

# Subspace correction methods and multigrid theory

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## Abstract

This introduction to the modern theory of iterative subspace correction methods for solving symmetric positive definite variational problems in a Hilbert space will appear as an appendix in the forthcoming book *Multigrid Methods: Basics, Parallelism and Adaptivity* by U. Trottenberg, C. W. Oosterlee, A. Schüller. The basics of stable space splittings and the convergence properties of additive and multiplicative Schwarz methods are given in form of a discussion rather than a rigorous mathematical treatment. The standard applications to multigrid algorithms and domain decomposition schemes are covered. The examples are based on finite difference discretizations of Poisson's equation, and are adapted to the main material of the book which is oriented towards a more practically oriented user of multigrid methods in large-scale applications in the engineering sciences.

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1. Introduction
2. Space splittings
3. Convergence theory
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## 1 Introduction

This monograph concentrates on the efficient parallel implementation of multigrid methods for PDE discretizations and emphasizes *quantitative* multigrid theory. The present chapter is complementary, and provides a bridge to the state-of-the-art *qualitative* multigrid theory. By *qualitative* we mean the analysis of optimality of multigrid algorithms and other methods used in scientific computing in the asymptotic range, i.e., if the mesh parameter  $h$  in the discretization of the problem tends to 0 and, consequently, the number of equations tends to infinity. Even though we believe that the distinction between *quantitative* and *qualitative* multigrid theories is more of a philosophical nature, both appear to have their merits and shortcomings. E.g., using the qualitative theory, the optimal  $O(N)$  operation count of a FMG-cycle to solve a finite difference or finite element discretization of a second order elliptic PDE problem within discretization error can be rigorously justified under conditions that are much more general than required by the quantitative theory. Non-uniform grids and nested refinement can easily be treated. However, no specification of the size of the constants in the  $O(N)$  estimate can be given. Both approaches cannot give a satisfactory treatment of the robustness problem at large, they do not provide reliable performance estimates for problems with strongly varying coefficients or predominantly nonsymmetric or indefinite behavior such as advection-diffusion problems or Helmholtz equations. Some recent developments in the qualitative theory have led to a unified treatment of seemingly different types of iterative solution methods for operator equations, including multigrid algorithms, domain decomposition methods, fictitious domain methods, but also 'old-fashioned' block-iterative solvers. A simplified version of this modern theory of *subspace correction methods* will be outlined below, together with examples for multigrid and domain decomposition algorithms.

The basic idea of subspace correction methods consists in the following (in later places, we will replace the *matrix notation* used at this moment by an *operator notation* which will make it easier to see connections with other concepts, e.g., from applied Fourier analysis). Given a linear system

$$Lu = f \quad (1)$$

of large dimension  $N$ , in a subspace correction method we use a finite number of *auxiliary problems*

$$\tilde{L}_j \tilde{u}_j = \tilde{f}_j, \quad j = 1, \dots, J, \quad (2)$$

of usually smaller dimension  $N_j$  ( $N_1 + \dots + N_J \geq N$ ). Note that in some applications the  $\tilde{L}_j$  are just diagonal submatrices of  $L$  and the  $\tilde{u}_j$  subvectors of  $u$  which has led to the synonym *subproblems* or *subspace problems* for (2). However, the main implicit assumption is that any of the  $\tilde{L}_j$  is invertible, and that the auxiliary problems (2) can be solved fast, typically, in  $O(N_j)$  operations. Finally, simple *prolongation matrices*  $P_j$  of dimension  $N \times N_j$  and *restriction matrices*  $R_j$  of dimension  $N_j \times N$  are necessary to link the subproblems (2) to the original system (1). In analogy with classical iterative methods such as Jacobi and Gauss-Seidel iterations, we can now define the two prototypes of algorithms for the solution of (2) based on the subproblems (2) and the given set of prolongation and restriction matrices.

**Additive subspace correction method**  $u^{k+1} = AS(\omega, u^k, L, f; \tilde{L}_j, P_j, R_j)$

**(1) Residual computation**

- Compute  $r^k = f - Lu^k$ .

**(2) Restriction and solution of independent subspace problems**

- For  $j = 1, \dots, J$ , compute  $\tilde{v}_j^k = \tilde{L}_j^{-1} R_j r^k$ .

**(3) Prolongation and update**

- Compute  $u^{k+1} = u^k + \omega \sum_{j=1}^J P_j \tilde{v}_j^k$ .

The error iteration matrix for this AS iteration becomes

$$M_{AS} = I - \omega BL, \quad B = \sum_{j=1}^J P_j \tilde{L}_j^{-1} R_j, \quad (3)$$

where  $B$  can be considered as *preconditioner* for  $L$  associated with the given choice of subspace problems (2), more precisely, with the choice of  $\{\tilde{L}_j, P_j, R_j : j = 1, \dots, J\}$ . The relaxation parameter  $\omega$  is introduced for convenience, it could be replaced by individual relaxation parameters  $\omega_j$ ,  $j = 1, \dots, J$ , and can be interpreted as a way of correctly *scaling* the subspace problems (2) with respect to the original system (1). Formally, the multiplication by  $B$  looks very suitable for parallelization, however, the parallel efficiency truly depends on the choices for  $\tilde{L}_j, P_j, R_j$ . Obviously, the iteration AS as detailed above generalizes the  $\omega$ -Jacobi relaxation (for details, see section 3).

As it may be anticipated from its description, the AS method is usually not very fast. A more efficient way to use the subproblems to compose an iterative method for (1) seems to be the following analog of a SOR relaxation:

**Multiplicative subspace correction method**  $u^{k+1} = MS(\omega, u^k, L, f; \tilde{L}_j, P_j, R_j)$

**(1) Initialization**

- Set  $v^1 = u^k$ .

**(2) Loop through subspace problems** For  $j = 1, \dots, J$ ,

- Compute  $r^j = f - Lv^j$ .

- Restrict and solve a subspace problem  $\tilde{v}_j^j = \tilde{L}_j^{-1} R_j r^j$ .

- Compute prolongation and update  $v^{j+1} = v^j + \omega P_j \tilde{v}_j^j$ .

**(3) Exit**

- Set  $u^{k+1} = v^{J+1}$ .

The error iteration matrix for the MS method possesses a product representation as follows:

$$M_{\text{MS}} = (I - \omega P_J \tilde{L}_J^{-1} R_J L) \dots (I - \omega P_1 \tilde{L}_1^{-1} R_1 L) . \quad (4)$$

At the first glance, the sequential nature of the MS iteration and the computation of residuals involving  $L$  in each step of the inner loop seem to make the algorithm less attractive for parallel implementations. However, a closer look at the implementation of the MS method reveals that such a statement is again dependent on the particular setting. The MS algorithm can be modified in several important directions, most importantly, the ordering of the subspace problems now matters (this is in analogy with differences between GS-LEX and GS-RB) and subspaces can be used repeatedly in one loop. Some of these variations will be mentioned in section 3.

In the subsequent sections 2 and 3 of this chapter, we present the abstract theory of subspace correction methods for the case of *symmetric positive definite systems* (1). This ‘soft’ theory requires only knowledge about the basics of Hilbert spaces and classical numerical linear algebra. We provide examples, partly with finite element background, that should help to understand the concepts and illustrate the results. In section 4, we exemplify the application of this theory to multigrid methods by deriving a qualitative V-cycle convergence result for the finite difference discretization of the Poisson equation. Domain decomposition methods, the other mainstream application of the theory of subspace correction methods, are touched in section 5.

Since this chapter is, in a certain sense, complementary to the main contents of this book, the presentation is kept on an informal level. The reader interested in more mathematical details and recent developments is recommended to consult the literature cited below. The same comment applies to the missing of information on the history of the theory presented below. Since the idea of transforming and splitting a large problem into a number of similar (sub-)problems is so obvious, analogous algorithms and attempts to formalize and treat them have been around in most areas of applied and numerical mathematics, e.g., in applied harmonic analysis and optimization. We specifically recommend the survey articles and books [17, 2, 4, 5, 7, 10, 11, 15, 18, 19] for further reading.

## 2 Space splittings

In this section, we change our notation slightly. Let  $V$  denote a finite-dimensional Hilbert space, with scalar product  $(u, v)_V$  and norm  $\|u\|_V = \sqrt{(u, u)_V}$ ,  $u, v \in V$ . For simplicity, everything is assumed to be real-valued. A function  $\ell \equiv \ell(u, v)$  with arguments  $u, v \in V$  and values in  $\mathbb{R}$  is called  $V$ -elliptic if it is linear in each argument (thus, representing a *bilinear form* on  $V$ ) and satisfies the inequalities

$$|\ell(u, v)| \leq \bar{\gamma} \|u\|_V \|v\|_V , \quad \gamma \|u\|_V^2 \leq \ell(u, u) \quad \forall u, v \in V .$$

The first inequality is also called *continuity* or *boundedness* of the bilinear form  $\ell$ , the second *stability* or *coercivity* of  $\ell$ . The best possible values of the positive constants  $0 < \gamma \leq \bar{\gamma} < \infty$  represent the *ellipticity constants* of  $\ell$ . If in addition  $\ell(u, v) = \ell(v, u)$  for all  $u, v \in V$  then  $\ell$  is called *symmetric*.

**Example 1** This example is related to the study of linear systems (1). Set  $V = \mathbb{R}^N$ , where the Euclidean scalar product

$$(u, v) = v^T u = \sum_{n=1}^N v_n u_n , \quad u = (u_1, \dots, u_N)^T , \quad v = (v_1, \dots, v_N)^T ,$$

defines the Hilbert space structure (to make the notation simpler, we will use  $(\cdot, \cdot)$  whenever the Euclidean scalar product in some  $\mathbb{R}^N$  is meant). Set

$$\ell(u, v) = (Lu, v) = \sum_{m,n=1}^N l_{m,n} u_n v_m .$$

Obviously, this is a symmetric  $\mathbb{R}^N$ -elliptic bilinear form if and only if  $L$  is symmetric positive definite; in this case the ellipticity constants  $\gamma, \bar{\gamma}$  coincide with the smallest and largest eigenvalues of  $L$ , respectively.

**Example 2** Another example of importance arises if elliptic boundary value problems are treated by *Galerkin methods*, in particular, by *finite element methods* (FEM). We outline the details for the homogeneous Dirichlet problem for the Poisson equation and linear finite elements on a two-dimensional domain  $\Omega$ . Recall that the underlying continuous problem takes the form

$$\begin{aligned} -\Delta u(x, y) &= f^\Omega(x, y), & (x, y) \in \Omega, \\ u(x, y) &= 0, & (x, y) \in \Gamma, \end{aligned} \tag{5}$$

where  $f^\Omega$  is a given function. Suppose that (5) possesses a (sufficiently smooth) solution  $u(x, y)$ . Formally, by multiplying the differential equation by any (sufficiently smooth) function  $v(x, y)$  that vanishes on the boundary  $\Gamma$ , integrating over  $\Omega$ , and applying Green's formula, we have

$$\begin{aligned} \int_{\Omega} f(x, y)v(x, y) dx dy &= - \int_{\Omega} \Delta u(x, y)v(x, y) dx dy \\ &= \int_{\Omega} u_x(x, y)v_x(x, y) + u_y(x, y)v_y(x, y) dx dy - \int_{\Gamma} u_n(x, y)v(x, y) d\Gamma(x, y) \\ &= \int_{\Omega} \nabla u(x, y) \cdot \nabla v(x, y) dx dy, \end{aligned}$$

where  $\nabla$  denotes the gradient operator. The integral with respect to  $\Gamma$  has been dropped since  $v(x, y) = 0$  on  $\Gamma$ . Thus, if we denote

$$\ell(u, v) = (u, v)_1 \equiv \int_{\Omega} \nabla u(x, y) \cdot \nabla v(x, y) dx dy, \quad f(v) = (f, v)_0 \equiv \int_{\Omega} f(x, y)v(x, y) dx dy,$$

then necessarily

$$\ell(u, v) = f(v) \tag{6}$$

for all sufficiently smooth  $v(x, y)$  vanishing on  $\Gamma$ . The derivation of this so-called *variational formulation* (6) associated with (5) can be made mathematically precise, if we introduce the concept of *weak solutions* of (5) in the Sobolev space  $H_0^1(\Omega)$  (see [12, 16]). For the purpose of this informal introduction into subspace correction methods, it suffices to switch immediately to Galerkin methods based on (6). Let us take any finite-dimensional space  $V$  ( $\dim V = N$ ) of functions on  $\Omega$  that vanish on  $\Gamma$  and such that both  $(\cdot, \cdot)_1$  and  $(\cdot, \cdot)_0$  make sense as scalar product on  $V$  (this essentially reduces to requiring that  $\|v\|_k^2 = (v, v)_k = 0$  and  $v \in V$  implies  $v = 0$  for either  $k = 0$  or  $k = 1$ ). With this assumption, it is clear that  $\ell(u, v)$  when considered as a bilinear form on that  $V$  is symmetric and  $V$ -elliptic with respect to either of the two scalar products (if  $V$  is equipped with the scalar product  $(\cdot, \cdot)_1$  then the ellipticity constants are simply  $\gamma = \bar{\gamma} = 1$ , in the other case their ratio  $\kappa(\ell) = \bar{\gamma}/\gamma$  is typically very large). In the following, we will sometimes use the more explicit notation  $\{V; (\cdot, \cdot)_V\}$  to indicate which scalar product is meant. For instance, if  $\ell$  is a symmetric  $V$ -elliptic form then  $\{V; \ell\}$  itself is a possible choice.

As a consequence, the finite-dimensional variational problem of finding  $u \in V$  such that

$$\ell(u, v) = f(v) \quad \forall v \in V \tag{7}$$

has a unique solution which is a minimizer of the *energy functional*  $J(u) = \ell(u, u) - 2f(u)$  associated with (7). This solution is, roughly speaking, a projection of the exact solution of (5) into the  $N$ -dimensional space  $V$ . There is a well-understood procedure to estimate the discretization error associated with this *Galerkin projection* which we will not go into. To find the solution  $u \in V$  of it computationally, typically a basis  $\Phi = \{\varphi^i, i = 1, \dots, N\}$  in  $V$  will be chosen, and (7) turns into an equivalent system of linear equations (1), where coefficient matrix and right-hand side are given by

$$L = ((l_{m,n} = \ell(\varphi^n, \varphi^m)))_{m,n=1}^N, \quad f = (f_m = f(\varphi^m))^T. \tag{8}$$

The matrix  $L$  is necessarily symmetric positive definite; its properties sensitively depend on both the bilinear form  $\ell$  and the choice of the basis  $\Phi$  in  $V$ .

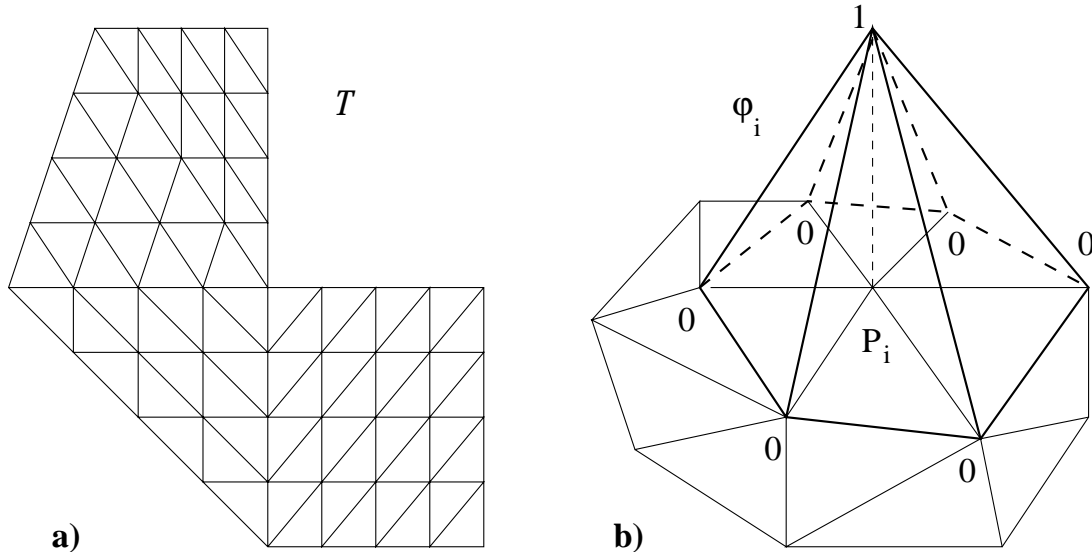


Figure 1: a) Triangulation and b) linear finite element nodal basis function

Here is the finite element example we wish to discuss. Assume for simplicity that  $\Omega$  is a polygonal domain and equipped with a quasi-uniform and regular triangulation  $\mathcal{T}$  (this means that all triangles are well-shaped (i.e., the smallest angle is bounded from below by a fixed value) and have approximately the same diameter  $\approx h$ , and that neighboring triangles share or a common vertex or a common edge). Figure 1a shows a typical triangulation. The space  $V = V(\mathcal{T})$  of *linear finite elements* associated with  $\mathcal{T}$  consists of all continuous functions on  $\Omega$  that vanish on  $\Gamma$  and are *linear* (i.e., take the form  $u(x, y) = a + bx + cy$  for some  $a, b, c \in \mathbb{R}$ ) when restricted to any of the triangles in  $\mathcal{T}$ . Any function  $u \in V(\mathcal{T})$  is uniquely determined by its values  $u^i$  at the interior vertices or *nodal points*  $P^i = (x^i, y^i)$  of  $\mathcal{T}$ , and can be recovered by linear interpolation on each triangle (clearly, values at boundary vertices are set to 0). The dimension  $N$  of  $V(\mathcal{T})$  coincides with the number of nodal points, and due to the assumptions on  $\mathcal{T}$ , satisfies  $N \approx h^{-2}$ . The standard basis functions  $\varphi^i$ , known as nodal basis or simply hat functions, are defined as the Lagrange functions for this local, piecewise linear interpolation scheme, i.e.,  $\varphi^i \in V(\mathcal{T})$ ,  $i = 1, \dots, N$ , is given by requiring

$$\varphi^i(x^s, y^s) = \begin{cases} 1 & , \quad s = i, \\ 0 & , \quad s \neq i, \end{cases} \quad s = 1, \dots, N.$$

The support of any  $\varphi^i$  is very small; it consists of the union of triangles adjacent to  $P^i$ . Figure 1b pictures a typical nodal basis function. Let us mention in by-passing that under some assumptions on  $\Omega$  and  $f^\Omega$  (which are weaker than the corresponding conditions for finite difference schemes), this choice of  $V$  as finite-dimensional space in the Galerkin formulation (7) leads to a discretization error of  $O(h^2)$  in the  $\|\cdot\|_0$  norm and of  $O(h)$  in the  $\|\cdot\|_1$  norm, respectively. The associated matrix  $L$  is sparse, with  $\approx N \approx h^{-2}$  nonzero entries, but has a deteriorating condition number  $\kappa(L) \approx h^{-2}$  if  $h \rightarrow 0$ . A drawback of the finite element approach is that the computation of the scalar products in the formula for  $l_{m,n}$  and  $f_m$  (see (8)) requires numerical integration, and leads to generally more work in the assembly process of the linear system.

