 + m_{-1}}{2} + \frac{m_1 - m_{-1}}{8c}(4 + c^2) \; .$$

Moreover, for I^0 we have

$$m_0 = p_{\mathrm{med}}(c \pm 1/2) = \frac{a}{4} + b = \frac{m_1 + m_{-1}}{2} + \frac{m_1 - m_{-1}}{8c}(\frac{15}{4} + c^2) \; .$$

For future use, let us set $q^+ := (m_1 - m_0)/\delta_2$ and $q^- := (m_0 - m_{-1})/\delta_2 = q^+ - 1$. Then from the last equality we derive

$$q^+ + q^- = \frac{m_1 - m_{-1}}{\delta_2} = -\frac{4c}{15/4 + c^2} = -\frac{16c}{15 + 4c^2} = -\frac{16c}{16 - \epsilon_0} . \tag{43}$$

This equality can be used to express $c \in I_A$ as a function of the interpolation data m_i, $i = -1, 0, 1$. In particular, from (43) it follows that $c \in I_A$ if and only if $q^+ + q^- \in I_A$. Moreover, from (43) and the above expression for a we have

$$p_{\text{med}}(0) - m_0 = p_{\text{med}}(0) - p_{\text{med}}(c \pm 1/2) = a(c^2 - 1/4) = \frac{(m_1 - m_{-1})\epsilon_0}{32c} = -\frac{\epsilon_0 \delta_2}{32 - 2\epsilon_0} ,$$

as well as $a = 2\delta_2/(16 - \epsilon_0)$. The latter formula gives (15) for $c \in I_A$. To see (14) observe that according to the above the difference $p_{\text{med}}(x) - p_{\text{mid}}(x)$ coincides with the quadratic midpoint interpolant for the data $(0, -\epsilon_0\delta_2/(32 - 2\epsilon_0), 0)$ instead of (m_{-1}, m_0, m_1). Now (12) gives the result.

Finally, we treat the case $c \in I_B := (3/2, 5/2)$ (the remaining case $c \in -I_B$ follows by symmetry). Here, from (11) we see that

$$m_{-1} = p_{\text{med}}(-2) = a(c+2)^2 + b , \quad m_0 = p_{\text{med}}(0) = ac^2 + b ,$$

which implies

$$a = -\frac{m_0 - m_{-1}}{4(1+c)} , \quad b = m_0 + \frac{m_0 - m_{-1}}{4(1+c)} c^2 ,$$

and that for I^1 we have

$$\begin{aligned} m_1 &= p_{\text{med}}(c \pm 1/2) = \frac{a}{4} + b = m_0 + \frac{m_0 - m_{-1}}{4(1+c)}(c^2 - \frac{1}{4}) \\ &= 2m_0 - m_{-1} + \frac{m_0 - m_{-1}}{4(1+c)}((c-2)^2 - \frac{33}{4}) . \end{aligned}$$

From this we get the analog of (43),

$$q^- = -\frac{16(1+c)}{32 + \epsilon_2} , \tag{44}$$

which, in turn, implies

$$p_{\text{med}}(2) - m_1 = -\frac{\epsilon_2 \delta_2}{32 + \epsilon_2} , \quad a = \frac{4\delta_2}{32 + \epsilon_2} .$$

This gives (14) and (15) for $c \in I_B$. Lemma 1 is established.

Proof of (20) in Theorem 1. Applying (19) twice gives $\Delta_2(\tilde{\tilde{m}}) \le \Delta_2(m)/4$ which is weaker than (20). To establish the stronger (20), we use the following argument. Consider the subpattern (m_{-1}, m_0, m_1). The associated interpolation/imputation step is governed by the parameter c. The first fact we need is that $c \notin I'_A := (-3/7, 3/7) \subset I_A$ implies

$$\max(\alpha_0, \alpha_1, \beta_0, \beta_1) \le \frac{125}{32 \cdot 257} < \frac{1}{64} . \tag{45}$$

