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to the reader to construct the switch following the above considerations. It is not so clear whether the preconditioner of section 2 can be useful for situations where the mappings T_j resp. $\tilde{T}_j$ are not available, at least, are not sparse. This is the case for some of the most popular finite element methods involving continuous pressure elements.

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"edge" functions $\hat{\mathbf{N}}_{e,J}$ which are similar to $\mathbf{N}^1_{P,J}$ and $\mathbf{N}_{e,J}$, resp.), and when using the same path system for defining T^*_J as for T_J, the matrices T^*_J, $\tilde{T}^*_J$, will be submatrices of T_J, $\tilde{T}_J$, resp..

In analogy with section 2.3, Theorem 2, we can finally avoid working with the nodal basis in $\hat{Z}_J$ and $\hat{A}_J$, $\hat{f}_J$ in an explicit way, and use in the computations the convential stiffness matrix A^*_J of $a_J(\cdot,\cdot)$ with respect to the nodal basis in $\hat{V}_J$. The reasoning for establishing the next assertion is completely the same as given in section 2.3 for Theorem 2, we therefore state only the final result for the case of a preconditioned Richardson iteration, the generalization to the pcg-setting is straightforward.

Theorem 3 *The preconditioned Richardson iteration for (45) with starting vector $\hat{c}^{(0)}_J$ in $\mathbf{R}^{\hat{m}_J}$ is equivalent to the Richardson iteration*

$$(c^*_J)^{(m+1)} = (c^*_J)^{(m)} + \omega(B^*_J + Q^*_J\tilde{C}_J(Q^*_J)^T)(f^*_J - A^*_J(c^*_J)^{(m)}) , \quad m \geq 0 \qquad (49)$$

with starting vector $(c^*_J)^{(0)} = \tilde{T}^*_J c^{(0)}_J$ for solving the linear system

$$A^*_J c^*_J = f^*_J , \qquad (50)$$

in the subspace $\mathcal{R}\tilde{T}^*_J$ of $\mathbf{R}^{\hat{n}_J}$. The preconditioner $\tilde{C}_J$ is defined by the recursion (37), and

$$B^*_j = \tilde{T}^*_J(\tilde{T}^*_J)^T\tilde{T}^*_J(\tilde{T}^*_J)^T . \qquad (51)$$

Thus, except for the choice of a starting vector $(c^*_J)^{(0)} \in \mathcal{R}\tilde{T}^*_J \subset \mathbf{R}^{\hat{n}_J}$ and the precomputation of the matrix (51), there is no need to specifically use information with respect to the nonconforming divergence-free subspaces $\hat{Z}_j$. With a proper choice of $\omega > 0$, the number of iterations (49) to get a fixed error reduction is independent of J. The operation count per iteration is bounded by $\mathrm{O}(\hat{n}_J)$.

One might think of several modifications. E.g., $\tilde{C}_J$ might be modified as discussed in subsection 2.3. Instead of reducing the $\hat{Z}_J$-$\hat{V}_J$-problem formally to a Z_J-V_J-problem on the same triangulatition which leads to a certain increase of storage, one might switch to a Z_{J-1}-V_{J-1}-problem. Here, the definition of $\hat{Q}_J, Q_J$ needs some more care.

We briefly indicate the extension to other discretizations using divergence-free elements. For $k \geq 4$, the combination of continuous P_k velocity elements and discontinuous P_{k-1} pressure elements is known to yield a local basis and to satisfy the discrete inf-sup condition, see [23] for details and the precise conditions on the triangulation. Since this basis can be obtained via the curl-operator from a basis construction for $X_h = S^1_{k+1}(\mathcal{T}_h)\cap H^2_0(\Omega)$ [18, 23], one may apply the switching procedure in the context of fourth-order problems (see [21, 10]) to define a multilevel preconditioner, or use it for the construction of suitable quasi-interpolant operators acting between the Z-spaces (of the P_k and the above introduced modified conforming P1 elements). Some low-order elements with discontinuous pressure elements of degree 0 resp. 1 have been discussed in [12], we leave it as an exercise with vertices and opposite edges denoted by P_i and e_i, resp. (the simpler case of estimating $\mathbf{u}_J - Q_J\mathbf{u}_J$ is left to the reader). By definition of $\hat{Q}_J$, we have