**Example 3** If  $\Omega$  is a rectangle (or composed of several rectangles), partitions  $\mathcal{R}$  into rectangles can be used instead of triangulations, and a completely similar setup leads to the space of *bilinear finite elements*  $V(\mathcal{R})$  with analogous properties of the associated Galerkin formulation (7) resp. (8). Clearly, the restriction of any  $u \in V(\mathcal{R})$  to any subrectangle of the partition will be a bilinear function:  $u(x, y) = a + bx + cy + dxy$ .

There are many other choices (higher-order finite element and spectral element spaces, wavelet spaces, linear combinations of radial basis functions or Gaussians) that appear in connection with

data approximation and the solution of various operator equations and can be used within a Galerkin scheme. However, for the purpose of this paper we will restrict ourselves to the above examples.

We will now introduce the notion of *stable space splittings* which allows us to give a unified treatment of subspace correction methods as methods based on properly representing Hilbert spaces by sums of other Hilbert spaces. The notation is chosen such that the analogy with the introduction becomes obvious. Fix the  $N$ -dimensional Hilbert space  $V$ , and consider the problem (7) where  $\ell$  is a symmetric  $V$ -elliptic bilinear form. For  $j = 1, \dots, J$ , let  $\tilde{V}_j$  be a  $N_j$ -dimensional Hilbert space and  $\tilde{\ell}_j$  symmetric  $\tilde{V}_j$ -elliptic bilinear forms. We do not assume that  $\tilde{V}_j \subset V$ , instead we require that a link between  $\tilde{V}_j$  and  $V$  is established by linear mappings (embeddings or *prolongations*)  $P_j : \tilde{V}_j \rightarrow V$ .

**Definition 4** *We call the formal decomposition*

$$\{V; \ell\} \cong \sum_{j=1}^J P_j \{\tilde{V}_j; \tilde{\ell}_j\} \quad (9)$$

a *stable space splitting* of  $\{V; \ell\}$  using the spaces  $\{\tilde{V}_j; \tilde{\ell}_j\}$  and the embeddings  $P_j$ ,  $j = 1, \dots, J$ , if any  $v \in V$  admits at least one representation

$$v = \sum_{j=1}^J P_j \tilde{v}_j, \quad \tilde{v}_j \in \tilde{V}_j, \quad j = 1, \dots, J, \quad (10)$$

and

$$\|v\| = \inf \left( \sum_{j=1}^J \tilde{\ell}_j(\tilde{v}_j, \tilde{v}_j) \right)^{1/2}, \quad (11)$$

satisfies a *two-sided inequality*

$$\eta \ell(v, v) \leq \|v\|^2 \leq \bar{\eta} \ell(v, v) \quad \forall v \in V. \quad (12)$$

with positive constants  $\eta, \bar{\eta}$ . The infimum in (11) is taken with respect to all admissible representations (10). The optimal constants  $\eta, \bar{\eta}$  in (12) are called *lower and upper stability constants* of the splitting (9), respectively, their ratio  $\kappa = \bar{\eta}/\eta$  is the *condition of the splitting*.

Note that in the finite-dimensional setting described here, the assumption (10) automatically implies (12). The definition can be extended to countably many spaces  $\tilde{V}_j$  ( $J = \infty$ ), and  $V$  as well as  $\tilde{V}_j$  could be separable Hilbert spaces. Then (12) becomes a real assumption. This extension is useful to connect the discrete theory outlined here with general recipes from approximation theory and the theory of function spaces. The importance of Definition 4 will become clear in section 3, let us just mention that keeping the size of  $\kappa$  small and independent of  $J$  will be critical for the fast convergence of subspace correction methods. We present now some examples of stable splittings which illustrate the flexibility of the abstract concepts and are a preparation for the subsequent sections of this chapter.

**Example 5** This example is related to classical *block-iterative solvers* for (1). Consider the situation of example 1. Split the index set  $\Lambda = \{1, \dots, N\}$  into pairwise disjoint non-empty sets  $\Lambda_j$  and set  $N_j = \#\Lambda_j$ . Without loss of generality, assume  $\Lambda_1 = \{1, \dots, N_1\}$ ,  $\Lambda_2 = \{N_1 + 1, \dots, N_1 + N_2\}$ , and so on. By  $\tilde{L}_j$  we denote the (symmetric positive definite) submatrices of  $L$  of dimension  $N_j$  corresponding to  $\Lambda_j$ . Set

$$\tilde{V}_j = \mathbb{R}^{N_j}, \quad \tilde{\ell}_j(\tilde{v}_j, \tilde{v}_j) = (\tilde{L}_j \tilde{v}_j, \tilde{v}_j),$$

and introduce the prolongations  $P_j : \mathbb{R}^{N_j} \rightarrow \mathbb{R}^N$  by

$$P_j : \tilde{v}_j \equiv (\tilde{x}_1, \dots, \tilde{x}_{N_j})^T \rightarrow v \equiv (x_1, \dots, x_N)^T, \quad x_l = \begin{cases} \tilde{x}_{l-N_1-\dots-N_{j-1}}, & l \in \Lambda_j \\ 0, & l \notin \Lambda_j \end{cases},$$

$j = 1, \dots, J$ . This gives a stable space splitting since (10) holds for exactly one choice of  $\tilde{v}_j$ :

$$\tilde{v}_j = R_j v, \quad R_j : (x_1, \dots, x_N)^T \rightarrow (x_{N_1+\dots+N_{j-1}+1}, \dots, x_{N_1+\dots+N_j})^T, \quad j = 1, \dots, J.$$

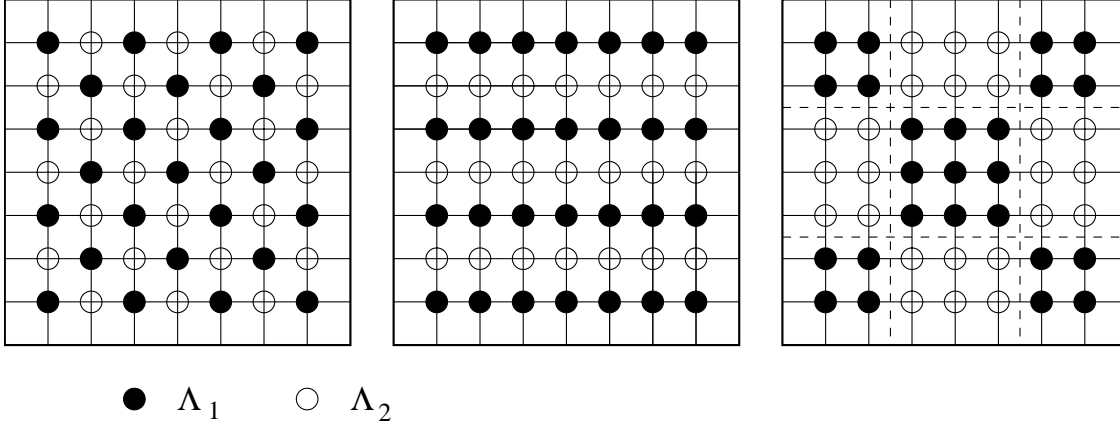


Figure 2: Choices of  $\{\Lambda_j\}$  for the 5-point discretization

Thus, the infimum in (11) can be dropped, and it is easy to see that

$$\|v\|^2 = \sum_{j=1}^J (\tilde{L}_j R_j v, R_j v) = (\tilde{L} v, v),$$

where  $\tilde{L} = \text{diag}(\tilde{L}_1, \dots, \tilde{L}_J)$  is a block-diagonal matrix of dimension  $N$  consisting of the diagonal submatrices of  $L$  corresponding to the index sets  $\Lambda_j$ . The stability constants of the splitting and its condition coincide with the extremal eigenvalues and the spectral condition number of the matrix  $\tilde{L}^{-1}L$ , respectively.

An extremal case occurs if we choose one-element sets  $\Lambda_j = \{j\}$ ,  $j = 1, \dots, N$ . Then  $N_j = 1$  and  $J = N$ . The  $\tilde{L}_j$  are of size 1 and given by the diagonal entry  $l_{j,j}$  of  $L$ . Thus,  $\tilde{L} = \text{diag}(L)$ , and the space splitting has to do with *diagonal preconditioning*. In the general case, the splitting is associated to *block-diagonal preconditioning*. Since the other, trivial extremal case would be to choose  $J = 1$  and  $\Lambda_1 = \Lambda$  (then  $\tilde{L}^{-1}L$  is just the identity matrix) there arises the interesting design problem of finding the right balance between the size of the submatrices  $N_j$  and their overall number  $J$ . In Figure 2 we show the grid points associated with several choices of index sets  $\Lambda_j$ ,  $j = 1, 2$ , for the model problem I, the 5-point discretization of the Dirichlet problem for Laplace's equation on a uniform square grid. In each case, we could have further split the two index sets. While the first two examples are related to smoothers with RB-ordering and line smoothing (the condition of the associated splitting is  $\approx h^{-2}$  and of the same order as for diagonal preconditioning and the condition number of  $L$  itself), the last one relates to the domain decomposition approach which will be discussed in section 5. In the latter case, the condition of the associated splitting is  $\approx h^{-1}H^{-1}$  where  $H < h$  is the stepsize parameter of the underlying choice of  $\Lambda_j$ . The reader is encouraged to establish these bounds, the extremal vectors that give the order of the constants  $\eta$  and  $\bar{\eta}$  in (12) are unit vectors, on the one hand, and vectors associated with the grid values of 'smooth' functions such as  $\varphi_h^{1,1}$ , on the other.

There are a lot of useful generalizations of this example: the sets  $\Lambda_j$  may overlap and the matrices  $\tilde{L}_j$  need not be the corresponding diagonal submatrices of  $L$ . The examples below (even though they are cast in a different language) are of this type.

**Example 6** Here is an example related to the *fictitious space method*. Consider three finite difference meshes of meshsize  $\approx h$  on three different domains  $\Omega \approx \hat{\Omega} \subset \bar{\Omega}$  as schematically shown in Figure 3. The first mesh which is not a square partition is assumed to be a slight distortion of the square partition of the L-shaped domain  $\hat{\Omega}$  shown in the center. I.e., we assume that there is a discrete one-to-one mapping between the grid points of the two meshes which is close to a linear mapping restricted to this set (such meshes are sometimes called *topologically equivalent*). The third

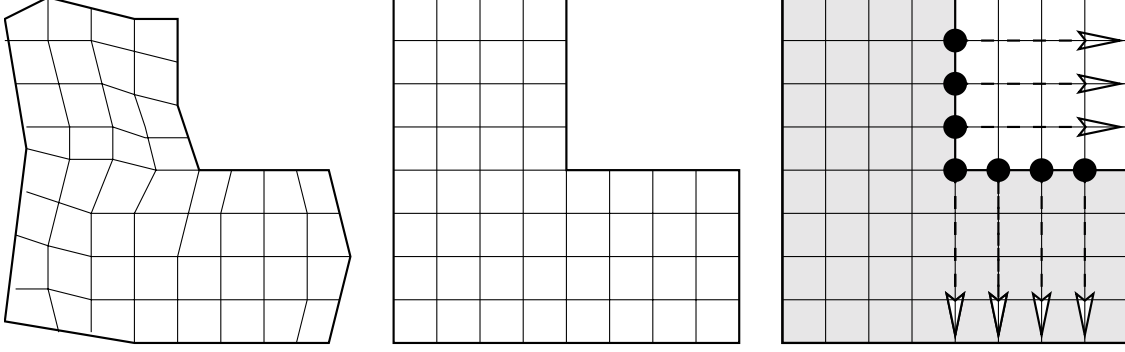


Figure 3: Meshes for the fictitious space example

mesh is a square mesh of a unit square  $\tilde{\Omega}$  which contains the mesh of the L-shaped domain (the further details shown in the right picture will be explained below).

Suppose that we have to solve a linear problem (1) associated with a finite difference or finite element discretization of (5) on the first domain and mesh (although this has not been detailed in this monograph, there are standard methods to derive finite difference approximations on unstructured meshes). By  $\hat{L}$  and  $\tilde{L}$  we will denote the stiffness matrices of the finite difference discretization of (5) with respect to the meshes on  $\hat{\Omega}$  and  $\tilde{\Omega}$ , respectively. Note that  $\hat{L}$  is a submatrix of  $\tilde{L}$ . Set  $V = \hat{V} = \mathbb{R}^N$  and  $\tilde{V} = \mathbb{R}^{\tilde{N}}$  where  $N$  and  $\tilde{N} > N$  denote the number of interior grid points of the meshes for  $\hat{\Omega}$  and  $\tilde{\Omega}$ , respectively. Define the bilinear forms  $\ell$ ,  $\hat{\ell}$ , and  $\tilde{\ell}$  as in Example 1. With proper assumptions on the discretization on the first mesh and on its one-to-one mapping onto the second mesh, we have

$$\ell(u, u) \approx \hat{\ell}(u, u) = h^{-2} \sum_e |\Delta_e u|^2 \quad (13)$$

for all vectors  $u \in \mathbb{R}^N$ , where the summation is with respect to all interior edges of the meshes on  $\Omega$  resp.  $\tilde{\Omega}$ , and  $\Delta_e u$  is the difference of the values of the grid function associated with  $u$  at the endpoints of  $e$ . A similar relationship holds for  $\tilde{\ell}$ . Let  $R : \mathbb{R}^N \rightarrow \mathbb{R}^{\tilde{N}}$  correspond to the natural extension-by-zero operator of grid functions on the first to meshes (which can be identified by assumption) to the larger square mesh, and  $P = R^T : \mathbb{R}^{\tilde{N}} \rightarrow \mathbb{R}^N$  the natural restriction operator. Then

$$\{V; \ell\} \cong \{\hat{V}; \hat{\ell}\} \cong P\{\tilde{V}; \tilde{\ell}\} \quad (14)$$

can be viewed as stable space splittings with  $J = 1$ . The condition of the first splitting is bounded, independently of  $h$ , as follows from the *spectral equivalence* of  $L$  and  $\hat{L}$  expressed by (13).

Let us show that the condition of the second splitting is of the order  $\approx h^{-1}$  as  $h \rightarrow 0$  (this behavior can be improved if a better  $R$  and  $P = R^T$  based on discrete harmonic extension are used). Consider any  $\tilde{u} \in \mathbb{R}^{\tilde{N}}$  such that  $u = P\tilde{u}$ . By (13) we have

$$\ell(u, u) \approx \hat{\ell}(u, u) = h^{-2} \sum_{e \subset \hat{\Omega}} |\Delta_e u|^2 \leq h^{-2} \sum_{e \subset \tilde{\Omega}} |\Delta_e Ru|^2 = \tilde{\ell}(Ru, Ru),$$

since  $Ru = u$  on  $\hat{\Omega}$  and  $Ru = 0$  on  $\tilde{\Omega} \setminus \hat{\Omega}$  by definition of  $R$ . Thus, since  $PRu = u$  we obtain

$$\|u\|^2 = \inf_{\tilde{u} : u = P\tilde{u}} \tilde{\ell}(\tilde{u}, \tilde{u}) \leq \tilde{\ell}(Ru, Ru) = \hat{\ell}(u, u) \leq \bar{\eta} \ell(u, u)$$

for some  $\bar{\eta} > 0$ .