Let the maximum in the left-hand side of (45) be denoted by $\phi(c)$. Then, obviously $\phi(-c) = \phi(c)$, and we need only to look at non-negative c. By examining the formulas for α_i, β_i, we find that

$$\phi(c) = \frac{13\epsilon_2}{32(32+\epsilon_2)} \le \frac{13}{32 \cdot 33}\,, \quad c \in I_B\,, \qquad \phi(c) = \frac{\hat{\epsilon}_{1/2}}{4 \cdot 32} \le \frac{1}{4 \cdot 32}\,, \quad c \in [1/2, 3/4]\,.$$

For all other $c \in [3/4, \infty)\backslash I_B$, we obviously have $\phi(c) = 0$. Finally, consider $c \in [1/4, 1/2]$. Then

$$\phi(c) = \frac{15\epsilon_0 + 4\hat{\epsilon}_{1/2}}{16(32 - 2\epsilon_0)} = \frac{3 + 64c - 124c^2}{32(15 + 4c^2)}\,.$$

It is easy to check that this expression is monotonously decreasing on $[3/7, 1/2]$, thus,

$$\phi(c) \le \phi(\frac{3}{7}) = \frac{125}{32 \cdot 257}\,, \quad c \in [\frac{3}{7}, \frac{1}{2}]\,.$$

Comparing all estimates, we arrive at (45).

Consider an arbitrary $\tilde{\tilde{\delta}}_{2,i''}$. In its estimation, two local interpolation/imputation steps, with parameters $\tilde{c}^*$, and $\tilde{c}$, and two second order differences of $\widetilde{m}$ are involved. Let $\tilde{\delta}_{2,i'}$ be the one with maximal absolute value. If we now assume that at least one of the two parameters $\tilde{c}^*$, $\tilde{c}$ does not belong to I'_A then by (45) and the same estimation procedure we have

$$|\tilde{\tilde{\delta}}_{2,i''}| \le \frac{31}{64} |\tilde{\delta}_{2,i'}|\,.$$

The same argumentation can now be applied to $\tilde{\delta}_{2,i'}$, i.e., we find c^*, c, and $\delta_{2,i}$ such that

$$|\tilde{\delta}_{2,i'}| \le \frac{31}{64} |\delta_{2,i}|\,,$$

if at least one of the parameters c^*, c does not belong to I'_A. Combining the two statements into one, we arrive at the following intermediate result: *For each $\tilde{\tilde{\delta}}_{2,i''}$, there exists a well-defined "chain" $\tilde{\tilde{\delta}}_{2,i''} \to \tilde{\delta}_{2,i'} \to \delta_{2,i}$, with associated parameters $\tilde{c}^*, \tilde{c}, c^*, c$, such that*

$$|\tilde{\tilde{\delta}}_{2,i''}| \le \frac{31}{128} |\tilde{\delta}_{2,i}|\,, \tag{46}$$

if at least one of the parameters $\tilde{c}^*, \tilde{c}, c^*, c$ does not belong to I_A. The bound in (46) corresponds to the worst situation when only one parameter does not belong to I'_A, giving one inequality with constant $31/64$, the other with $1/2$ from (19). The constant can be slightly improved upon by replacing $1/64$ by the sharper bound from (45) or by optimizing the size of I'_A but we will not pursue this here.

Thus, (20) is established if we can show that having $\tilde{c}^*, \tilde{c}, c^*, c \in I'_A$ simultaneously is impossible. After an appropriate index shift, we can assume that both indices i and i' are in $\{-1, 0\}$, i.e., the parameters c^*, c are associated with the data (m_{-2}, m_{-1}, m_0), (m_{-1}, m_0, m_1), respectively. Observe that $c^*, c \in I'_A$ is equivalent to

$$\hat{q} := \frac{m_0 - m_{-1}}{m_0 - m_1},\ \hat{q}^* := \frac{m_0 - m_{-1}}{m_{-2} - m_{-1}} \in (q_0, q_0^{-1})\,, \qquad q_0 = \frac{145}{369}\,. \tag{47}$$

Indeed, since $I'_A \subset I_A$, we can use the definition of q^+, q^- and (43) to find that $\hat{q} = -q^+/q^-$ satisfies

$$\xi := \frac{16c}{15+4c^2} = -\frac{q^+ + q^-}{q^+ - q^-} = \frac{1-\hat{q}}{1+\hat{q}} .$$

Since $c \in I'_A$ is equivalent to $|\xi| < 112/257$ and $\hat{q} = (1-\xi)/(1+\xi)$, we see that (47) hgolds for $\hat{q}$. The statement for $\hat{q}^*$ follows by symmetry.