$$\begin{aligned} \mathbf{w}_j(P_1) &= (\hat{\mathbf{u}}_{e_2} + \hat{\mathbf{u}}_{e_3} - \hat{\mathbf{u}}_{e_1}) - \mathbf{u}_{P_1} \ , \\ \mathbf{w}(M_{e_1}) &= \hat{\mathbf{u}}_{e_1} - (2\hat{\mathbf{u}}_{e_1} - \frac{1}{2}(\mathbf{u}_{P_2} + \mathbf{u}_{P_3})) = \frac{1}{2}(\mathbf{u}_{P_2} + \mathbf{u}_{P_3}) - \hat{\mathbf{u}}_{e_1} \ , \end{aligned}$$

analogously for the other vertices and edge midpoints of Δ (all these values represent the limits of $\mathbf{w}_J$ from the interior of Δ). Note that by the first step of the definition of $\hat{Q}_J$ the values $\mathbf{u}_i = \hat{Q}_J\hat{\mathbf{u}}_J(P_i)$ are convex combinations of the nodal vectors of $\hat{\mathbf{u}}_J$ at the midpoints of edges emenating from P_i, thus, associated with one of the neighbored triangles. This gives, with some absolute constant C,

$$2^{2J}\|\hat{\mathbf{u}}_J - \hat{Q}_J\hat{\mathbf{u}}_J\|^2_{L_2(\Delta)} \le \sum_{e,e'} |\hat{\mathbf{u}}_J(M_e) - \hat{\mathbf{u}}_J(M_{e'})|^2 \ ,$$

where the sum is over all pairs (e, e') of edges in $\mathcal{E}_J$ with a common endpoint and at least touching Δ. Now, sum over all triangles Δ and compare with the definition of the bilinear forms in (39). This establishes Lemma 6.

We wish to mention that $\hat{Q}_J$, Q_J are examples of specific quasi-interpolants in spaces of piecewise polynomial functions, and that the proof of Lemma 6 may be given in terms which generalize to more complicated situations (see [21, Section 3.2] for more information).

Lemma 5 and 6 give an optimal preconditioner for the nonconforming P1 velocity approximation, i.e., for the linear system

$$\hat{A}_J\hat{c}_J = \hat{f}_J \ , \tag{45}$$

where the right-hand side vector $\hat{f}_J$ is composed of the scalar products $(\mathbf{f}, \hat{\mathbf{N}}_{J,i})_0$. This is because all matrices involved are sparse, with $\mathrm{O}(\hat{m}_J)$ non-zero elements (note that $\hat{m}_J \asymp \hat{n}_J \asymp m_J \asymp n_J$). If we denote by $\tilde{T}^*_J$ and T^*_J in analogy to Lemma 1 the transformation matrices between $\hat{Z}_J$ and $\hat{V}_J$ then by obvious properties of the L_2-norm on $\hat{V}_J$

$$b(\hat{\mathbf{u}}_J, \hat{\mathbf{u}}_J) = 2^{2J}\|\hat{\mathbf{u}}_J\|^2_{L_2} \asymp (\tilde{T}^*\hat{c}_J, \tilde{T}^*\hat{c}_J) = ((\tilde{T}^*)^T\tilde{T}^*\hat{c}_J, \hat{c}_J) \ , \tag{46}$$

which leads to

$$\hat{B}_J = (\tilde{T}^*_J)^T\tilde{T}^*_J \tag{47}$$

as a reasonable proposal for the matrix $\hat{B}_J$, compare (43). Recall that $\tilde{T}^*_J$ transforms the nodal basis vector $\hat{c}_J$ of $\hat{\mathbf{u}}_J \in \hat{Z}_J$ into its coefficient vector c^*_J with respect to the basis in $\hat{V}_J$. Also, $\hat{Q}_J$ and Q_J can be expressed as