On the other hand, fix an arbitrary  $u \in \mathbb{R}^N$  and consider any  $\tilde{u} \in \mathbb{R}^{\tilde{N}}$  such that  $u = P\tilde{u}$ . Then

$$\hat{\ell}(u, u) = \tilde{\ell}(Ru, Ru) = h^{-2} \sum_{e \subset \tilde{\Omega}} |\Delta_e Ru|^2$$

$$\begin{aligned}
&\leq h^{-2} \left( \sum_{e \subset \hat{\Omega}} |\Delta_e \tilde{u}|^2 + 3 \sum_{\tilde{P}^s \in \partial \hat{\Omega}} |\tilde{u}(x^s, y^s)|^2 \right) \\
&\leq \tilde{\ell}(\tilde{u}, \tilde{u}) + 3h^{-2} \|\tilde{u}\|_{\partial \hat{\Omega}}^2.
\end{aligned}$$

The second term in the last expression (the Euclidean norm of the subvector of  $\tilde{u}$  corresponding to the grid points  $\tilde{P}^s$  on the boundary  $\partial \hat{\Omega}$  of the L-shaped domain  $\hat{\Omega}$ ) is bounded from above by  $Ch^{-1} \tilde{\ell}(\tilde{u}, \tilde{u})$ . For our example, this discrete Poincare type estimate can be verified as follows. Observe that we can connect each of those  $\tilde{P}^s$  with a grid point  $\tilde{P}^{s'}$  on the boundary of  $\tilde{\Omega}$  on a separate set  $E^s$  of  $\leq Ch^{-1}$  interior edges from the mesh on  $\tilde{\Omega}$  (see the dashed lines in Figure 3). Since  $\tilde{u}(x^{s'}, y^{s'}) = 0$ , we have

$$|\tilde{u}(x^s, y^s)|^2 \leq \left( \sum_{e \in E^s} |\Delta_e \tilde{u}| \right)^2 \leq Ch^{-1} \sum_{e \in E^s} |\Delta_e \tilde{u}|^2.$$

Summation with respect to  $s$  (recall that the sets  $E^s$  are pairwise disjoint) gives the above bound. Altogether, we have

$$\hat{\ell}(u, u) \leq (1 + 3Ch^{-1}) \tilde{\ell}(\tilde{u}, \tilde{u})$$

for all  $\tilde{u}$  satisfying  $u = P\tilde{u}$ . It remains to take the infimum with respect to  $\tilde{u}$  which yields

$$\ell(u, u) \approx \hat{\ell}(u, u) \leq 1/\eta \|u\|^2 \quad (\eta = (1 + Ch^{-1})^{-1} \approx h).$$

Thus, we have proved the upper bound  $O(h^{-1})$  for the condition, that it cannot be improved is clear from looking at unit vectors  $u$ , on the one hand, and a  $u$  obtained from the grid function  $\varphi_h^{1,1}$  by restricting it to  $\hat{\Omega}$ , on the other.

**Example 7** This example introduces to the *multilevel splittings of finite element spaces* which are central for the applications to multigrid theory. A more complete account of the underlying concepts and their roots in spline approximation and function space theory, see [15]. The standard setting is to start with a sequence of partitions  $\{\mathcal{T}_j\}$  of a polyhedral domain in  $\mathbb{R}^d$  obtained by some sort of regular refinement, and such that a fixed type of finite element construction leads to an increasing sequence

$$V_1 \subset V_2 \subset \dots \subset V_j \subset \dots$$

of finite element spaces on these partitions. Not all finite element constructions share this property but there are a number of worked examples. For instance, linear finite element spaces on triangulations and tetrahedral partitions in two and three dimensions, respectively, which are of importance for the numerical solution of second order elliptic boundary value problems satisfy the above nestedness assumption. For simplicity, let  $\Omega$  be the unit square. Consider a sequence of uniform triangulations  $\mathcal{T}_j$  of diameter  $\approx 2^{-j}$ ,  $j = 1, 2, \dots$ , as shown in Figure 4. Note that  $\mathcal{T}_{j+1}$  is obtained from  $\mathcal{T}_j$  by subdividing all triangles into four (equal) subtriangles. This procedure is called *regular dyadic refinement*. Note that the triangulation shown in Figure 1 a) is an example of a triangulation on a polygonal domain obtain by two regular dyadic refinements from an initial, coarse triangulation into 5 triangles. More general refinement procedures are possible (bisection algorithms, nested local refinement) but will not be discussed here.

Let  $\tilde{V}_j = V(\mathcal{T}_j)$  be the corresponding finite element spaces of Example 2, and set  $\tilde{\ell}_j(\tilde{u}_j, \tilde{v}_j) = 2^{2j}(\tilde{u}_j, \tilde{v}_j)_0$ . Note that  $\tilde{V}_j$  is a proper subspace of  $\tilde{V}_{j+1}$ , with scalar products that are identical up to a constant scaling factor, and that the dimensions  $N_j = \dim \tilde{V}_j \approx 2^{2j}$  grow exponentially with  $j$ . Denote by  $I_j^{j+1} : \tilde{V}_j \rightarrow \tilde{V}_{j+1}$  the natural embedding operators, and let  $I_j^j = I_{j-1}^j \dots I_1^{j+1}$ ,  $j < J$ , be their iterates. Assume now that we have to solve (6) with respect to the space  $V_J = V(\mathcal{T}_J)$ . Therefore, we set  $\ell_J(u_J, v_J) = (u_J, v_J)_1$ ,  $u_J, v_J \in V_J$ .

**Theorem 8** *The space splitting*

$$\{V_J; \ell_J\} \cong \sum_{j=1}^J I_j^J \{\tilde{V}_j; \tilde{\ell}_j\} \quad (15)$$

is stable, with stability constants  $\eta_J, \bar{\eta}_J$  and condition  $\kappa_J$  that remain bounded, independently of  $J$ :

$$0 < \eta \leq \eta_J \leq \bar{\eta}_J \leq \bar{\eta} < \infty, \quad \kappa_J \leq \kappa = \bar{\eta}/\eta. \quad (16)$$

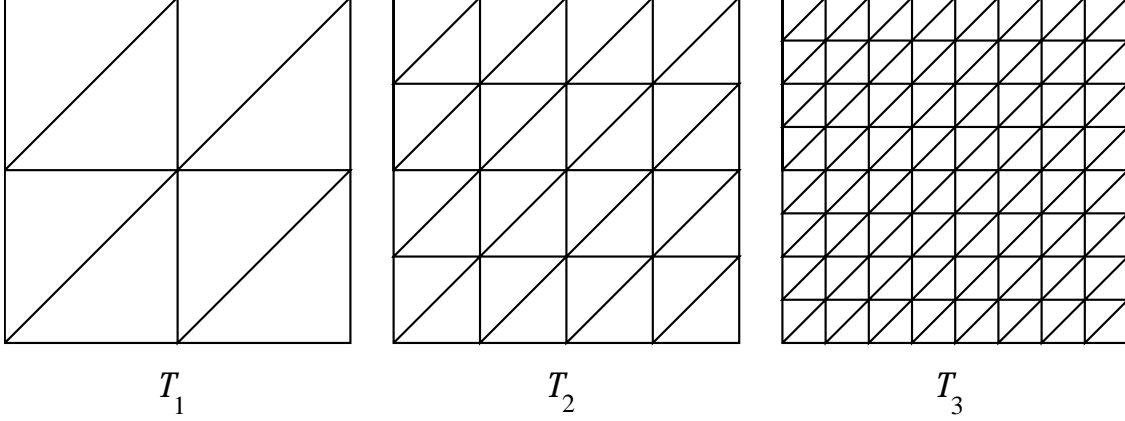


Figure 4: Dyadically refined triangulations for Example 7

Proofs of this result can be found in the literature cited in the introduction. There are a lot of variations connected with it. For instance, if

$$\tilde{u}_j = \sum_{i=1}^{N_j} \tilde{u}_j^i \varphi_j^i$$

is the unique representation of  $\tilde{u}_j$  with respect to the nodal basis  $\Phi_j$  in the finite element space  $\tilde{V}_j$  then

$$\tilde{\ell}_j(\tilde{u}_j, \tilde{u}_j) \approx 2^{2j} \sum_{i=1}^{N_j} (\tilde{u}_j^i)^2 \|\varphi_j^i\|_0^2 \approx \sum_{i=1}^{N_j} (\tilde{u}_j^i)^2. \quad (17)$$

To see this so-called *L<sub>2</sub>-stability of the nodal basis*  $\Phi_j$ , use the fact that for a function  $u$  which is linear on a triangle  $\Delta$  and takes values  $\alpha, \beta, \gamma$  at its three vertices, we have

$$\int_{\Delta} u(x, y)^2 dx dy \approx |\Delta|(\alpha^2 + \beta^2 + \gamma^2)$$

with constants independent of  $\Delta$  (as usual,  $|\Delta|$  denotes the area of  $\Delta$ ). Application to the piecewise linear function  $\tilde{u}_j$  and each triangle in  $\mathcal{T}_j$  leads to (17). This relationship allows us to conclude that the splitting into one-dimensional spaces  $\tilde{V}_j^i$  associated with the basis functions  $\varphi_j^i$

$$\{V_J; \ell_J\} \cong \sum_{j=1}^J \sum_{i=1}^{N_j} I_j^i \{\tilde{V}_j^i; \tilde{\ell}_j^i\} \quad (18)$$

is also stable, with stability constants and condition satisfying again (16) (with possibly different values for  $\eta, \bar{\eta}$ ). The only requirement on  $\tilde{\ell}_j^i$  is

$$\tilde{\ell}_j^i(\varphi_j^i, \varphi_j^i) \approx 1, \quad (19)$$

uniformly in  $j$  and  $i$ , which is satisfied for choosing it as the restriction of either  $\ell_J$  or  $\tilde{\ell}_j$  to  $\tilde{V}_j^i$ . To get the necessary estimates for the triple bar norm associated with (18), take the corresponding result for the splitting (15), substitute (17), together with the scaling assumption (19) on  $\tilde{\ell}_j^i$  (note that  $\tilde{u}_j^i \in \tilde{V}_j^i$  means that  $\tilde{u}_j^i$  can be written as a scalar multiple of the basis function  $\varphi_j^i$ ).

We conclude with some general statements on stable space splittings. First, let us mention the following equivalent formulation of (6) if a stable splitting (9) is available. Define linear operators  $T_j : V \rightarrow \tilde{V}_j$  and elements  $\tilde{f}_j \in \tilde{V}_j$ ,  $j = 1, \dots, J$ , by requiring

$$\tilde{\ell}_j(T_j u, \tilde{v}_j) = \ell(u, P_j \tilde{v}_j) \quad \forall \tilde{v}_j \in \tilde{V}_j, \quad (20)$$

and

$$\tilde{\ell}_j(\tilde{\phi}_j, \tilde{v}_j) = f(P_j \tilde{v}_j) \quad \forall \tilde{v}_j \in \tilde{V}_j. \quad (21)$$

For any given  $u \in V$ , these are well-defined Galerkin formulations on the spaces  $\tilde{V}_j$ . The operator

$$\mathcal{P} = \sum_{j=1}^J P_j T_j : V \rightarrow V \quad (22)$$

is called *additive Schwarz operator* associated with (9). Also, define  $\phi = \sum_{j=1}^J P_j \tilde{\phi}_j$ .

**Theorem 9** *Assume that the space splitting (9) is stable. Then  $\mathcal{P}$  is symmetric positive definite with respect to  $\{V; \ell\}$ , its spectral condition number coincides with the condition of the splitting, and its extremal eigenvalues with the values  $\bar{\eta}^{-1}$  and  $\eta^{-1}$ . The operator equation*

$$\mathcal{P}u = \phi, \quad (23)$$

has the same solution as the variational problem (6).

For the elementary proof, see [15, section 4.1]. In some cases,  $\mathcal{P}$  can be written explicitly. E.g., in example 5, we have  $\mathcal{P} = \tilde{L}^{-1}L$ . The additive Schwarz operator associated with the splitting (18) takes the form

$$\mathcal{P}u = \sum_{j=1}^J \sum_{i=1}^{N_j} \frac{\ell(u, \varphi_j^i)}{\tilde{\ell}_j^i(\varphi_j^i, \varphi_j^i)} \varphi_j^i. \quad (24)$$

To see this, note that the problems (20) are one-dimensional and, therefore, can be solved explicitly, and that the prolongations are given by natural embeddings  $I_j^J$  which are omitted in the above formula.

The formula (24) has a very familiar appearance, it reminds us of the Fourier series representation (with the difference that the system

$$\Phi = \cup_{j=1}^J \Phi_j = \{\phi_j^i : i = 1, \dots, N_j, j = 1, \dots, J\} \quad (25)$$

is neither orthogonal nor a basis in  $V$ ). This connection is very useful, especially for proving stability of certain space splittings by using known results from applied harmonic analysis and function space theory [15] but also to see the benefits and drawbacks of emerging wavelet algorithms for solving PDE discretizations [6].

The last remark is about verifying the stability of a given splitting. The *upper bound* for  $\|v\|^2$  requires to find a *good decomposition* of arbitrary elements  $u \in V$  with components  $\tilde{v}_j \in \tilde{V}_j$  such that  $v = \sum_j P_j \tilde{v}_j$ . If there is only one admissible representation (10) then we have no choice but to consider this decomposition (thus, to ‘guess’ a good set of auxiliary spaces  $\tilde{V}_j$  is the important part of proving anything about the splitting). Otherwise, we have some choice in (10), and suitable decompositions are constructed by using various projections onto the spaces  $\tilde{V}_j$ . E.g., for deriving Theorem 8, one often relies on the  $L_2$ -orthoprojection operators  $Q_j : L_2(\Omega) \rightarrow \tilde{V}_j$  given by

$$(Q_j u, \tilde{v}_j)_0 = (u, \tilde{v}_j)_0 \quad \forall \tilde{v}_j \in \tilde{V}_j, \quad (26)$$

$j \geq 1$ . For them, the two-sided inequality

$$(u_J, u_J)_1 \approx \|Q_1 u_J\|_0^2 + \sum_{j=2}^J 2^{2j} \|Q_j u_J - Q_{j-1} u_J\|_0^2 \quad \forall u_J \in V_J, \quad (27)$$

can be proved (e.g., using approximation-theoretic and elliptic regularity results), again with constants uniformly bounded with respect to  $J$ .

Let us show that (27) implies Theorem 8. By setting  $\tilde{u}_j = Q_j u_J - Q_{j-1} u_J$  for  $j = 2, \dots, J$ , and  $\tilde{u}_1 = Q_1 u_J$ , we have  $\tilde{u}_j \in \tilde{V}_j$  and  $u_J = \sum_{j=1}^J \tilde{u}_j$ . This implies the upper bound for the stability of the splitting (15):

$$\begin{aligned} \|u_J\|^2 &= \inf_{\tilde{v}_j \in \tilde{V}_j : u_J = \sum_{j=1}^J \tilde{v}_j} \sum_{j=1}^J 2^{2j} \|\tilde{v}_j\|_0^2 \leq \|Q_1 u_J\|_0^2 + \sum_{j=2}^J 2^{2j} \|Q_j u_J - Q_{j-1} u_J\|_0^2 \\ &\leq C(u_J, u_J)_1 = C \ell_J(u_J, u_J). \end{aligned}$$

On the other hand, for an arbitrary decomposition (10), using the fact that the spaces  $\tilde{V}_j$  form an increasing sequence and that the  $Q_j$  are linear projections, we have

$$Q_j u_J = Q_j \left( \sum_{l=1}^J \tilde{v}_l \right) = \sum_{l=1}^j \tilde{v}_l + \sum_{l=j+1}^J Q_j \tilde{v}_l$$

and

$$\begin{aligned} \|Q_j u_J - Q_{j-1} u_J\|_0^2 &= \|\tilde{v}_j + \sum_{l=j+1}^J Q_j \tilde{v}_l - \sum_{l=j}^J Q_{j-1} \tilde{v}_l\|_0^2 \\ &\leq \left( 2 \sum_{l=j}^J \|\tilde{v}_l\|_0 \right)^2 = 4 \left( \sum_{l=j}^J 2^{-l/2} (2^{l/2} \|\tilde{v}_l\|_0) \right)^2 \\ &\leq 4 \left( \sum_{l=j}^J 2^{-l} \right) \left( \sum_{l=j}^J 2^l \|\tilde{v}_l\|_0^2 \right) \leq 2^{-j+3} \sum_{l=j}^J 2^l \|\tilde{v}_l\|_0^2. \end{aligned}$$

An analogous estimation works for  $\|Q_1 u_J\|_0^2$ . Thus,

$$\begin{aligned} \ell_J(u_J, u_J) &\leq C \left( \|Q_1 u_J\|_0^2 + \sum_{j=2}^J 2^{2j} \|Q_j u_J - Q_{j-1} u_J\|_0^2 \right) \\ &\leq C \left( \sum_{j=1}^J 2^{2j} 2^{-j+3} \sum_{l=j}^J 2^l \|\tilde{v}_l\|_0^2 \right) = C \left( \sum_{l=1}^J 2^l \|\tilde{v}_l\|_0^2 \left( \sum_{j=1}^l 2^{j+3} \right) \right) \\ &\leq 16C \left( \sum_{l=1}^J 2^{2l} \|\tilde{v}_l\|_0^2 \right). \end{aligned}$$

Taking the infimum with respect to all admissible representations (10), we see the lower bound for the stability of the splitting (15).

