Guided Research Project Options for Mathematics and Computational Sciences Undergraduates and Beginning Graduate Students

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1 Generalities

The projects for guided undergraduate research I am offering focus on modern aspects of data and function representation, multi-scale schemes, connections to practical numerical algorithms, and others. Prerequisites are core second year courses in Mathematics such as Analysis I/II, Linear Algebra I, Numerical Methods (or their equivalents in the ESM I-IV offerings), programming skills (Matlab or C(++)) and, last but not least, independent thinking. If you are not sure, please talk to me. Most of them are suitable for incoming graduate students as well, and could serve as topic for a talk in the Graduate Research Seminar, or to get into touch with my research area.

Projects typically contain a reading part (to catch up on new terminology and scientific background), a theoretical problem, and computational challenge, with a lot of freedom to emphasize one or the other aspect. Some projects are loosely connected, and invite for organizing work in groups of 2-3 students (however, every student will still have his own problem to work on). Most of them could lead to a bachelor thesis. For students majoring in Computational Sciences I will "shape" the projects below to emphasize the computational part, and (on request) offer more choices. The same applies to graduate students, i.e., it is possible to adapt a topic to the individual interests and experience.

Some of the topics are offered every year, some are phased out, new ones are added. This year, projects 2 and 7 are substantially modified resp. new.

2 Approximation of Lie-group-valued functions

There are various applications of special matrix groups to communication theory (e.g., to multi-port channel signal transmission). Over the last two-three years, I ran into some applied problems, where SO(3) resp. SU(2) played a role. E.g., we needed to find special parametrizations of SU(2), or approximate a measured transfer matrix function $f : [a, b] \longrightarrow \mathrm{SU}(2)$ for a power-complementary 2-port channel by adequate FIR (described by matrix-valued polynomials) resp. IIR filters (matrix-valued rational functions). Student(s) with some knowledge about Lie groups/algebras, representation theory, etc. might be quick on contributing to these seemingly simple problems (not simple for me!), and try to generalize them. However, any other student is welcome to try as well! Tanya Shingel has written a nice M.Sc. thesis (August 2005) on the approximation part where one could start from.

If you want to get into touch with the engineering part of this topic, I can set up a collaboration with Prof. Christi Madsen (Texas A&M University, College Station TX) or Prof. Reinhold Noe (University of Paderborn) who are electrical engineers and world-leaders in optical fiber communication systems. Talk to me if you are ready for such an adventure!

3 Regularity of refinable functions

Refinable functions (= solutions of a refinement equation) are at the heart of several widely used multi-scale algorithms such as discrete wavelet transform, subdivision, and multi-grid methods.

Under simplifying assumptions (mainly, shift-invariance of the underlying grid structure), properties of refinable functions such as the convergence of the cascade algorithm used to construct them, the approximation power of trial spaces built from their shifts, their smoothness in Hoelder and Sobolev scales, etc., are well-understood. To use these theoretical results in practice, we have built some Matlab code which still needs a lot of improvement. One project is related to understanding and speeding up the computation of upper and lower bounds for L_p-regularity exponents which is based on the concept of joint spectral radius of matrix families. Often, symmetries can be taken into account to reduce the amount of computational work. A second project would start from another approach (Cohen/Daubechies, Pollicott/Weiss) based on Fredholm theory for trace class operators which leads to alternative ways of computing regularity exponents.

4 Multilevel finite element and spline constructions

Finite elements are piecewise polynomial defined over partitions of domains in $\mathbb{R}^d$, and used for discretizing partial differential equations in Engineering and Science. Recently, in connection with adaptive finite element schemes, there has been considerable interest in building some theory for less structured grid topologies. Sample projects are

- the construction and properties of semi-orthogonal (pre-)wavelets for irregularly refined triangular grid sequences,
- the dependence of the L_p-norms of projection operators onto finite element spaces on the properties of the partition,

etc.

5 Subdivision

Subdivision schemes are a generic tool to create fine-scale discrete surfaces from a coarse model for use in computer graphics (animation) and other geometric design applications. The recursively used basic step, a coarse-to-fine step of refining a topology and creating additional geometric information (point locations, normals, etc.), is present in many other computational schemes, for instance, in multi-grid schemes. Projects could be

- theory and computational verification of properties of subdivision schemes on grids consisting of quadrilaterals and triangles (2D manifolds or functions of two variables),
- custom-designing prolongation operators for multi-grid algorithms (e.g., for the Morley finite element case or nonconforming P1 finite elements in 3D),
- nonlinear schemes and their stability (this was taken by C.Kühn (Class of 2005) for his B.Sc thesis but lots of questions remain),

and more. The last subtopic will be co-advised by our graduate student Stanislav Harizanov.

6 Frames, with applications

Frames are redundant systems for representing objects in Hilbert spaces that are useful in signal processing, numerical linear algebra, and other areas. The defining property is

$$\|f\|^2 \approx \sum_k |(f_k, f)|^2/\|f_k\|^2$$

for all f from the closed linear span of the frame system $\{f_k\}$ (here, $A \approx B$ is a short-hand notation for the two-sided inequality $c \cdot A \leq B \leq C \cdot A$ which holds with absolute constants $0 < c \leq C < \infty$ that are independent on the variables in A and B, i.e., on f in the above definition). Obviously, frames generalize orthogonal systems.

Several projects can be pursued, e.g.,

- Characterization/construction of finite thight frames in $\mathbf{R}^n$ and $\mathbf{C}^n$ (this has some connections with geometric extremal problems and quantum information theory).
- Asymptotic convergence results of greedy algorithms are quite different for orthonormal and general systems. It is not known how they behave for (tight) frames.
- Frames (systems of elements) can be generalized to systems of subspaces, with interesting applications to solving large linear systems, e.g., those arising from PDEs. Two possibilities for guided research projects come to mind: adaptivity and the selection of good sub-frames, or the comparison of the performance of Gauss-Seidel, Jacobi, and some other standard iterative methods.
- Quarkonial frames are highly redundant systems of functions that generalize both Gabor and wavelet-type systems. Despite the their high redundancy, they form frames in Besov-Sobolev classes (Triebel, 1997-2003). The study of the frame property of their subsystems is an open and intriguing problem, as positive results would have immediate impact on the justification of modern computational methods such as hp- and meshless discretizations for PDEs.

7 Applied nonlinear dynamics

People from DoCoMo Euro-Lab (the research group of the European arm of a Japanese communication technology company located in Munich) who collaborate with prof. Harald Haas visited IUB in July, and asked for assistance with the mathematical modeling of a particular communication system. After discussions, the problem boiled down to the classification of the generic limit behavior of a discrete dynamical system defined on the n-dimensional torus $\mathbb{T}^n$, and its dependence on some input parameters. Students with interest in nonlinear analysis and probability could study idealized versions of this dynamical system, prove statements on its asymptotic behavior, or run systematic simulations. A visit to DoCoMo Euro-Lab (or even an internship) is a distant possibility.