$$Q_J = T^*_J Q^*_J \tilde{T}_J \ , \qquad \hat{Q}_J = T_J\hat{Q}^*_J\tilde{T}^*_J \ , \tag{48}$$

where Q^*_J and $\hat{Q}^*_J$ are the matrices corresponding to the natural extensions of Q_J and $\hat{Q}_J$, resp., to the pair of spaces V_J and $\hat{V}_J$. With a proper definition and scaling of the basis functions $\hat{\mathbf{N}}_{J,i}$ (consisting of "vertex" functions $\hat{\mathbf{N}}_{P,J}$ and (an explicit choice for $\hat{B}_J$ will be provided below). Then, by the above-mentioned results of [20], we have

Lemma 5 *Let the forms and matrices satisfy the definitions and properties (39)-(43). If the Z_J-preconditioner C_J is defined by the recursion (32) then*

$$\hat{C}_J = \hat{B}_J + Q_J C_J Q_J^T \tag{44}$$

is a symmetric positive definite preconditioner for $\hat{A}_J$. The condition number of the preconditioned linear system $\kappa(\hat{A}_J^{1/2}\hat{C}_J\hat{A}_J^{1/2})$ is uniformly bounded in J whenever the constants in the assumptions are.

Let us define suitable Q_J and $\hat{Q}_J$ by local operations as follows. We explain them by using the nodal vectors (coefficients with respect to the nodal bases in $\hat{V}_J$ and V_J) which will be useful also for the final implementation. The mapping Q_J is essentially given by an edge operation which has to be carried out for all $e \in \mathcal{E}_J$. If $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_e$ denote the nodal vectors of $\mathbf{u}_J \in Z_J$ at the endpoints P_1, P_2, and the midpoint M_e of the edge e, resp., then

$$(Q_J\mathbf{u}_J)(M_e) = \frac{1}{4}(\mathbf{u}_1 + 2\mathbf{u}_e + \mathbf{u}_2)\ .$$

Note that this definition preserves the $I_{\Delta,e}$-values for all e. Thus, the mapping maps Z_J into $\hat{Z}_J$, compare (13), the same edge integrals describe the discrete divergence-free conditions in $\hat{V}_J$.

The mapping $\hat{Q}_J$ which does not enter the algorithm is given by the following two-step procedure. First, nodal vectors $\mathbf{u}_P \equiv (\hat{Q}_J\hat{\mathbf{u}}_j)(P)$ are assigned to all $P \in \mathcal{V}_J$ by taking averages (convex combinations) of the nodal vectors of $\hat{\mathbf{u}}_J$ associated with edges emenating from P (e.g., it would be sufficient to copy the nodal vector $\hat{\mathbf{U}}_e$ corresponding to the midpoint M_e of one of those edges). In a second step, the remaining nodal values $\mathbf{u}_e \equiv (\hat{Q}_J\hat{\mathbf{u}}_J)(M_e)$ are now determined by another edge operation:

$$\mathbf{u}_e = 2\hat{\mathbf{u}}_e - \frac{1}{2}(\mathbf{u}_{P_1} + \mathbf{u}_{P_2}) \qquad \forall\, e \in \mathcal{E}_J\ ,$$

note that the nodal vectors $\mathbf{u}_{P_i}$ are available from the first step. Again, the second step is designed such that $I_{\Delta,e}$-values are preserved such that $\hat{Q}_J$ maps from $\hat{Z}_J$ into Z_J.