Speaking in practical terms, the orthoprojections  $Q_j$  are still too involved (the solution of (26) is not straightforward), and one would like to replace them by more explicit constructions. Finite element interpolation operators  $I_j u : C(\tilde{\Omega}) \rightarrow \tilde{V}_j$  defined by requiring the interpolation condition  $(I_j u)(x^i, y^i) = u(x^i, y^i)$  at all (boundary and interior) vertices of  $\mathcal{T}_j$  come to mind but do not necessarily lead to the 'optimal' decomposition to prove Theorem 8, more recently, *quasi-interpolants* have been proposed. A simple set of quasi-interpolant operators which could be used for the above linear finite element spaces is given by

$$\tilde{Q}_j u = \sum_{i=1}^{N_j} \frac{(u, \varphi_j^i)_0}{(1, \varphi_j^i)_0} \varphi_j^i, \quad j \geq 1. \quad (28)$$

Although these  $\tilde{Q}_j$  are not projections onto  $\tilde{V}_j$ , they at least reproduce constant functions locally in the interior of the triangulation  $\mathcal{T}_j$  which is often enough (*local preservation of polynomials of a certain degree* is one of the characteristics of quasi-interpolant operators). More importantly, the  $\tilde{Q}_j$  are well-defined and uniformly bounded with respect to  $L_2(\Omega)$ , and they can be computed by fast algorithms.

The typical method to establish the *lower bound* in the stability requirement (12) of a splitting (9) is the proof of so-called *strengthened Cauchy-Schwarz inequalities* [18, 19]. The simplest version is as follows: Introduce a matrix  $E = ((\epsilon_{j,i}))_{j,i=1}^J$  where the entries are defined as the positive constants for which

$$\ell(P_j \tilde{v}_j, P_i \tilde{v}_i)^2 \leq \tilde{\ell}_j(\tilde{v}_j, \tilde{v}_j) \tilde{\ell}_i(\tilde{v}_i, \tilde{v}_i) \quad \forall \tilde{v}_j \in \tilde{V}_j, \tilde{v}_i \in \tilde{V}_i. \quad (29)$$

Without loss of generality, we may assume that  $E$  is symmetric. Let  $\lambda_{\max}(E)$  denote the largest eigenvalue of the matrix  $E$ .

**Lemma 10** For an arbitrary space splitting (9) we have

$$\ell(u, u) \leq \lambda_{\max}(E) \|u\|^2 \quad \forall u \in V, \quad (30)$$

where the matrix  $E$  is determined from the strengthened Cauchy-Schwarz inequalities (29) as described above.

The proof of this lemma is straightforward: Considering an arbitrary admissible representation (10), we obtain

$$\begin{aligned} \ell(u, u) &= \ell\left(\sum_{j=1}^J P_j \tilde{u}_j, \sum_{j=1}^J P_j \tilde{u}_j\right) = \sum_{j,l=1}^J \ell(P_j \tilde{u}_j, P_l \tilde{u}_l) \\ &\leq \sum_{j,l=1}^J \epsilon_{j,l} \tilde{\ell}_j(\tilde{v}_j, \tilde{v}_j)^{1/2} \tilde{\ell}_l(\tilde{v}_l, \tilde{v}_l)^{1/2} \leq \lambda_{\max}(E) \sum_{j=1}^J \tilde{\ell}_j(\tilde{v}_j, \tilde{v}_j). \end{aligned}$$

Since the representation was arbitrary, we arrive at (30).

By properly scaling the auxiliary bilinear forms  $\tilde{\ell}_j$  we can ensure that  $\epsilon_{j,j} = 1$ . As a consequence, we have  $0 \leq \epsilon_{j,l} \leq 1$  for all nondiagonal entries. This implies  $1 \leq \lambda_{\max}(E) \leq J$ , both extremes are possible. Going through our above examples, we see that  $\eta = 1$  in Example 5 because in this case  $E$  can be chosen the identity matrix. For examples where  $J$  is small (such as Example 6), we can use the trivial bound  $\lambda_{\max}(E) \leq J \max \epsilon_{j,j}$ . A nontrivial situation arises in Example 7. By carefully applying Green's formula on each triangle of the underlying triangulations, one can show that

$$(\tilde{v}_j, \tilde{v}_l)_1 \leq C 2^{j-l/2} (2^j \|\tilde{v}_j\|_0) (2^l \|\tilde{v}_l\|_0) \quad \forall \tilde{v}_j \in \tilde{V}_j, \tilde{v}_l \in \tilde{V}_l, \quad j, l \geq 1. \quad (31)$$

Thus, we can choose  $E = ((C 2^{j-l/2})_{j,l=1}^J)$ , and because of the exponential decay of the  $\epsilon_{j,l}$  away from the diagonal, we obtain  $\lambda_{\max}(E) \leq C$  for some absolute constant  $C$ , independently of  $J$ .

**Example 11** We conclude with an appendix to Example 7. Depending on the application, it may happen that the same spaces  $V$  and  $\tilde{V}_j$  are equipped with different choices of bilinear forms. E.g., if the Poisson problem (5) is modified by adding a source term  $q \cdot u(x, y)$ , where, for simplicity,  $q > 0$  is a constant, then the appropriate bilinear form takes the form

$$\ell^q(u, v) = (u, v)_1 + q(u, v)_0 \quad \forall u, v \in V.$$

For large  $q$  one definitely should take into consideration the term associated with the  $L_2$ -scalar product. Therefore, if we again take the finite element spaces of Example 7 then the following results are of interest.

**Lemma 12 a)** *The splitting*

$$\{V_J; (\cdot, \cdot)_0\} \cong \sum_{j=1}^J \{\tilde{V}_j; (\cdot, \cdot)_0\} \quad (32)$$

is stable and has condition  $\kappa = J$ . The following stability bounds for (32) are sharp:

$$J^{-1} \|u_J\|_0^2 \leq \|u_J\|^2 \leq \|u_J\|_0^2 \quad \forall u_J \in V_J. \quad (33)$$

**b)** *The splitting*

$$\{V_J; \ell^q\} \cong \sum_{j=J_0}^J \{\tilde{V}_j; 2^{2j}(\cdot, \cdot)_0\} \quad (34)$$

is stable with condition  $\kappa = O(1)$ , independently of  $q \geq 0$  and  $J$  if  $J_0 = J_0(q)$  is chosen according to the following rules: If  $q \leq 1$  or  $q \geq 2^{2J}$  then  $J_0 = 1$  or  $J_0 = J$ , respectively, while in the intermediate range  $1 < q < 2^{2J}$  the choice  $J_0 = \lceil \log_2 q/2 \rceil + 1$  is appropriate.

**Proof.** Since by definition of the orthoprojections  $Q_j$  we have

$$\|u_J\|_0^2 = \|Q_{J_0}u_J\|_0^2 + \sum_{j=J_0+1}^J 2^{2j}\|Q_ju_J - Q_{j-1}u_J\|_0^2, \quad 1 \leq J_0 \leq J, \quad (35)$$

the upper bound in (33) is obvious (set  $J_0 = 1$  and look at the definition of  $\|u_J\|$  for the splitting (32)). The lower bound follows from Lemma 10 in conjunction with the trivial estimate  $\lambda_{\max}(E) \leq J$ . That the bounds cannot be improved follows by considering special  $u_J$ . E.g, take any  $u_J \neq 0$  which belongs to the  $L_2$ -orthogonal complement space  $W_J = V_J \ominus V_{J-1}$ . By applying  $Q_J - Q_{J-1}$  to any admissible representation  $u_J = \sum_{j=1}^J \tilde{v}_j$ , we get  $u_J = (Q_J - Q_{J-1})\tilde{v}_J$ , and since  $Q_J - Q_{J-1}$  is an orthoprojection onto  $W_J$ , we get

$$\|u_J\|_0^2 = \|(Q_J - Q_{J-1})\tilde{v}_J\|_0^2 \leq \|\tilde{v}_J\|_0^2 \leq \sum_{j=1}^J \|\tilde{v}_j\|_0^2$$

which leads to  $\|u_J\|_0^2 \leq \|u_J\|^2$  for such  $u_J$ , and to the sharpness of the upper bound. Concerning the lower bound, pick  $u_J = \varphi_1^1 \in \tilde{V}_1 \subset V_J$ , and look at the admissible representation given by  $\tilde{v}_j = J^{-1}u_J$ ,  $j = 1, \dots, J$ . Then

$$\|u_J\| \leq \sum_{j=1}^J J^{-2}\|u_J\|_0^2 = J^{-1}\|u_J\|_0^2.$$

This establishes the sharpness of the upper bound.

As to the stability of (34), we will concentrate on the intermediate range  $1 < q < 2^{2J}$  (the reader will be able to deal with the remaining cases). By definition of  $J_0$ , we have  $2^{2(J_0-1)} \leq q < 2^{2J_0}$ . Thus, by (27) and (35) we can estimate

$$\begin{aligned} \ell^q(u_J, u_J) &\geq c(\|Q_1u_J\|_0^2 + \sum_{j=2}^J 2^{2j}\|Q_ju_J - Q_{j-1}u_J\|_0^2) \\ &\quad + q(\|Q_{J_0}u_J\|_0^2 + \sum_{j=J_0+1}^J \|Q_ju_J - Q_{j-1}u_J\|_0^2) \\ &\geq c(2^{2J_0}\|Q_{J_0}u_J\|_0^2 + 2^{-2J_0}\|Q_1u_J\|_0^2 + \sum_{j=2}^{J_0} 2^{2(j-J_0)}\|Q_ju_J - Q_{j-1}u_J\|_0^2) \\ &\quad + \sum_{j=J_0+1}^J (2^{2j} + 2^{2J_0})\|Q_ju_J - Q_{j-1}u_J\|_0^2 \\ &\geq c(2^{2J_0}\|Q_{J_0}u_J\|_0^2 + \sum_{j=J_0+1}^J 2^{2j}\|Q_ju_J - Q_{j-1}u_J\|_0^2). \end{aligned}$$

This gives the upper stability estimate for the splitting (34), with a constant  $\bar{\eta} \leq C$ .

For the lower estimate, we complement the Cauchy-Schwarz inequalities (31) by their trivial counterparts for the  $L_2$ -scalar product

$$(\tilde{v}_j, \tilde{v}_i)_0 \leq 2^{-(j+i)}(2^j\|\tilde{v}_j\|_0)(2^i\|\tilde{v}_i\|_0) \quad \forall \tilde{v}_j \in \tilde{V}_j, \tilde{v}_i \in \tilde{V}_i.$$

Multiplying here by  $q \approx 2^{2J_0}$ , and adding the result to (31) we obtain

$$\ell^q(\tilde{v}_j, \tilde{v}_i) \leq C(2^{lj-l/2} + 2^{-(j+l-2J_0)})(2^j\|\tilde{v}_j\|_0)(2^l\|\tilde{v}_i\|_0) \quad \forall \tilde{v}_j \in \tilde{V}_j, \tilde{v}_i \in \tilde{V}_i,$$

for all  $J_0 \leq j, l \leq J$ . Obviously, in this range of  $j, l$ , the first term  $2^{lj-l/2}$  dominates the second, therefore, again  $\lambda_{\max}(E) \leq C$ , independently on  $q$ , and  $J$ . Applying Lemma 10 concludes the proof of Lemma 12.

### 3 Convergence theory

After this extended introduction into the concept of stable space splittings, we are now deriving the convergence theory for the associated subspace correction methods. Let us briefly link the notation of the previous section to the AS and MS methods as defined in the introduction. All what we have to do is to fix basis systems in the spaces involved, and to identify elements of these spaces with coefficient vectors, and operators between them with matrices. Even though this might temporarily lead to some confusion, we will use the same notation for elements and vectors as well as for operators and matrices, respectively. Thus,  $P_j$  will denote an operator from  $\tilde{V}_j$  into  $V$ , and, at the same time, a rectangular  $N \times N_j$  matrix representing this operator with respect to the bases chosen in  $\tilde{V}_j$  and  $V$ , respectively. Assume that (9) is stable, we will use the notation  $L$  and  $\tilde{L}_j$  for the matrices associated with  $\ell$  (i.e.,  $\ell(u, u) = (Lu, u)$ ) and  $\tilde{\ell}$ . Thus, the matrix representation of the operators  $T_j$  can be derived from (20):

$$(\tilde{L}_j T_j u, \tilde{v}_j) = \tilde{\ell}_j(T_j u, \tilde{v}_j) = \ell(u, P_j \tilde{v}_j) = (Lu, P_j \tilde{v}_j) = (P_j^T Lu, \tilde{v}_j).$$

This gives  $T_j = \tilde{L}_j^{-1} P_j^T L$ , and

$$\mathcal{P} = \sum_{j=1}^J P_j \tilde{L}_j^{-1} P_j^T L \equiv BL, \quad B = \sum_{j=1}^J P_j \tilde{L}_j^{-1} P_j^T. \quad (36)$$

Thus, if we fix the restriction operators as the adjoint operators (transposed matrices) of the prolongations, i.e.,

$$R_j = P_j^T, \quad j = 1, \dots, J, \quad (37)$$

then everything falls into place: The *additive subspace correction method* defined in section 1 is nothing but the *extrapolated Richardson method* (or  $\omega$ -Richardson relaxation) applied to the reformulation (23) of the original variational problem (6). The assumption (37) is more or less natural since we are restricted to symmetric positive definite  $L$  and  $\tilde{L}_j$ ; it ensures that the preconditioner  $B$  is also symmetric positive definite. As a by-product, the *preconditioned conjugate gradient method* with preconditioner  $B$  can be applied for solving (1), and the design of stable space splittings for  $\{V; \ell\}$  with small condition can be viewed as a method of constructing good preconditioners  $B$  in a systematic way. We will give this pcg-method the descriptive name AS-CG.

Here is another useful representation which leads to a unified treatment of AS and MS methods in terms of classical iterative methods for a so-called *extended semi-definite problem*. Set  $\tilde{L}_{j,l} = \tilde{L}_j^{-1} P_j^T L P_l$ ,  $j, l = 1, \dots, J$ , and  $\tilde{N} = \sum_{j=1}^J N_j$ . Define the  $\tilde{N} \times \tilde{N}$  matrix  $\tilde{\mathcal{P}}$  as a  $J \times J$  block matrix whose entries are the  $N_j \times N_l$  matrices  $\tilde{L}_{j,l}$ . We will use the notation

$$\tilde{\mathcal{P}} = \tilde{\mathcal{L}} + \tilde{\mathcal{D}} + \tilde{\mathcal{U}} \quad (38)$$

for the standard decomposition of the block matrix into lower triangular, diagonal, and upper triangular block matrices. Let  $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_J)^T$  be the corresponding block representation of  $\mathbb{R}^{\tilde{N}}$ -vectors. Set  $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_J)^T$ , where the  $\tilde{\phi}_j$  are determined in (21).

**Lemma 13** *Assume that a stable space splitting (9) is given, and that (37) holds.*

a) *If  $\tilde{u}$  is a solution of the semi-definite problem*

$$\tilde{\mathcal{P}}\tilde{u} = \tilde{\phi}, \quad (39)$$

*then  $u = \tilde{P}\tilde{u} \equiv \sum_{j=1}^J P_j \tilde{u}_j$  is the (unique) solution of (6) and its reformulation (23).*

b) *For any fixed  $\tilde{N} \times \tilde{N}$  matrix  $\tilde{\mathcal{B}}$ , we consider the linear iteration*

$$\tilde{u}^{k+1} = \tilde{u}^k + \tilde{\mathcal{B}}(\tilde{\phi} - \tilde{\mathcal{P}}\tilde{u}^k), \quad k \geq 0, \quad (40)$$

*with a starting vector  $\tilde{u}^0$  given. The iteration (40) generates an iteration in  $V$  by the formula  $u^k = \tilde{P}\tilde{u}^k$ ,  $k \geq 0$ . If  $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_{\text{AS}} \equiv \omega \tilde{\mathcal{I}}$ , where  $\tilde{\mathcal{I}}$  is the  $\tilde{N} \times \tilde{N}$  identity matrix, then (40) generates the additive subspace correction method AS associated with the splitting. Analogously, if  $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_{\text{MS}} \equiv (\omega^{-1} \tilde{\mathcal{I}} + \tilde{\mathcal{L}})^{-1}$ , then (40) generates the multiplicative subspace correction method MS associated with the splitting.*

**Proof.** Part a) can be seen from applying  $\tilde{P}$  to both sides of (39) resulting in

$$\tilde{P}\tilde{\mathcal{P}}\tilde{u} = \sum_{j=1}^J \sum_{l=1}^J P_j \tilde{L}_j^{-1} P_j^T L P_l \tilde{u}_l = \mathcal{P}\tilde{P}\tilde{u} = \tilde{P}\tilde{\phi} = \phi .$$

Now compare with Theorem 9.