Next, we will check that the corresponding subpattern $(\widetilde{m}_{-2}, \widetilde{m}_{-1}, \widetilde{m}_0, \widetilde{m}_1)$ of $\widetilde{m}$ which solely depends on $(m_{-2}, m_{-1}, m_0, m_1)$ *cannot* satisfy the analog of (47). Without loss of generality, assume that $m_{-1} = 1$, $m_0 = 0$. A straightforward calculation along the lines of Lemma 1 and 2 shows that $\widetilde{m}_0$ resp. $\widetilde{m}_1$ is monotonously increasing resp. decreasing as a function of $c \in I'_A$, and that

$$\widetilde{m}_0 \in (-\frac{49}{580}, \frac{40}{369}) , \quad \widetilde{m}_1 \in (-\frac{49}{1476}, \frac{40}{145}) , \qquad c \in I'_A$$

(the tight interval bounds are computed from considering the extremal cases $c = \pm 3/7$)). We also note that $\widetilde{m}_1 - \widetilde{m}_{-1} > 0$ only if $c < 0$, since for $c = 0$ we have $\widetilde{m}_1 = \widetilde{m}_{-1} = 0$. Similarly, by symmetry

$$\widetilde{m}_{-1} \in (1 - \frac{40}{369}, 1 + \frac{49}{580}) , \quad \widetilde{m}_{-2} \in (1 - \frac{40}{145}, 1 + \frac{49}{1476}) , \qquad c^* \in I'_A .$$

In particular, $\widetilde{m}_{-1} \geq 329/369$, and $\widetilde{m}_{-1} - \widetilde{m}_0 \geq 279/369$ are always positive. We leave the details to the reader.

With these preparations, we can show that

$$\tilde{\hat{q}} := \frac{\widetilde{m}_{-1} - \widetilde{m}_0}{\widetilde{m}_1 - \widetilde{m}_0}, \; \tilde{\hat{q}}^* := \frac{\widetilde{m}_0 - \widetilde{m}_{-1}}{\widetilde{m}_{-2} - \widetilde{m}_{-1}} < q_0 = \frac{145}{369} . \tag{48}$$

We concentrate on $\tilde{\hat{q}}$, the similar result for $\tilde{\hat{q}}^*$ again follows by symmetry arguments. If $c \in [0, 3/7)$ then $\tilde{\hat{q}} \leq 0$ since $\widetilde{m}_{-1} - \widetilde{m}_0 \leq 0$ and the numerator is positive. For $c \in [-1/4, 0)$, we have

$$\widetilde{m}_{-1} - \widetilde{m}_0 \geq \widetilde{m}_{-1} \geq \frac{329}{369} , \quad \widetilde{m}_1 - \widetilde{m}_0 \leq \frac{1}{9} + \frac{1}{15} = \frac{8}{45} ,$$

where the latter inequality follows since $\widetilde{m}_1 - \widetilde{m}_0$ is decreasing, and $\widetilde{m}_1 = 1/9$, $\widetilde{m}_0 = -1/15$ for $c = -1/4$. Finally, for $c \in [-3/7, -1/4]$ we get in a similar way

$$\widetilde{m}_{-1} - \widetilde{m}_0 \geq \frac{329}{369} + \frac{1}{15} \geq \frac{1768}{1845} , \quad \widetilde{m}_1 - \widetilde{m}_0 \leq \frac{40}{145} + \frac{49}{580} = \frac{209}{580} .$$

Together, this gives

$$\tilde{\hat{q}} \leq \max(\frac{369 \cdot 8}{329 \cdot 45}, \frac{1845 \cdot 209}{1768 \cdot 580}) = 0.376... < q_0 = \frac{145}{369} = 0.392... ,$$

and (48) is proved.