Lemma 6 *The mappings Q_J and $\hat{Q}_J$ satisfy (42), even with respect to the larger spaces $\hat{V}_J$ and V_J.*

Proof. Since the E-norm on V_J resp. on $\hat{V}_J$ can be expressed by sums of local differences of nodal vectors, it is actually sufficient to look at the nodal vectors (on the set $\mathcal{V}_{J+1}$) of the W_J-functions $\hat{\mathbf{u}}_J - \hat{Q}_J\hat{\mathbf{u}}_J$ and $\mathbf{u}_J - Q_J\mathbf{u}_J$ and check that they are expressed by suitable linear combinations of such differences. The remaining steps are obvious. We deal with $\mathbf{w}_J \equiv \hat{\mathbf{u}}_J - \hat{Q}_J\hat{\mathbf{u}}_J$ on a generic triangle $\Delta \in \mathcal{T}_J$

asymptotic approximation order. It should be noted that the stiffness matrices associated with the nodal basis in this nonconforming P1 space are extremely sparse (as usual, the bilinear form $a(\cdot,\cdot)$ will be appropriately extended to the nonconforming space under consideration). To match homogeneous Dirichlet boundary conditions, nodal vectors vanish for boundary points.

We again consider the multilevel setting. For the sequence of triangulations (10) obtained by dyadic refinement, we denote the nonconforming P1-spaces by $\hat{V}_j$ and their solenoidal subspaces by $\hat{Z}_j$. The dimensions are $\dim \hat{V}_j \equiv \hat{n}_j = 2|\mathcal{E}_j|$, and $\dim \hat{Z}_j \equiv \hat{m}_j = |\mathcal{E}_j| + |\mathcal{V}_j|$, the standard basis in Z_j consists of $|\mathcal{E}_j|$ functions $\hat{\mathbf{N}}_{e,j}$ associated with edges and $|\mathcal{V}_j|$ functions $\hat{\mathbf{N}}_{P,j}$ associated with vertices. The nodal vector values for these two types of basis functions are the same as schematically shown in the upper half of Figure 2 (for $\mathbf{N}_P^1$ and $\mathbf{N}_e$). See [4, ch. VI.8] for more details. E.g., it is well-known that $\hat{Z}_j$ is closely connected with the nonconforming Morley space $\hat{X}_j$ via a discrete curl-operation, in analogy to the connection between Z_j and X_j which was explored in section 2. The difference is that the hierarchy of Morley spaces $\hat{X}_j$ does not possess nice multilevel splittings (see our recent paper [22] containing some negative results in this direction).

We propose an alternative along the lines of [21]: We provide a two-level switch from the $\hat{Z}_J$-problem to the Z_J-problem that uses the preconditioner of Theorem 1 (and, thus, the multilevel splittings of Z_J resp. X_J) as the approximate solver for the reference problem. Modifications as explained in Theorem 2 are possible, too. The same general idea can be explored for other element types.

We use Theorem 1 of [21] together with the fictitious space lemma of Nepomnjashchich (see [20, Theorem 17] or [21, Lemma 1]). Applied to our context, we first introduce the bilinear forms

$$\begin{array}{rcl} (\mathbf{u}_J,\mathbf{v}_J)_E & \equiv & a_J(\mathbf{u}_J,\mathbf{v}_J) = \sum_{\Delta\in\mathcal{T}_J} \int_\Delta \nabla\mathbf{u}_J : \nabla\mathbf{v}_J \, dx \ , \\ b(\mathbf{u}_J,\mathbf{v}_J) & = & 2^{2j}(\mathbf{u}_J,\mathbf{v}_J)_0 \ , \end{array} \tag{39}$$

for all $\mathbf{u}_J, \mathbf{v}_J \in W_J \equiv \hat{Z}_J + Z_J$. Evidently,

$$\begin{array}{l} a_J(\hat{\mathbf{u}}_J,\hat{\mathbf{u}}_J) = (\hat{\mathbf{u}}_J,\hat{\mathbf{u}}_J)_E \quad \forall\ \hat{\mathbf{u}}_J \in \hat{Z}_J \ , \\ a(\mathbf{u}_J,\mathbf{u}_J) = (\mathbf{u}_J,\mathbf{u}_J)_E \quad \forall\ \mathbf{u}_J \in Z_J \ , \end{array} \tag{40}$$

and

$$(\hat{\mathbf{u}}_J,\hat{\mathbf{u}}_J)_E \le Cb(\hat{\mathbf{u}}_J,\hat{\mathbf{u}}_J) \qquad \forall\ \hat{\mathbf{u}}_J \in W_J \ . \tag{41}$$