Analogously, applying  $\tilde{P}$  to both sides of the iteration (40) and using the relationship  $u^k = \tilde{P}\tilde{u}^k$ , we get

$$u^{k+1} = u^k + \tilde{P}\tilde{\mathcal{B}}(\tilde{L}_1^{-1}P_1^T, \dots, \tilde{L}_1^{-1}P_1^T)^T r^k \quad (41)$$

Here the explicit form of the  $\tilde{L}_{j,l}$  and  $\tilde{\phi}_j = \tilde{L}_j^{-1}P_j^T f$  have been utilized (for the latter formula, compare (21)). Thus, setting  $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_{\text{AS}}$ , we immediately arrive at  $u^{k+1} = u^k + B_{\text{AS}}Lr^k$  which coincides with the AS iteration.

To see the result for the MS method, some algebraic transformations are necessary. For convenience, denote  $B_j = \omega P_j \tilde{L}_j^{-1} P_j^T$ . Consider one loop of the MS method. Denote  $u = u^k$ ,  $r = f - Lu^k$ . By induction, we obtain

$$\begin{aligned} v^2 &= u + B_1 r , \\ v^3 &= u + ((B_1 + B_2) - B_2 L B_1) r , \\ v^4 &= u + ((B_1 + B_2 + B_3) - (B_3 L B_1 + B_2 L B_1 + B_3 L B_2) + B_3 L B_2 L B_1) r , \\ &\dots \\ v^{J+1} &= u + \left( \sum_{j=1}^J B_j - \sum_{1 \leq l < j \leq J} B_j L B_l + \dots + (-1)^{J+1} B_J L B_{J-1} L \dots B_2 L B_1 \right) r . \end{aligned}$$

Repeatedly using the formula  $B_j L B_l = \omega^2 P_j \tilde{L}_{j,l} \tilde{L}_l^{-1} P_l^T$ , we get

$$B_{j_s} L B_{j_{s-1}} L \dots B_{j_2} L B_{j_1} = \omega^s P_{j_s} (\tilde{L}_{j_s, j_{s-1}} \dots \tilde{L}_{j_3, j_2} \tilde{L}_{j_2, j_1}) \tilde{L}_{j_1}^{-1} P_{j_1}^T ,$$

for all  $1 \leq j_1 < \dots < j_s \leq J$ . After substitution into the previous representation for  $v^{J+1}$ , and returning to the notation of the MS iteration ( $v^{J+1} = u^{k+1}$ ,  $u = u^k$ ,  $r = r^k$ ), we have

$$u^{k+1} = u^k + \omega \left( \sum_{j=1}^J \sum_{l=1}^{j-1} P_j (I - \omega \tilde{L}_{j,l} + \omega^2 \sum_{m:l < m < j} \tilde{L}_{j,m} \tilde{L}_{m,l} - \dots + (-\omega)^{j-l} \tilde{L}_{j,j-1} \dots \tilde{L}_{l+1,l}) \tilde{L}_l^{-1} P_l^T \right) r^k .$$

This has to be compared with the formula (41) for

$$\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_{\text{MS}} = \omega (\tilde{\mathcal{L}} + \omega \tilde{\mathcal{L}})^{-1} = \omega (\tilde{\mathcal{L}} - \omega \tilde{\mathcal{L}} + \dots + (-1)^{J-1} \tilde{\mathcal{L}}^{J-1}) .$$

By computing the entries of the powers of the lower triangular block matrix  $\tilde{\mathcal{L}}$ , it is not hard to see from the latter expressions that the two iterations coincide. This gives the statement of part b) for the MS method. Lemma 13 is proved.

Thus, the AS and MS methods can be viewed as  $\omega$ -Richardson relaxation and Richardson-SOR iteration applied to the block matrix  $\tilde{\mathcal{P}}$ . Clearly, assuming that all diagonal blocks  $\tilde{L}_{j,j}$  are invertible, we could have chosen  $\tilde{\mathcal{B}} = \omega \tilde{\mathcal{D}}^{-1}$  to define the counterpart of the  $\omega$ -Jacobi relaxation.

By using this formalism, the convergence theory of subspace correction methods reduces to standard derivations as known for the case of block-Jacobi and block-SOR (see [9] or [15, Theorem 18]) Slightly different derivations can be found in [2, 18, 19]. To measure convergence behavior, we will use the energy norm  $\|e^k\|_\ell = \sqrt{\ell(e^k, e^k)} = \sqrt{\ell(L e^k, e^k)}$  of the error  $e^k = u - u^k$  after  $k$  iterations (consequently,  $e^0$  denotes the error of the starting guess). The convergence rate is defined below as the energy norm of the corresponding error iteration matrix.

**Theorem 14** *Let  $V$  be a finite-dimensional Hilbert space, and  $\ell$  a symmetric  $V$ -elliptic bilinear on  $V$ . Assume that (9) is a stable space splitting (with stability constants  $\eta, \bar{\eta}$  and condition  $\kappa$ ) given by the auxiliary spaces  $\{\tilde{V}_j; \ell_j\}$  and the prolongations  $P_j$ ,  $j = 1, \dots, J$ . The restrictions  $R_j$  are defined by (37).*

(i) The additive subspace correction method AS converges if and only if  $0 < \omega < 2/\eta$ . The optimal convergence rate is achieved for  $\omega^* = 2\eta\bar{\eta}/(\eta + \bar{\eta})$ , and equals

$$\rho_{\text{AS}}^* = \min_{0 < \omega < 2\eta} \lambda_{\max}(M_{\text{AS}}) = 1 - \frac{2}{1 + \kappa} . \quad (42)$$

Correspondingly, the AS-CG method converges with the guaranteed error bound

$$\|e^k\|_\ell \leq 2 \left( 1 - \frac{2}{1 + \sqrt{\kappa}} \right)^k \|e^0\|_\ell , \quad k \geq 1 . \quad (43)$$

(ii) Assume that, in addition to the above, strengthened Cauchy-Schwarz inequalities (29) hold such that  $E$  is symmetric and  $\epsilon_{j,j} = 1$  for all  $j = 1, \dots, J$ . Then the multiplicative algorithm subspace correction method MS converges for  $0 < \omega < 2$ . The analogously defined optimal convergence rate can be estimated by

$$(\rho_{\text{MS}}^*)^2 \leq 1 - \frac{1}{\bar{\eta}(2\lambda_{\max}(E) + 1)} . \quad (44)$$

Without assuming (29), one has

$$(\rho_{\text{MS}}^*)^2 \leq 1 - \frac{1}{\log_2(4(J+1)) \cdot \kappa} . \quad (45)$$

The results in (ii) remain valid for any reordering of the spaces in the splitting.

Intuitively, it might seem that the multiplicative algorithm MS should perform better than AS which can be observed in many applications, and parallels the experience with Jacobi- and Gauss-Seidel methods for specific classes of linear systems. However, this practical observation is not reflected by the upper estimates of Theorem 14. In fact, in [14] it was shown that the logarithmic factor in (45) cannot be removed. The counterexamples of [14] are based on some exotic Toeplitz matrices  $L$  and the splittings mentioned in Example 5.

There are numerous modifications of the multiplicative algorithm, and refined theories which serve some applications better, and lead to sharper estimates under special circumstances. We mention the *symmetric multiplicative subspace correction method* SMS, which is the abstract counterpart of the SSOR-method. It combines two steps of the MS method into one and, therefore, formally twice as expensive. The iteration operator takes the form

$$M_{\text{SMS}} = (I - \omega P_J T_J) \dots (I - \omega P_1 T_1) (I - \omega P_1 T_1) \dots (I - \omega P_J T_J) ,$$

and can be viewed as the MS method applied to the splitting

$$\{V; \ell\} \cong \{\tilde{V}_J; \tilde{\ell}_J\} + \dots + \{\tilde{V}_1; \tilde{\ell}_1\} + \{\tilde{V}_1; \tilde{\ell}_1\} + \dots + \{\tilde{V}_J; \tilde{\ell}_J\} .$$

In analogy to the situation with SOR and SSOR, one has  $\rho_{\text{SMS}}^* = (\rho_{\text{MS}}^*)^2$ , i.e., the convergence theory of the SMS method is covered by Theorem 14, (ii). An advantage is that we can now write  $M_{\text{SMS}} = I - BL$ , where  $B$  is symmetric. Thus, the application of the pcg-method is possible, this time with a *multiplicative preconditioner* rather than with the *additive preconditioner* associated with  $\mathcal{P}$ . The use of subspace correction methods as preconditioners in standard Krylov type iterative methods is becoming good practice and can considerably increase the robustness of a solver in comparison with applying subspace correction methods or Krylov space methods on their own.

A more general multiplicative version, the *variable symmetric multiplicative algorithm* has been proposed and analyzed by Bramble et al. (see [2, Algorithm III]). In a multigrid environment, the general recommendation is to allow for more subspace correction steps on the spaces  $\tilde{V}_j$  corresponding to coarse meshes, and, therefore, to the inexpensive subproblems in (20). The benefit is that weaker assumptions suffice to state optimal convergence estimates, without significantly increasing the arithmetic complexity of the iteration in the asymptotical range ( $J \rightarrow \infty$ ). Speaking in terms of subspace splittings, each auxiliary space  $\{\tilde{V}_j; \tilde{\ell}_j\}$  appears  $\nu_j$  times, with  $\nu_j$  not necessarily a fixed number (the MS and SMS methods correspond to  $\nu_j = 1$  and  $\nu_j = 2$ , respectively). Note that these

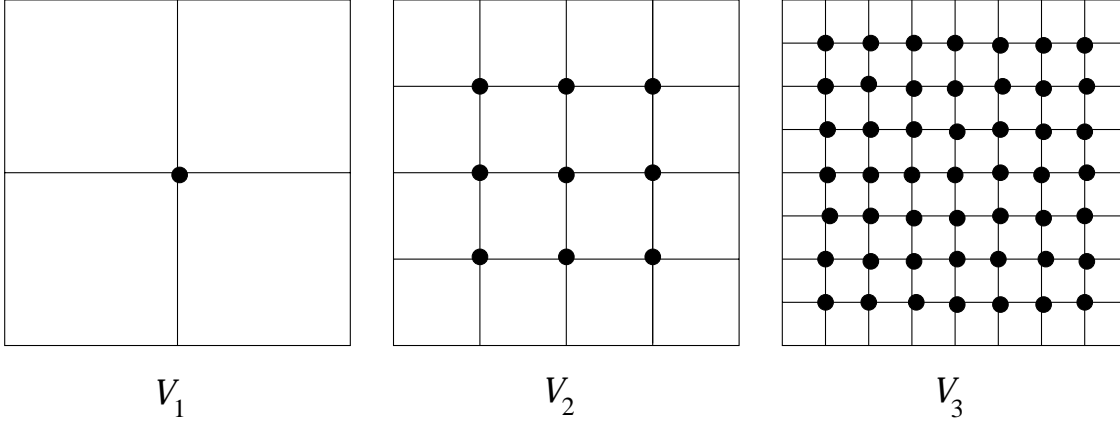


Figure 5: Grids  $\mathcal{V}_j$ ,  $j \leq 3$ , for FDM problems

modifications matter a lot for the multiplicative algorithms but have no immediate impact on the additive method (usually, the estimates for  $\kappa$  cannot be improved this way).

The reader is encouraged to look at the examples of the previous section 1, and to interpret the results on the conditioning of the splittings introduced so far as indicators for the convergence rates of the associated additive and multiplicative subspace correction methods. It is also recommended to derive the corresponding matrix representations and estimate the complexity of the implementation. As should be clear from the abstract theory, the ultimate goal is to obtain stable splittings with small condition (hopefully, independently of discretization and problem parameters) and a reasonable overall operation count for the components  $\tilde{L}_j$ ,  $P_j$ , and  $R_j (= P_j^T)$ ,  $j = 1, \dots, J$ , involved in the algorithms. Roughly speaking, the best we can hope for are so-called *asymptotically optimal algorithms*, the convergence rate of which stays well away from 1, and such that the computational work per iteration grows only linearly with the problem size. The examples of multigrid algorithms and domain decomposition methods discussed below are of this type. Relying mostly on the results for Example 7, we will give examples of such asymptotically optimal algorithms for finite difference discretizations.

## 4 Multigrid applications

In this section we will derive a V-cycle multigrid method for the 5-point discretization of the Poisson problem and justify its asymptotic optimality by interpreting it as a special instance of a multiplicative subspace correction method and using the general theory for the latter. The result is *qualitative*: Other than saying that the convergence factor of the method satisfies  $\rho_h < \rho^* < 1$  (independently of the mesh parameter  $h$ ), no concrete values of  $\rho_h$  resp. of the upper bound  $\rho^*$  can be predicted.

Consider the unit square  $\Omega$  and the sequence of uniform grids  $\mathcal{V}_j = \Omega_{2^{-j}, 2^{-j}}$ ,  $j \geq 1$ . Thus,  $\mathcal{V}_j$  is the set of all interior vertices of the triangulation  $\mathcal{T}_j$  (compare Figures 4 and 5). We will simultaneously speak about vectors in  $\mathbb{R}^{N_j}$  and grid functions on  $\mathcal{V}_j$  assuming that the connection between vector indices and grid points in  $\mathcal{V}_j$  is clear (e.g., given by the ordering discussed in chapter 1). All grid functions are extended to the boundary of  $\Omega$  by assuming zero values at the boundary grid points. Thus, grid functions on  $\Omega_j$  can be identified with finite element functions in  $V(\mathcal{T}_j)$ . Let

$$L_j u_j = f_j \tag{46}$$

be the linear system corresponding to the standard 5-point finite difference discretization of (5) with respect to the grid  $\mathcal{V}_j$ ,  $j \geq 1$ . For short, we will call (46) *FDM problem of level j*.

Our concern will be the construction of a multigrid algorithm for the solution of any of these systems, say, of the FDM problem of level  $J$ . Thus, we set  $V_J = \mathbb{R}^{N_J}$  and  $\ell_J(u_J, v_J) = (L_J u_J, v_J)$ ,

as done before. The simple key to making a ‘qualified’ guess for a suitable space splitting is the following observation: The FDM matrices  $L_j$  of level  $j$  and the Galerkin stiffness matrix (8) of the bilinear form

$$\ell(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx dy$$

with respect to the finite element nodal basis  $\Phi_j$  in  $\tilde{V}_j$  coincide up to the forefactor  $2^{2j} (= h^{-2})$  in  $L_j$ . Just compute the few different values of  $\ell(\varphi_j^n, \varphi_j^m)$  in (8) (only basis functions with nontrivial intersection of supports have to be considered). This is incidental, and does not generalize to more general domains, grids, differential operators or to the 3D case. Note, however, that *spectral equivalence* of the FDM problem of level  $J$  with a corresponding FEM problem would be enough to derive useful results in essentially the same way as detailed below.

All notation is explained in Example 7 or will be introduced below. Denote the transfer operator between finite element functions in  $V(\mathcal{T}_j)$  and vectors in  $V_j$  (grid functions on  $\mathcal{V}_j$ ) by  $\tilde{I}_j$ , and set  $\tilde{I}_j^J = \tilde{I}_j I_j^J$  for all  $1 \leq j \leq J$ . With the proper enumeration of the nodal basis functions in  $\Phi_j$ , the matrix representation of  $\tilde{I}_j$  is the identity matrix, and we will omit this  $\tilde{I}_j$  in later places. Note that the prolongations  $P_j = \tilde{I}_j^J$  have a simple meaning: Given any linear finite element function  $\tilde{u}_j \in \tilde{V}_j$ , the vector  $P_j \tilde{u}_j$  represents the values of  $\tilde{u}_j$  on the finest grid  $\mathcal{V}_J$ . In other words,  $P_j$  corresponds to consecutive *linear interpolation* of grid functions from  $\mathcal{V}_l$  onto  $\mathcal{V}_{l+1}$  along the edges of the triangulation  $\mathcal{T}_l$ ,  $l = j, \dots, J-1$ . As a consequence of Theorem 8 and (18), we have

**Theorem 15** *The following splittings are stable, with uniformly bounded stability constants and condition if  $J \rightarrow \infty$  (compare (16)):*

$$\{V_J; \ell_J\} \cong \sum_{j=1}^J \tilde{I}_j^J \{\tilde{V}_j; \tilde{\ell}_j\} \cong \sum_{j=1}^J \sum_{i=1}^{N_j} \tilde{I}_j^J \{\tilde{V}_j^i; \tilde{\ell}_j^i\}, \quad (47)$$

where  $\tilde{\ell}_j(\tilde{u}_j, \tilde{v}_j) = 2^{2j}(\tilde{u}_j, \tilde{v}_j)_0$ , and  $\tilde{\ell}_j^i$  satisfies (19). The MS method associated with the second splitting represents a V-cycle multigrid method for solving the FDM discretization (46) of level  $J$  while the AS method leads to a multilevel preconditioner. Both methods can be implemented with  $O(N_J)$  operations per iteration and converge at rates  $\leq \rho < 1$ , where  $\rho$  does not depend on  $J$ .

According to the material of section 3, the additive and multiplicative subspace correction methods based on the splittings in (47) should possess convergence rates

$$\rho_{J,AS}^* \leq \rho_1^* < 1, \quad \rho_{J,MS}^* \leq \rho_2^* < 1.$$

Recall that the stronger estimate (44) of Theorem 14 can be applied since strengthened Cauchy-Schwarz inequalities are available for the underlying finite element splitting. Provided that the relaxation parameter is well chosen, the iteration count to reach a given error reduction should therefore not grow with  $J$  in any significant way. Alternatively, pcg-methods such as AS-CG can be used, thereby avoiding the problem of choosing an appropriate  $\omega$ .