Since i' is either 0 or -1, the quotient $q' = (\widetilde{m}_{i'+1} - \widetilde{m}_{i'})/(\widetilde{m}_{i'-1} - \widetilde{m}_{i'})$ coincides with either $1/\tilde{\hat{q}}$ or $\tilde{\hat{q}}^*$. By (48) and using (47) in reverse direction (and with m replaced by $\widetilde{m}$), we conclude that at least one of the parameters $\tilde{c}^*$, $\tilde{c}$ does not belong to I'_A. Thus, it is impossible to have all four parameters in I'_A at the same time. This finishes the proof of (20).

References

[1] A. Barinka, S. Dahlke, N. Mulders, The IGPM Villemoes machine, Preprint No. 184, IGPM, RWTH Aachen, January 2000, `http://www.igpm.rwth-aachen.de/reports`.

[2] A. S. Cavaretta, W. Dahmen, C. A. Micchelli, Stationary subdivision, Memoirs AMS, vol. 93, AMS, Providence, RI, 1991.

[3] A. Cohen, N. Dyn, B. Matei, Quasilinear subdivision schemes with applications to ENO interpolation, manuscript, `http://www.ann.jussieu.fr/~cohen`.

[4] I. Daubechies, J. Lagarias, Two-scale difference equations II. Local regularity, infinite products of matrices and fractals, SIAM J. Math. Anal. 23, 4 (1992), 1031–1079.

[5] I. Daubechies, O. Runborg, W. Sweldens, Normal multiresolution approximation of curves, Tech. report, Bell Labs, April 2002, `http://cm.bell-labs.com/who/wim`.

[6] D. Donoho, T. P.-Y. Yu, Nonlinear pyramid transforms based on median interpolation, SIAM J. Math. Anal., 31, 5 (2000), 1030–1061.

[7] D. Donoho, T. P.-Y. Yu, Robust nonlinear wavelet transform based on median-interpolation, Preprint, Stanford University, 1997.

[8] N. Dyn, J. Gregory, D. Levin, Analysis of uniform binary subdivision schemes for curve design, Constr. Approx. 7 (1991), 127–147.

[9] T. N. T. Goodman, T. P.-Y. Yu, Interpolation of medians, Adv. Comput. Math. 11 (1999), 1–10.

[10] Q. Jiang, P. Oswald, Triangular $\sqrt{3}$-subdivision: The regular case, J. Comput. Appl. Math. (to appear), `http://cm.bell-labs.com/who/poswald`.

[11] F. Kuijt, R. van Damme, Shape preserving interpolatory subdivision schemes for nonuniform data, J. Approx. Th. 114, 1 (2002), 1–32.

[12] J.-L. Merrien, P. Sablonniere, Monotone and convex C^1 Hermite interpolants generated by an adaptive subdivision scheme, C.R. Acad. Sci. Paris, Ser. I, Math. 333, 5 (2001), 493–497.

[13] C. A. Micchelli, H. Prautzsch, Refinement and subdivision for spaces of integer translates of a compactly supported function, in Numerical Analysis, D. F. Griffith and G. A. Watson, eds. Acad. Press, New York, 1987, pp. 192–222.

[14] P. Oswald, Smoothness of nonlinear subdivision schemes, in Curves and Surfaces, St.-Malo 2002 (submitted), `http://cm.bell-labs.com/who/poswald`.

[15] O. Rioul, Simple regularity criteria for subdivision schemes, SIAM J. Math. Anal. 23 (1992), 1544–1576.

[16] L. Villemoes, Wavelet analysis of the refinement equation, SIAM J. Math. Anal. 25 (1994), 1433–1460.

[17] T. P.-Y. Yu, Developments in interpolating wavelet transforms, Ph.D. thesis, Program of Scientific Computing and Computational Mathematics, Stanford Univ., Stanford CA, 1997.