Suppose that we have linear mappings $Q_J : Z_J \to \hat{Z}_J$ and $\hat{Q}_J : \hat{Z}_J \to Z_J$ (for simplicity we use the same notation for their matrix representations with respect to the nodal bases in Z_J and $\hat{Z}_J$) satisfying

$$\begin{array}{ll} b(\mathbf{u}_J - Q_J\mathbf{u}_J, \mathbf{u}_J - Q_J\mathbf{u}_J) \le C(\mathbf{u}_J,\mathbf{u}_J)_E & \forall\ \mathbf{u}_J \in Z_J \ , \\ b(\hat{\mathbf{u}}_J - \hat{Q}_J\hat{\mathbf{u}}_J, \hat{\mathbf{u}}_J - \hat{Q}_J\hat{\mathbf{u}}_J) \le C(\hat{\mathbf{u}}_J,\hat{\mathbf{u}}_J)_E & \forall\ \hat{\mathbf{u}}_J \in \hat{Z}_J \ . \end{array} \tag{42}$$

Finally, let $\hat{A}_J$ denote the stiffness matrix of the form $a_J(\cdot,\cdot)$ with respect to the basis of $\hat{Z}_J$, and $\hat{B}_J$ any symmetric positive definite preconditioner for the matrix representation of the form $b(\cdot,\cdot)$ with respect to the basis in $\hat{Z}_J$, i.e., we assume

$$cb(\hat{\mathbf{u}}_J,\hat{\mathbf{u}}_J)_E \le (\hat{B}_J^{-1}\hat{c}_J,\hat{c}_J) \le Cb(\hat{\mathbf{u}}_J,\hat{\mathbf{u}}_J) \qquad \forall\ \hat{\mathbf{u}}_J = \sum_i \hat{c}_{J,i}\hat{\mathbf{N}}_{J,i} \in \hat{Z}_J \ . \tag{43}$$

chosen starting vector $\tilde{c}_J^{(0)} \neq 0$. Examples (b) and (c) correspond to exact solutions $\mathbf{u} = (x, -y)^T$ and $\mathbf{u} = (x^2y, -xy^2)$, resp., while (d) is the standard driven cavity problem (see [25]). For (b)-(d), the reduction to the case of homogeneous boundary conditions was based on an extension of the inhomogeneous boundary values to a function from Z_J supported in a thin boundary strip (which is possibly not the most favourable method). The starting vector for the iteration was 0. In all four cases, forces $\mathbf{f}$ were set to zero. Though the iteration numbers are not so impressive (compared to a scalar Poisson problem with about three times less iterations to reach the same error reduction) they support the results of Theorem 2. We expect significantly less iterations if the discretization of level J is solved in a full multigrid fashion.

J	n_J	(a)	(b)	(c)	(d)
3	450	37	37	37	42
4	1922	45	51	49	55
5	7938	51	61	58	65
6	32258	57	69	66	72
7	130050	61	73	71	77
8	522222	64	76	75	81

Table 1. Iteration numbers for the pcg-iteration.

We briefly comment on further possible modifications. As usual, we could have replaced S_0 by better approximations to A_0^{-1} than just the inverse of the diagonal part of A_0. More generally, the choice of S_j given by (25) which corresponds, roughly speaking, to Jacobi smoothing in the associated multiplicative Schwarz setting is not ultimative. See the proof of Theorem 1 from which it becomes clear that any replacement by other positive definite symmetric matrices $\hat{S}_j$ is allowed if they are uniformly spectrally equivalent to the S_j from (25). Other choices, like non-symmetric Gauss-Seidel type sweeps, require additional considerations. Obviously, the above described subspace splittings can be used in a multiplicative fashion as well (see [30], [29], and the monograph [2] for more information on multigrid theory based on appropriate subspace splittings).