Let us derive the details of the algorithms using the second splitting of (47). We will show that the MS method (applied in reverse ordering) is indeed equivalent to a standard *V-cycle multigrid methods*, with one Jacobi relaxation as pre-smoothing step and no post-smoothing step. The AS method is simpler but still reveals the structure of a V-cycle. Recall that  $\tilde{I}_j^J = \tilde{I}_j I_{j-1}^J \dots I_j^{j+1}$  and that the matrix representations of the  $\tilde{I}_j$  are identity matrices and can be omitted. The stencil notation of the finite element restriction operators  $I_{j+1}^j = (I_j^{j+1})^T$  (with respect to the finite element nodal bases) is as follows

$$I_{j+1}^j : \begin{bmatrix} & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & \end{bmatrix}.$$

These restrictions are intermediate to the Full and Half Weighting restriction operators discussed previously. Finally, the scaling of the  $\tilde{\ell}_j^i$  is fixed by setting

$$\tilde{\ell}_j^i(\varphi_j^i, \varphi_j^i) = (\varphi_j^i, \varphi_j^i)_1 = 4.$$

The inversion of  $\tilde{L}_j^i$  on the one-dimensional  $\tilde{V}_j^i$  corresponds to a scalar multiplication by  $1/4$ .

We start with the AS method. According to (3) and (36), it suffices to describe the matrix-vector multiplication for the preconditioner  $B_J$  associated with the splitting. From (36) (compare also (24)) we conclude

$$B_J = 2^{-2J-2} \sum_{j=1}^J I_{J-1}^J \dots I_j^{j+1} (I_j^{j+1})^T \dots (I_{J-1}^J)^T.$$

The factor  $2^{-2J}$  comes from the forefactor  $2^{2J}$  in the splitting while an additional  $1/4$  comes from the inversion of the  $\tilde{L}_j^i$  (see the above scaling for  $\tilde{\ell}_j^i$ ). We will incorporate a factor  $1/2$  into each  $I_{J-1}^J$ . Thus, we set

$$\hat{I}_j^{j+1} = 2^{-1} I_j^{j+1}, \quad \hat{I}_j^J = \tilde{I}_J \hat{I}_{J-1}^J \dots \hat{I}_j^{j+1}, \quad \hat{L}_j = \text{diag}(L_j) = 2^{2j+2} I. \quad (48)$$

The second splitting in (47) can be written in the equivalent form

$$\{V_J; \ell_J\} \cong \sum_{j=1}^J \hat{I}_j^J \{\tilde{V}_j; \hat{\ell}_j\}, \quad (49)$$

where  $\hat{\ell}_j(\tilde{u}_j, \tilde{v}_j) = (\hat{L}_j \tilde{u}_j, \tilde{v}_j) = 2^{2j+2}(\tilde{u}_j, \tilde{v}_j)$  for all  $\tilde{u}_j, \tilde{v}_j \in \tilde{V}_j$ ,  $j = 1, \dots, J$ . As a result, we can simplify the formula for  $B_J$  to

$$B_J = \sum_{j=1}^J \hat{I}_{J-1}^J \dots \hat{I}_j^{j+1} \hat{L}_j^{-1} (\hat{I}_j^{j+1})^T \dots (\hat{I}_{J-1}^J)^T,$$

which can be written in a recursive way:

$$B_1 = \hat{L}_1^{-1}, \quad B_{j+1} = \hat{I}_j^{j+1} B_j (\hat{I}_j^{j+1})^T + \hat{L}_{j+1}^{-1}, \quad j = 1, \dots, J-1. \quad (50)$$

Before we can interpret (50) as a simplified multigrid V-cycle, we will look at the MS method for the second splitting in (47) or, what is the same, for (49). To get an efficient algorithm, reverse ordering is the right choice (i.e., the inner loop of the MS iteration will start with the  $J$ -th subproblem and end with the first one). Set

$$K_j = \omega \hat{I}_{J-1}^J \dots \hat{I}_j^{j+1} \hat{L}_j^{-1} (\hat{I}_j^{j+1})^T \dots (\hat{I}_{J-1}^J)^T, \quad j = 1, \dots, J,$$

and show by induction that the matrices

$$L_j = (\hat{I}_j^{j+1})^T \dots (\hat{I}_{J-1}^J)^T L_J \hat{I}_{J-1}^J \dots \hat{I}_j^{j+1} \quad (51)$$

indeed coincide with the FDM matrices  $L_j$  of level  $j = 1, \dots, J$ , respectively (for general  $L_J$  based on 5-point FDM discretizations, these matrices are usually different from the corresponding FDM discretization on the coarser grid  $\mathcal{V}_j$  but still preserve the 5-point stencil property (see [2, Section 7])). The induction step can be performed in stencil notation. With the stencils for

$$L_{j+1} : \quad 2^{2j+2} \begin{bmatrix} & -1 & & & \\ -1 & 4 & -1 & & \\ & & & & \\ & & & & \\ & -1 & & & \end{bmatrix}, \quad \hat{I}_{j+1}^j : \quad \begin{bmatrix} & & 1/4 & 1/4 & \\ 1/4 & 1/8 & 1/4 & & \\ & & & & \\ 1/4 & 1/4 & & & \\ & & & & \end{bmatrix},$$

at hand, one computes

$$L_{j+1} \hat{I}_{j+1}^j : \quad 2^{2j+2} \begin{bmatrix} & & -1/4 & -1/4 & \\ & -1/2 & 1/4 & 1/2 & -1/4 \\ -1/4 & 1/4 & 1 & 1/4 & -1/4 \\ -1/4 & 1/2 & 1/4 & -1/2 & \\ & -1/4 & -1/4 & & \end{bmatrix},$$

and

$$\hat{I}_j^{j+1} L_{j+1} \hat{I}_{j+1}^j : \quad 2^{2j+2} \begin{bmatrix} & -1/4 & & & \\ -1/4 & 1 & -1/4 & & \\ & & & & \\ & -1/4 & & & \end{bmatrix} = 2^{2j} \begin{bmatrix} & -1 & & & \\ -1 & 4 & -1 & & \\ & & & & \\ & -1 & & & \end{bmatrix}.$$

The last stencil is with respect to  $\mathcal{V}_j$  while the other ones are with respect to  $\mathcal{V}_{j+1}$ .

Using the above notation, the inner loop of one (reverse) MS step takes the form

$$z^j = z^{j+1} + K_j(f_j - L_j z^{j+1}), \quad j = J, \dots, 1, \quad (z^{J+1} = u^k, u^{k+1} = z^1).$$

Thus, the defect iteration of the MS method is given by

$$r^{k+1} = (I - L_J K_1)(I - L_J K_2) \dots (I - L_J K_J) r^k. \quad (52)$$

Let us look at the defect iteration of a V(1,0)-cycle using the  $L_j$  as coarse grid stiffness matrices and  $S_j = I - \omega \hat{L}_j^{-1} L_j$  as (pre-)smoothing. This is nothing but  $\omega$ -Jacobi relaxation used as smoother on all levels. Denote the error propagation matrix for this V-cycle for the FDM problem of level  $j$  by

$$M_j = I - C_j L_j, \quad j = 1, \dots, J,$$

By Theorem 2.4.1 with  $\nu_1 = 1, \nu_2 = 0$ , and  $\gamma = 1$ , we have

$$\begin{aligned} M_{j+1} &= (I - \hat{I}_j^{j+1} C_j (\hat{I}_j^{j+1})^T L_{j+1}) (I - \omega \hat{L}_{j+1}^{-1} L_{j+1}) \\ &= I - (\omega \hat{L}_{j+1}^{-1} + \hat{I}_j^{j+1} C_j (\hat{I}_j^{j+1})^T (I - \omega L_{j+1} \hat{L}_{j+1}^{-1})) L_{j+1}. \end{aligned}$$

To start, set formally  $M_0 = 0$  or, directly,  $M_1 = S_1 = I - \omega \hat{L}_1^{-1} L_1$ . In our particular case,  $\hat{L}_1 = L_1$ . From this a recurrence for  $C_j$  can be derived:

$$C_{j+1} = \omega \hat{L}_{j+1}^{-1} + \hat{I}_j^{j+1} C_j (\hat{I}_j^{j+1})^T - \omega \hat{I}_j^{j+1} C_j (\hat{I}_j^{j+1})^T L_{j+1} \hat{L}_{j+1}^{-1}, \quad j = 1, \dots, J-1, \quad (53)$$

where  $C_0 = 0$  resp.  $C_1 = \omega \hat{L}_1^{-1}$ . In this relation, multiply by  $\hat{I}_{j-1}^j \dots \hat{I}_{j+1}^{j+2}$  from the left and by  $(\hat{I}_{j+1}^{j+2})^T \dots (\hat{I}_{j-1}^j)^T$  from the right, and recall the above expressions for  $K_j$  and  $L_j$ :

$$\hat{K}_{j+1} \equiv \hat{I}_{j-1}^j \dots \hat{I}_{j+1}^{j+2} C_{j+1} (\hat{I}_{j+1}^{j+2})^T \dots (\hat{I}_{j-1}^j)^T = K_{j+1} + \hat{K}_j - \hat{K}_j L_j K_{j+1}.$$

Obviously,  $\hat{K}_1 = K_1$  and  $\hat{K}_J = C_J$ . From this relation, we see that

$$I - L_J \hat{K}_{j+1} = (I - L_J \hat{K}_j)(I - L_J K_{j+1}), \quad j = 1, \dots, J-1,$$

which results in

$$I - L_J C_J = I - L_J \hat{K}_J = (I - L_J K_1)(I - L_J K_2) \dots (I - L_J K_J).$$

What we have shown is that the defect iteration of the MS method (52) and of the above V(1,0) multigrid cycle are identical which implies that the two iterations are identical, too. More importantly, according to Theorem 15 we have proved the optimality of this algorithm (that each iteration requires only  $O(N_j) = O(2^{2j})$  operations was shown for general multigrid cycles). I.e., for any  $0 < \omega < 2$ , convergence is guaranteed and the convergence rate will be bounded away from 1, independently of  $J$ . The same holds true for the AS method (with small enough  $\omega$ ) and the AS-CG algorithm. To a certain extent, we have obtained a strong result, since it guarantees optimality for the simplest multigrid V-cycle algorithm making the optimality of more advanced V-cycle and W-cycle algorithms a very likely matter (one could argue that the AS method is a yet simpler V-cycle method, see the following remarks).

The only but important difference between the AS and MS methods in the multigrid context can be seen from comparing the recursions for  $B_j$  (50) and for  $C_j$  (53): In a multiplicative algorithm, additional smoothing operations involving the coarse grid matrices  $L_j$  on all levels are incorporated, whereas in the additive method the matrices  $L_j$  ( $j < J$ ) are even not required. The recursion for  $C_j$  ‘degenerates’ to the recursion for  $B_j$  if we set  $L_j = 0$ ,  $j = 1, \dots, J$ . Thus, both algorithms can be implemented in essentially the same way. This observation is helpful if a multigrid method is used as preconditioner for  $L_J$  in a Krylov space iteration. The reader is recommended to derive the details for the SMS method which leads (in contrast to the above described MS method) to a symmetric multigrid preconditioner  $C_j$ .

The reader is encouraged to check the few changes that are necessary to adapt the above considerations to Example 11. This example reveals one possibility of modifying the standard

multigrid V-cycle to obtain a robust solution method for the linear problems that arise at each time step when parabolic problems such as the heat equation are solved by implicit schemes with variable time steps.

As can be concluded from the above derivation, the abstract theory of subspace correction methods covers only a certain part of multigrid theory. In particular, the *coarse grid matrices*  $L_j$  have to satisfy (51), i.e., they are defined from  $L_J$  by *Galerkin projection* and depend on the set of prolongation/restriction operators. E.g., if we change the above interpolation scheme inherited from the natural embeddings of the linear finite element spaces to Full Weighting (bilinear interpolation) than the associated Galerkin coarse grid matrices (51) would be defined by compact 9-point stencils, and depend on the difference  $J - j$ . The matrices  $\tilde{L}_j$  resp. the bilinear forms  $\tilde{\ell}_j$  which describe the auxiliary problems on the spaces  $\tilde{V}_j$  of the splitting are essentially responsible for the smoothers. Here, we have restricted our attention to *symmetric positive definite smoothers* and, of course, to *symmetric positive definite problems* (1) from the very beginning. Extensions to cover a broader spectrum of multigrid applications are discussed in [2], see also [18, 19, 11]. For expositions which emphasize *multilevel preconditioning*, (i.e., the AS method in a multigrid context) in connection with finite element and wavelet space decompositions for operator equations, we refer to [15, 5, 6].

## 5 A domain decomposition example

Domain decomposition is a natural concept for the parallelization of large scale computations for the solution of partial differential equations. Introductions to this field can be found in [17, 4]. We will sketch some of the basic algorithmic ideas and the convergence theory, again using the Poisson equation (5) on the unit square  $\Omega$  discretized by a 5-point FDM scheme or, equivalently, by linear finite elements. For simplicity, we fix a grid  $\mathcal{V}_J$  of dyadic stepsize  $h = 2^{-J}$  as our computational grid  $\mathcal{V}$  and, correspondingly,  $\mathcal{T} = \mathcal{T}_J$  as the triangulation of the finite element space. Consider the linear system (1), where  $L = L_J$  is the FDM matrix of level  $J$ .

The basic idea of a domain decomposition method is illustrated in Figure 6, where in a) a decomposition into 4 *nonoverlapping domains* and an *interface*  $\Gamma$  while b) shows a decomposition into 2 *overlapping domains*. On each of the domains, local problems are defined, e.g., by restricting the partial differential equation to the subdomain and complementing it by some boundary conditions. Solving (in parallel) the local problems and glueing them together leads to an approximation of the global problem on  $\Omega$ . Obviously, this procedure defines a preconditioner (i.e., an approximative inverse) for  $L$ , and represents one step of an iterative domain decomposition method. Since it is based on defining subproblems, it should fit into the framework of subspace correction methods and allow for the same modifications as the abstract methods (e.g., cg-accelerations and multiplicative versions are possible).

The reader can imagine that in realistic applications much more general subdomain patterns than shown in Figure 6 can arise, and that the design of suitable decompositions is subject to many side conditions (e.g., the physical nature of the underlying problem, load balancing, and minimization of communication are typical issues). Decompositions into *strips*, where any grid point belongs to at most 2 subdomains, are somewhat easier to handle, and reduce essentially to the situation of two subdomains (such as shown for the overlapping case in Figure 6 b) ). Interior vertices such as in Figure 6 a), where more than two subdomains touch each other, cause theoretical and practical problems. For both basic versions, subdomains are denoted by  $\tilde{\Omega}_m$ ,  $m = 1, \dots, M$ . We introduce the subgrids  $\tilde{\mathcal{V}}_m$  as the part of  $\mathcal{V}$  interior to  $\tilde{\Omega}_m$ , analogously,  $\tilde{\mathcal{T}}_m$  denotes the restriction of  $\mathcal{T}$  to  $\tilde{\Omega}_m$ . The sets of all grid functions on  $\mathcal{V}$  and  $\tilde{\mathcal{V}}_m$  (or, equivalently, linear finite element functions on  $\mathcal{T}$  and  $\tilde{\mathcal{V}}_m$ ) will be denoted by  $V$  and  $\tilde{V}_m$ , respectively. To avoid confusion with the notation used in the previous subsection, we will not make any notational difference between spaces, matrices, and operators for grid functions and finite element functions of different levels  $j = 1, \dots, J$ , assuming that the reader is aware of the identification process and the formal differences. In particular, we will consistently use  $V_j, V_j^i, \ell_j, \ell_j^i$  for the spaces and bilinear forms  $\tilde{V}_j, \tilde{V}_j^i, \tilde{\ell}_j, \tilde{\ell}_j^i$  defined above. The same applies to the prolongations  $\tilde{I}_j^J$ .

As auxiliary problems on  $\tilde{V}_m$  we will consider 5-point FDM discretizations of the same Poisson problem (5) with respect to the domains  $\tilde{\Omega}_m$  instead of  $\Omega$ . In particular, homogeneous Dirichlet boundary conditions are assumed along  $\partial\tilde{\Omega}_m$  (there are a lot of variations such as imposing Neumann

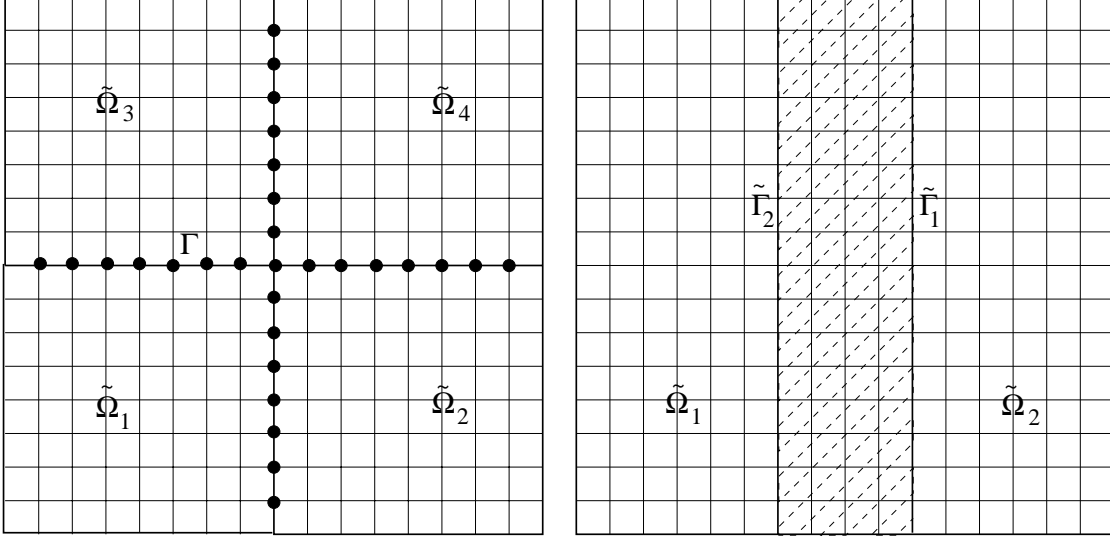


Figure 6: a) Nonoverlapping and b) overlapping domain decompositions

or Robin boundary conditions which have successfully been used [4, 17] but we will not discuss them here). Thus,  $\tilde{L}_m$  is the submatrix of  $L$  associated with the grid points in  $\mathcal{V}_m$ , the associated bilinear form will be denoted by  $\tilde{\ell}_m$ . In the nonoverlapping case, where

$$\mathcal{V}_\Gamma \equiv \mathcal{V} \setminus \cup_{m=1}^M \tilde{\mathcal{V}}_m \neq \emptyset,$$

we also need to create an auxiliary problem for the unknowns associated with the interface  $\Gamma$ . This so-called *interface problem* should in one way or the other approximate the *Schur complement matrix*

$$S_\Gamma = L_\Gamma - \sum_{m=1}^M L_{m,\Gamma}^T \tilde{L}_m^{-1} L_{m,\Gamma} \quad (54)$$

which represents the stiffness matrix for the reduced problem with respect to  $V_\Gamma$ , the set of grid functions on  $\mathcal{V}_\Gamma$  (the finite element counterpart of  $V_\Gamma$  is the trace space of  $V$  onto the interface which consists of linear spline functions interpolating the grid functions defined on  $\mathcal{V}_\Gamma$ ). In (54), the notation comes from rewriting the linear system  $Lx = f$  in a block form related to the subgrids  $\tilde{\mathcal{V}}_m$ :

$$\begin{pmatrix} \tilde{L}_1 & \dots & 0 & L_{1,\Gamma} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \tilde{L}_M & L_{M,\Gamma} \\ L_{1,\Gamma}^T & \dots & L_{M,\Gamma}^T & L_\Gamma \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_M \\ x_\Gamma \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_M \\ f_\Gamma \end{pmatrix}.$$

Thus,  $\tilde{x}_m = \tilde{L}_m^{-1}(\tilde{f}_m - L_{m,\Gamma} x_\Gamma)$ ,  $m = 1, \dots, M$ , and

$$S_\Gamma x_\Gamma = f_\Gamma - \sum_{m=1}^M L_{m,\Gamma}^T \tilde{L}_m^{-1} \tilde{f}_m$$

represents the reduced problem for determining the grid values  $x_\Gamma$  on  $\mathcal{V}_\Gamma$ . The solution of (1) can

formally be written as

$$x_\Gamma = S_\Gamma^{-1} \left( f_\Gamma - \sum_{m=1}^M L_{m,\Gamma}^T \tilde{L}_m^{-1} \tilde{f}_m \right), \quad \tilde{x}_m = \tilde{L}_m^{-1} (\tilde{f}_m - L_{m,\Gamma} x_\Gamma), \quad m = 1, \dots, M.$$

Since  $S_\Gamma$  represents a dense matrix, the explicit computation and storage of which should be avoided, we look for an approximate substitute  $\tilde{S}_\Gamma$  the inverse of which is easy to compute, i.e., we look for a symmetric positive definite preconditioner  $B_\Gamma = \tilde{S}_\Gamma^{-1} \approx S_\Gamma^{-1}$ . Introduce the associated bilinear forms by

$$s_\Gamma(x_\Gamma, y_\Gamma) = (S_\Gamma x_\Gamma, y_\Gamma), \quad \tilde{s}_\Gamma(x_\Gamma, y_\Gamma) = (\tilde{S}_\Gamma x_\Gamma, y_\Gamma).$$

As the above formulas reveal, the extension  $P_\Gamma$  of grid functions on  $\mathcal{V}_\Gamma$  to  $\mathcal{V}$  needs special attention.

Below, we will briefly discuss choices for the components and the stability question of the resulting splitting

$$\{V; \ell\} \cong \sum_{m=1}^M P_m \{\tilde{V}_m; \tilde{\ell}_m\} + P_\Gamma \{V_\Gamma; \tilde{s}_\Gamma\}, \quad (55)$$

where the rectangular matrices  $P_m$  correspond to the extension-by-zero of grid functions on  $\tilde{\mathcal{V}}_m$  to  $\mathcal{V}$  (consequently,  $P_m^T$  represents the natural restriction of grid functions on  $\mathcal{V}$  to  $\tilde{\mathcal{V}}_m$ ). For interpretation of later results, note that the choices

$$\tilde{S}_\Gamma = S_\Gamma, \quad P_\Gamma^T = (-L_{1,\Gamma}^T \tilde{L}_1^{-1}, \dots, -L_{M,\Gamma}^T \tilde{L}_M^{-1}, I_\Gamma), \quad (56)$$

would give rise to a *tight splitting* in (55), with  $\eta = \bar{\eta} = \kappa = 1$  ( $I_\Gamma$  is the identity matrix in the subspace  $V_\Gamma$ ). This fact is expressed by the identity

$$(Lu, u) = \sum_{m=1}^M (\tilde{L}_m \tilde{u}_m, \tilde{u}_m) + (S_\Gamma u_\Gamma, u_\Gamma), \quad u = \sum_{m=1}^M P_m \tilde{u}_m + P_\Gamma u_\Gamma. \quad (57)$$

Clearly,

$$L^{-1} = \sum_{m=1}^M P_m \tilde{L}_m^{-1} P_m^T + P_\Gamma S_\Gamma^{-1} P_\Gamma^T. \quad (58)$$

In the case of overlapping domain decompositions, the introduction of a special interface problem can be avoided, and one directly looks at

$$\{V; \ell\} \cong \sum_{m=1}^M P_m \{\tilde{V}_m; \tilde{\ell}_m\}. \quad (59)$$

As we will see, in both cases the results may depend on the number of domains  $M$ . For obtaining  $M$ -independent convergence results, a so-called *coarse grid problem* has to be included into the definition of  $B_\Gamma$  resp. into the splitting (59). Another question is the systematic replacement of  $\tilde{L}_m^{-1}$  by *inexact solves*, both for the solution of the subproblems associated with the subdomains  $\tilde{\Omega}_m$ , and in the application of  $P_\Gamma$ , see (56). This becomes particularly important if the dimension of the subproblems  $\tilde{N}_m = \dim \tilde{V}_m$  is large, and makes the use of direct solvers prohibitive.

We will now derive a realization of the above concepts by applying Theorem 8 resp. Theorem 15 following essentially [15, 13]. Although the assumptions of this derivation are a bit restrictive, the results are typical and can be used as a guideline in other, more realistic situations. In addition, since the only thing we will do is to regroup the one-dimensional spaces  $V_j^i$  forming the multigrid splittings of Theorem 15 with respect to the subdomains  $\Omega_m$  and the interface  $\Gamma$ , the resulting domain decomposition algorithms could be viewed as a specific way to parallelize a multigrid method. This provides another link between the basic theme of this monograph and domain decomposition methods.

Fix some integer  $J^* = 1, \dots, J-1$ , and let the domains  $\tilde{\Omega}_m$ ,  $m = 1, \dots, 2^{2J^*}$ , form a uniform partition of the unit square  $\Omega$  into squares of sidelength  $H = 2^{-J^*}$ . Figure 6 a) corresponds to the case  $J^* = 1$ . The *interface*  $\Gamma$  consists of the horizontal and vertical grid lines associated with  $\mathcal{V}_{J^*}$ . To start with, let us assume that the linear systems with the coefficient matrices  $\tilde{L}_m$  can be solved

by a *direct method*, i.e., we assume that  $\tilde{L}_m^{-1}$  is available (e.g., in form of a  $LU$ -factorization). This means, that of all components in the representation (58) only  $S_\Gamma^{-1}$  needs some cheap replacement (in other words, we look for a preconditioner for  $S_\Gamma$ ). We will provide this preconditioner by regrouping the components of the multilevel splittings mentioned above. From the definition of  $S_\Gamma$ , we have

$$(S_\Gamma u_\Gamma, u_\Gamma) = \inf_{u: u_\Gamma = u|_\Gamma} \ell(u, u),$$

we leave this as an exercise to the reader. From Theorem 15 we have that

$$\ell(u, u) \approx \inf_{u = I_{J^*}^J u_{J^*} + \sum_{j=J^*+1}^J I_j^J \sum_i u_j^i} \ell_{J^*}(u_{J^*}, u_{J^*}) + \sum_{j=J^*+1}^J \sum_i \ell_j^i(u_j^i, u_j^i), \quad u \in V. \quad (60)$$

To prove (60), use the stability estimate for the first splitting in (47) with  $J$  replaced by  $J^*$  to substitute back  $\ell_{J^*}(u_{J^*}, u_{J^*})$  for the components with  $j < J^*$  in the second splitting of (47). Together this gives

$$(S_\Gamma u_\Gamma, u_\Gamma) \approx \inf_{u_\Gamma = (I_{J^*}^J u_{J^*} + \sum_{j=J^*+1}^J \sum_i I_j^J u_j^i)|_\Gamma} \ell_{J^*}(u_{J^*}, u_{J^*}) + \sum_{j=J^*+1}^J \sum_i \ell_j^i(u_j^i, u_j^i).$$

Since  $\ell_j^i(u_j^i, u_j^i) \geq 0$ , the infimum will not change if we omit all those terms for which  $\varphi_j^i|_\Gamma = 0$  (for  $j > J^*$  this is equivalent to  $\text{supp } \varphi_j^i \subset \tilde{\Omega}_m$  for some  $m$ ). For convenience, for each  $j = J^*, \dots, J$  we denote by  $\hat{V}_j \subset V_j$  the set of all

$$\hat{u}_j = \sum_{i: \varphi_j^i|_\Gamma \neq 0} u_j^i \equiv \sum_{i: \varphi_j^i|_\Gamma \neq 0} c_j^i \varphi_j^i.$$

Note that  $\hat{V}_{J^*} = V_{J^*}$ . Obviously, any such  $\hat{u}_j$  is uniquely determined by its values on  $\Gamma$  (more precisely, by the grid values  $c_j^i$  at the points in  $\mathcal{V}_j^\Gamma = \mathcal{V}_j \cap \Gamma$ ), and can be recovered from its trace  $\hat{u}_j|_\Gamma$  by the discrete *extension-by-zero operator*  $E_j : V_j^\Gamma \equiv \mathcal{V}_j|_\Gamma \rightarrow \hat{V}_j \subset V_j$  of level  $j$  defined by

$$E_j u_j^\Gamma = \begin{cases} u_j^\Gamma & \text{on } \mathcal{V}_j^\Gamma \\ 0 & \text{on } \mathcal{V}_j \setminus \mathcal{V}_j^\Gamma \end{cases}, \quad u_j^\Gamma \in V_j^\Gamma.$$

As before, in all these definitions we identify grid functions on  $\mathcal{V}_j$  and  $\mathcal{V}_j^\Gamma$  with the corresponding linear finite element functions on  $\Omega$  and  $\Gamma$ , respectively. Observe finally that

$$2^{2j} \|\hat{u}_j\|_0^2 \approx \sum_{i: \varphi_j^i|_\Gamma \neq 0} (c_j^i)^2 \approx 2^j \|\hat{u}_j|_\Gamma\|_{0,\Gamma}^2, \quad \hat{u}_j \in \hat{V}_j \quad (61)$$

(the notation  $(\cdot, \cdot)_{0,\Gamma}$  resp.  $\|\cdot\|_{0,\Gamma}$  stands for the scalar product resp. the norm in  $L_2(\Gamma)$ ). Taking all this into consideration, we can continue with

$$\begin{aligned} (S_\Gamma u_\Gamma, u_\Gamma) &\approx \inf_{u_\Gamma = (I_{J^*}^J u_{J^*} + \sum_{j=J^*+1}^J \sum_i I_j^J u_j^i)|_\Gamma} \ell_{J^*}(u_{J^*}, u_{J^*}) + \sum_{j=J^*+1}^J \sum_{i: \varphi_j^i|_\Gamma \neq 0} \ell_j^i(u_j^i, u_j^i) \\ &\approx \inf_{u_\Gamma = (I_{J^*}^J u_{J^*} + \sum_{j=J^*+1}^J I_j^J \hat{u}_j)|_\Gamma} \ell_{J^*}(u_{J^*}, u_{J^*}) + \sum_{j=J^*+1}^J 2^{2j} \|\hat{u}_j\|_0^2 \\ &\approx \inf_{u_\Gamma = u_{J^*}|_\Gamma + \sum_{j=J^*+1}^J u_j^\Gamma} \ell_{J^*}(u_{J^*}, u_{J^*}) + \sum_{j=J^*+1}^J 2^j \|u_j^\Gamma\|_0^2. \end{aligned}$$

The constants in the above two-sided inequalities are independent of  $J^*$  and  $J$ .

The last relationship represents nothing but the stability assertion of a splitting for the Schur complement problem  $\{V_\Gamma; s_\Gamma\}$  with respect to the hierarchy of spaces  $V_{J^*}^\Gamma \subset \dots \subset V_J^\Gamma = V_\Gamma$ . To follow the mathematical formalities, introduce  $\ell_j^\Gamma(u_j^\Gamma, v_j^\Gamma) = 2^j (u_j^\Gamma, v_j^\Gamma)_{0,\Gamma}$  as the auxiliary scalar products on  $V_j^\Gamma$ ,  $j = J^* + 1, \dots, J$ , and denote the natural restriction of  $I_j^J$  to the interface  $\Gamma$  by  $I_j^{J,\Gamma}$ . Formally, we can write  $I_j^{J,\Gamma} = \hat{I}_j^{J,\Gamma} E_j : V_j^\Gamma \rightarrow V_\Gamma$ , where  $\hat{I}_j^{J,\Gamma} = |_\Gamma \circ I_j^J$ . We have proved

**Theorem 16** *Under the above restrictions on  $\{\Omega_m\}$ , the space splitting*

$$\{V_\Gamma; s_\Gamma\} \cong \hat{I}_{J^*}^{J,\Gamma} \{V_{J^*}; \ell_{J^*}\} + \sum_{j=J^*+1}^J I_j^{J,\Gamma} \{V_j^\Gamma; \ell_j^\Gamma\} \quad (62)$$

for the Schur complement problem governed by  $S_\Gamma$  is stable, with stability constants and condition that remain bounded, independently of  $J^*$  and  $J$ .

It is straightforward to realize that the resulting AS and MS methods based on (62) represent modified multigrid V-cycles for the levels  $J^*, \dots, J$  if the bilinear forms  $\ell_j^\Gamma(\cdot, \cdot)$  are discretized using the  $L_2(\Gamma)$ -stability of the basis  $\{\varphi_j^i|_\Gamma\}$  expressed by the second relation in (61). The first modification in comparison with the V-cycles of section 5 consists in the *coarse grid problem* associated with  $\{V_{J^*}; \ell_{J^*}\}$  which requires the solution of a FDM discretization of level  $J^*$ . The second difference is that the prolongation/restriction operations are now performed only with respect to the values on  $\Gamma$ . Therefore, the operation count of the preconditioning step (without multiplication by  $S_\Gamma$  and costs for solving the coarse grid problem) will be proportional to the number of unknowns on  $\Gamma$  which is  $\approx 2^{J+J^*}$ .

The coarse grid problem which arose naturally in the above derivation from the components with  $j \leq J^*$  of the multilevel splitting (47) represents a bottleneck in the parallelization of a domain decomposition code. Historically, the first algorithms that used decompositions with many domains did not include a coarse grid problem, at the cost of reduced convergence rates. In our derivation, the *no-coarse-grid-problem* case can be mimiced as follows: Instead of starting from (60), we could have dropped all components with  $j < J^*$  in (47), and considered the reduced splitting

$$\{V; \ell\} \cong \sum_{j=J^*}^J I_j^J \{V_j; \ell_j\} \cong \sum_{j=J^*}^J \sum_i I_j^J \{V_j^i; \ell_j^i\} \quad (63)$$

as the starting point. This modification leads to a deterioration of the upper stability constant from  $\approx 1$  for the splittings in (47) to  $\approx 2^{2J^*}$  for (63). Indeed, going back to the finite element interpretation, for any  $u \in V = V_J$ , by definition of the triple bar norm for (47), there are  $v_j \in V_j$  such that

$$u = \sum_{j=1}^J v_j, \quad \sum_{j=1}^J 2^{2j} \|v_j\|_0^2 \leq C \ell(u, u).$$

To simplify notation, we have dropped the natural embeddings  $I_j^J$ . Setting  $u_{J^*} = \sum_{j=1}^{J^*} v_j$ , we have by an application of the Cauchy-Schwarz inequality

$$\|u_{J^*}\|_0^2 \leq \left( \sum_{j=1}^{J^*} 2^{-j} (2^j \|v_j\|_0) \right)^2 \leq \sum_{j=1}^{J^*} 2^{2j} \|v_j\|_0^2,$$

which results in

$$u = u_{J^*} + \sum_{j=J^*+1}^J v_j, \quad 2^{2J^*} \|u_{J^*}\|_0^2 + \sum_{j=J^*+1}^J 2^{2j} \|v_j\|_0^2 \leq 2^{2J^*} \sum_{j=1}^J 2^{2j} \|v_j\|_0^2 \leq C 2^{2J^*} \ell(u, u).$$

Thus, the deterioration is no more than by a factor  $\approx 2^{2J^*}$ . To see that this factor can be attained, consider a function from  $V_1$  such as  $u = \varphi_1^1$  which has norms  $\ell(u, u) \approx \|u\|_0^2 \approx 1$  but is not well-represented with respect to the functions  $v_j$ ,  $j \geq J^*$ , allowed in the reduced splittings. The lower bound will remain  $\approx 1$ . The reader will easily verify these facts. If we now proceed as before, we would arrive at a splitting of the form

$$\{V_\Gamma; s_\Gamma\} \cong \sum_{j=J^*}^J I_j^{J,\Gamma} \{V_j^\Gamma; \ell_j^\Gamma\} \quad (64)$$

This splitting does not involve a coarse grid problem, in exchange it inherits the worse condition number  $\kappa \approx 2^{2J^*} = H^{-2}$  from (63).

If  $J^*$  and  $J$  increase, the dimension of the interface problem may become considerably large. For this (and other) reasons, a lot of attempts have been made to further enhance parallelization. A very popular idea is to extend the domain decomposition principle to the interface problem, and to decompose  $\Gamma$  into ‘subdomains’ of its own. The first thing which comes to mind is a decomposition

$$\Gamma = \sum_{m,n} \Gamma_{m,n}, \quad \Gamma_{m,n} = \partial\Omega_m \cup \partial\Omega_n.$$

where the summation extends over all  $m, n$  for which  $\Gamma_{m,n} \neq \emptyset$ . In our example, the  $\Gamma_{m,n}$  are edges associated with the grid  $\mathcal{V}_{J^*}$  which leads to the name *edge spaces* for the sets of grid functions  $V_{m,n}^\Gamma = V_\Gamma|_{\Gamma_{m,n}}$  to be introduced as additional auxiliary spaces. The appealing part of this choice is that potential subproblems associated with these local interfaces are truly one-dimensional and all similar to each other. Problems should be expected at the interior vertices of the domain decomposition which has triggered the introduction of additional *vertex spaces*. The reader who has followed our considerations to this point will be able to introduce local problems on the respective  $\Gamma$ -components by further regrouping the subspaces  $V_j^i$  associated with  $\Gamma$  appearing in the above derivation of Theorem 16. This will lead to potentially better parallelizable  $S_\Gamma$ -preconditioners (compare [17, p. 140]). See [4] for a more comprehensive and systematic discussion of the interface problems arising in connection with non-overlapping domain decompositions, and [17] for numerical support. We have left out many other aspects such as the definition of infinite-dimensional trace spaces, the construction of approximate harmonic extension operators (replacements for  $P_\Gamma$ ), and the connection with boundary integral equations and boundary element methods.

As mentioned before, it is often prohibitive to solve the subproblems  $\tilde{L}_m \tilde{u}_m = \tilde{f}_m$ ,  $m = 1, \dots, M$ , by a direct method (or by an iterative method within machine accuracy). Instead, one would like to replace the action of  $\tilde{L}_m^{-1}$  by a simpler preconditioner and use *inexact solves*. However, this is by no means a trivial task since the  $\tilde{L}_m^{-1}$  enter both  $S_\Gamma$  and  $P_\Gamma$  in a complicated way (see [4, Section 5] and [17, Section 4.4]). Some specific proposals, however, come for free if one reviews our derivation for Theorem 16 carefully. Let us begin with an rearrangement of the splitting associated with (60):

$$\begin{aligned} \{V; \ell\} &\cong I_{J^*}^J \{V_{J^*}; \ell_{J^*}\} + \sum_{m=1}^{2^{2J^*}} \left( \sum_{j=J^*+1}^J \sum_{i: \text{supp } \varphi_j^i \subset \tilde{\Omega}_m} I_j^J \{V_j^i; \ell_j^i\} \right) + \sum_{j=J^*+1}^J \sum_{i: \varphi_j^i|_\Gamma \neq 0} I_j^J \{V_j^i; \ell_j^i\} \\ &\cong I_{J^*}^J \{V_{J^*}; \ell_{J^*}\} + \sum_{m=1}^{2^{2J^*}} \left( \sum_{j=J^*+1}^J \sum_{i: \text{supp } \varphi_j^i \subset \tilde{\Omega}_m} I_j^J \{V_j^i; \ell_j^i\} \right) + \sum_{j=J^*+1}^J I_j^J E_j \{V_j^\Gamma; \ell_j^\Gamma\}. \end{aligned}$$

The stability constants of these splittings are uniformly bounded, independently of  $J^*$  and  $J$ . The replacement of the last group of components is admissible due to the properties of the extension operators  $E_j$  as discussed above. This last group (considered together with the coarse grid problem) is the exact counterpart of the splitting (62). The groups of components associated with the subdomains  $\tilde{\Omega}_m$  represent a replacement of  $\{\tilde{V}_m; \tilde{\ell}_m\}$  by a local multigrid splitting. If the AS method associated with the above splitting are considered then this results in a replacement of  $\tilde{L}_m^{-1}$  by the corresponding local multilevel preconditioner based on an application of Theorem 15 on  $\tilde{\Omega}_m$ . Analogously,  $P_\Gamma S_\Gamma^{-1} P_\Gamma^T$  is replaced by some multilevel preconditioner associated with the values on  $\Gamma$  which is similar in structure to the above preconditioner for  $S_\Gamma$  but also involves the extension operators  $E_j$  and their transposes  $E_j^T$ . As a result, the exact solution of subproblems with  $\tilde{L}_m$ , i.e., the multiplication by  $\tilde{L}_m^{-1}$ , is avoided by replacing it with one iteration step of a multilevel preconditioned iterative method for the subproblem on  $\tilde{\Omega}$ . The reader is encouraged to work out the details.

After this discussion of the nonoverlapping case, we will present the standard result for the *overlapping case* in an analogous setting. In addition to  $1 \leq J^* < J$ , let us fix another integer  $\hat{J}$  such that  $J^* \leq \hat{J} \leq J$ . Set  $\delta = 2^{-\hat{J}}$ , and define the  $\tilde{\Omega}_m$ ,  $m = 1, \dots, 2^{2\hat{J}^*}$ , by extending the dyadic squares of sidelength  $H = 2^{-J^*}$  used above by a corridor of width  $\delta$  in both coordinate directions in the interior of  $\Omega$ . Thus, any two neighboring  $\tilde{\Omega}_m$  overlap in a small strip of width  $2\delta$ . All other specifications are the same as in the nooverlapping case.

**Theorem 17** *For the overlapping decomposition  $\{\Omega_m\}$  just defined, the stability constants and condition of the space splitting*

$$\{V; \ell\} \cong I_{J^*}^J \{V_{J^*}; \ell_{J^*}\} + \sum_{m=1}^{2^{2J^*}} P_m \{\tilde{V}_m; \tilde{\ell}_m\} \quad (65)$$

behave like

$$0 < c \leq \eta \leq \bar{\eta} \leq C 2^{J-J^*} = C \frac{H}{\delta}, \quad \kappa \approx 2^{J-J^*} = \frac{H}{\delta}. \quad (66)$$

The constants in these estimates are independent of  $J^*$ ,  $\hat{J}$ , and  $J$ .

Before we sketch the proof of Theorem 17, we will comment on the practical consequences of it. From (66) we see that only *sufficient overlap*, i.e., when the overlap parameter  $\delta$  becomes proportional to  $H$ , and the inclusion of the coarse grid problem lead to the optimal preconditioning effect ( $\kappa = O(1)$ ). Clearly, this means more work per local problem (e.g., if  $\delta = H$  then a local problem is up to 9 times larger, and the solution of all subproblems would at least take tenfold the time needed for the subproblems associated with a comparable nonoverlapping domain partition). However, as a practical observation, already small overlap  $\delta \approx 2h \dots 4h$  often leads to reasonably good convergence rates, at little extra cost. For the splitting (59) which does not contain a coarse grid problem, the condition number may further increase, at most by a factor  $\approx H^{-2}$ ). In an overlapping environment, the replacement of the direct solves (involving  $\tilde{L}_m^{-1}$ ) by inexact solves on the subdomains is no obstacle: Any spectral equivalent replacement  $B_m \approx \hat{L}_m^{-1}$  would do. A drawback is the increased amount of data communication between neighboring subdomains.

To avoid unnecessary technicalities, let us sketch the argument for the finite element version of Theorem 17. We will again omit the mappings  $I_{J^*}^J$ . The proof of (66) relies on two essential observations. First,

$$\{\tilde{V}_m; \tilde{\ell}_m\} \cong \sum_{j=J^*}^J \sum_{i: \text{supp } \varphi_j^i \subset \tilde{\Omega}_m} \{V_j^i; \ell_j^i\} \quad (67)$$

is stable with  $0 < c \leq \eta \leq \bar{\eta} \leq C < \infty$  with  $c, C$  independent of all parameters. For the domains  $\tilde{\Omega}_m$  under consideration, this is a rather standard consequence of the basic results of Theorem 15 which gives the same result for the domain  $\Omega$ . The reduction is by observing that (67) can be viewed as the *trivial localization* of the splitting

$$\{V_J; \ell_J\} \cong \sum_{j=1}^J \sum_i \{V_j^i; \ell_j^i\} \quad (68)$$

to the subdomain  $\tilde{\Omega}_m$ , where trivial means that all components of the splitting with support at least partially outside  $\tilde{\Omega}_m$  are omitted. Be aware that trivial localization of multilevel splittings to a general subdomain may lead to very poorly conditioned splittings (the above subdomains are among the nice ones in this respect). Since, by the same Theorem 15,

$$\{V_{J^*}; \ell_{J^*}\} \cong \sum_{j=1}^{J^*} \sum_i \{V_j^i; \ell_j^i\} \quad (69)$$

with uniform bounds for the stability constants, we can substitute these splittings for the components of the splitting (65). This results in the splitting

$$\{V; \ell\} \cong \sum_{j=1}^{J^*} \sum_i \{V_j^i; \ell_j^i\} + \sum_{j=J^*}^J \sum_{i: \text{supp } \varphi_j^i \subset \tilde{\Omega}_m} \{V_j^i; \ell_j^i\}, \quad (70)$$

which should have essentially the same stability constants and condition number as (65). These simple manipulations with stable splittings have been introduced and analyzed in [15, p. 82/83] under the names *refinement* and *clustering* of stable splittings.

Thus, it suffices to find estimates for the stability constants of (70). This can be done by comparing the triple bar norms of the splittings (70) and (68) with each other. Let us denote them by  $\| \| u \| \|_{mod}$  and  $\| \| u \| \|$ , respectively. Analogous notation is introduced for the stability constants. The differences between the two splittings are as follows. On the one hand, some of the components  $\{V_j^i; \ell_j^i\}$  occur several times (but no more than 5 times) in (70). On the other, (70) represents a *subsplitting* of (68), i.e., all components in (70) are contained also in (68), the latter splitting contains some more components for  $J^* < j < \hat{J}$  associated with the interface  $\Gamma$  which is defined as in the nonoverlapping case. This immediately gives

$$5 \| \| u \| \|_{mod}^2 \geq \| \| u \| \|^2 \geq \eta \ell(u, u) \implies \eta_{mod} \geq \frac{\eta}{5}.$$

However, in the other direction, we can only prove

$$\| \| u \| \|_{mod}^2 \leq C 2^{\hat{J}-J^*} \| \| u \| \|^2 \leq C 2^{\hat{J}-J^*} \bar{\eta} \ell(u, u).$$

Although this is technically involved, we try to convey the idea. Take any close-to-optimal decomposition of  $u \in V$  with respect to the splitting (68),

$$u = \sum_{j=1}^J v_j \equiv \sum_{j=1}^J \sum_i c_j^i \varphi_j^i : \quad \sum_{j=1}^J \sum_i 2^{2j} \|v_j\|_0^2 \leq C \| \| u \| \|^2,$$

and modify it such that it matches the decompositions admissible in the splitting (70). The only problematic terms are those for which  $\varphi_j^i|_\Gamma \neq 0$  and  $J^* < j < \hat{J}$  (there is nothing to prove in the cases of sufficient overlap  $\hat{J} = J^*$  or  $\hat{J} = J^* + 1$ ). Summing all these terms with the same  $j$ , we define functions  $\hat{v}_j \in \hat{V}_j$ ,  $J^* < j < \hat{J}$ , associated with  $\Gamma$  (see the definition before (61)). Obviously,

$$\|\hat{v}_j\|_0^2, \|v_j - \hat{v}_j\|_0^2 \leq C \|v_j\|_0^2.$$

Setting  $\hat{u}_{J^*+1} = 0$ , we will recursively define

$$\hat{w}_j = \hat{v}_j + \hat{u}_j, \quad \hat{u}_{j+1} = E_{j+1}(\hat{w}_j)|_\Gamma, \quad u_{j+1} = \hat{w}_j - \hat{u}_{j+1}, \quad j = J^* + 1, \dots, \hat{J} - 1.$$

Note that the functions  $u_j \in V_j$  as well as  $\hat{u}_j \in \hat{V}_j$  are linear combinations of terms admissible in (70), and that  $\hat{v}_{J^*+1} + \dots + \hat{v}_{\hat{J}-1} = u_{J^*+2} + \dots + u_j + \hat{u}_j$ . Thus,

$$u = \sum_{j=1}^{J^*} v_j + \sum_{j=J^*}^{\hat{J}-1} (v_j - \hat{v}_j + u_j) + (v_j + u_j + \hat{u}_j) + \sum_{j=\hat{J}}^J v_j \equiv \sum_{j=1}^J w_j$$

is an admissible decomposition in the definition of the triple bar norm associated with (70) which yields

$$\| \| u \| \|_{mod}^2 \leq \sum_{j=1}^J 2^{2j} \|w_j\|_0^2 \leq C \sum_{j=1}^J 2^{2j} \|v_j\|_0^2 + \sum_{j=J^*+1}^{\hat{J}} 2^{2j} \|u_j\|_0^2 + 2^{2\hat{J}} \|\hat{u}_j\|_0^2.$$

If we can show that

$$\sum_{j=J^*+2}^{\hat{J}} 2^{2j} \|u_j\|_0^2 + 2^{2\hat{J}} \|u_j\|_0^2 \leq C 2^{\hat{J}-J^*} \| \| u \| \|^2,$$

then things fall into place. By definition of the recursion we have

$$\begin{aligned} \| \| u_{j+1} \| \|_0^2 &= \| \hat{w}_j - E_{j+1} \hat{w}_j |_\Gamma \|_0^2 \approx \| \hat{w}_j \|_0^2 \approx 2^{-j} \| \hat{w}_j |_\Gamma \|_{0,\Gamma}^2 = 2^{-j} \| (\hat{v}_{J^*+1} + \dots + \hat{v}_j) |_\Gamma \|_{0,\Gamma}^2 \\ &\leq 2^{-j} \left( \sum_{l=J^*+1}^j \| \hat{v}_l |_\Gamma \|_{0,\Gamma} \right)^2 \leq C 2^{-j} \left( \sum_{l=J^*+1}^j 2^{l/2} \| \hat{v}_l \|_0 \right)^2 \\ &\leq C 2^{-j} \left( \sum_{l=J^*+1}^j 2^{-l/2} (2^l \| \hat{v}_l \|_0) \right)^2 \leq C 2^{-J^*-j} \sum_{l=J^*+1}^j 2^{2l} \| \hat{v}_l \|_0^2. \end{aligned}$$

This yields

$$\begin{aligned} \sum_{j=J^*+2}^J 2^{2j} \|u_j\|_0^2 &\leq C 2^{-J^*} \sum_{l=J^*+1}^{J-1} 2^{2l} \|\hat{v}_l\|_0^2 \sum_{j=l}^{J-1} 2^j \leq C 2^{J-J^*} \sum_{l=J^*+1}^{J-1} 2^{2l} \|\hat{v}_l\|_0^2 \\ &\leq C 2^{J-J^*} \sum_{l=1}^J 2^{2l} \|v_l\|_0^2 \leq C 2^{J-J^*} \|u\|^2. \end{aligned}$$

Since  $\hat{u}_j = E_j(\hat{w}_{j-1}|_\Gamma)$  and, thus,  $\|\hat{u}_j\|_0^2 \leq C \|\hat{w}_{j-1}\|_0^2$ , the estimate for the last term is the same. This finishes the derivation of the upper bound

$$\bar{\eta}_{mod} \leq C 2^{J-J^*} \bar{\eta},$$

and Theorem 17, (66), follows from the uniform bounds for  $\eta, \bar{\eta}$  obtained in Theorem 15.

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