3 Nonconforming P1-P0 elements

This element is possibly the simplest low-order element satisfying the discrete inf-sup condition for regular triangulations with constants uniform in h. It consists of nonconforming linear Crouzeix-Raviart elements for the velocity and piecewise constant pressure elements. The local interpolation problem prescribes nodal vectors for the velocity only at the edge midpoints M_e of the edges of the triangulation (or, equivalently, edge averages). This looks more economical compared to the modified conforming P1-element of section 2. Both elements yield the same *where*

$$\tilde{S}_j = \tilde{T}_j S_j \tilde{T}_j^T \ , \quad j \geq 0 \ . \tag{38}$$

Thus, except for the choice of a starting vector $\tilde{c}_J^{(0)} \in \mathcal{R}\tilde{T}_J \subset \mathbf{R}^{n_J}$ *and the pre-computation of the matrices* $\tilde{S}_j$*, there is no need to specifically use information with respect to the discretely divergence-free subspaces* Z_j*. The convergence properties of (35) are completely the same as the convergence properties of (34). The conclusions of Theorem 1 remain valid for the corresponding pcg-method.*

Proof. By the introduced notation and (33), we have

$$\tilde{c}_J^{(m+1)} = \tilde{c}_J^{(m)} + \omega \tilde{T}_J C_J \tilde{T}_J^T (\tilde{f}_J - \tilde{A}_J \tilde{c}_J^m) \ .$$

Denote $\tilde{C}_j \equiv \tilde{T}_j C_j \tilde{T}_j^T$. We will show that these matrices satisfy (37). For $j = 0$ this is obvious since $C_0 = S_0$, compare with (38) and (37). Using the first relation in (17) in Lemma 1, we can write

$$\tilde{T}_j S_j \tilde{T}_j^T = \tilde{T}_j T_j \tilde{S}_j T_j^T \tilde{T}_j^T \ , \quad j \geq 0 \ .$$

Together with the recursion (32) this gives

$$\tilde{C}_j = \tilde{T}_j T_j (\tilde{I}_j \tilde{C}_{j-1} \tilde{I}_j^T + \tilde{S}_j) T_j^T \tilde{T}_j^T \ , \quad j \geq 1 \ .$$

The following reasoning shows that we can drop the outer multiplications. To this end, we use the second relation in (17). Observe that the ranges of both $\tilde{S}_j$ and $\tilde{I}_j \tilde{C}_{j-1} \tilde{I}_j^T$ are contained in the range of $\tilde{T}_j$. Thus, $\tilde{T}_j T_j$ is the identity on the range of $\tilde{I}_j \tilde{C}_{j-1} \tilde{I}_j^T + \tilde{S}_j$, and can be dropped. As a basic fact of linear algebra, the imbedding of the above ranges implies the imbedding of the null-space of $\tilde{T}_j^T$ into the null-space of $(\tilde{I}_j \tilde{C}_{j-1} \tilde{I}_j^T + \tilde{S}_j)^T = \tilde{I}_j \tilde{C}_{j-1} \tilde{I}_j^T + \tilde{S}_j$. Since the range of $Id - T_j^T \tilde{T}_j^T$ is in the null-space of $\tilde{T}_j^T$ (apply again (17) to see this), we can omit also the matrix product $T_j^T \tilde{T}_j^T$. This proves the assertion.

Note that the computation of $\tilde{S}_j$ might be done in the setup phase, and our suggestion is to store its elements. Since the computation of the $O(n_j)$ non-zero entries of $\tilde{S}_j$ is a local procedure (with respect to the nodal points in $\mathcal{T}_{j+1}$), it might also be done in each iteration step. The same remarks are in order for the prolongation operators $\tilde{I}_j$ which involve the local geometry to a certain extent. We will deal with these questions in connection with the practical testing of the proposed preconditioner in an adaptive code.

The following table contains iteration numbers for some tests with a pcg-method based on Theorem 2 as stand-alone iterative solver. All calculations are for the unit square, with an initial triangulation into two triangles. The iteration was stopped when the error (measured by the l_2-norm of the preconditioned residual) was reduced by a factor of 10^{-6}. The largest problem ($J = 8$) contains about 500000 unknowns. The four runs are for problems specified as follows. Example (a) is a problem with exact solution $\mathbf{u} = \mathbf{u}_h = (0,0)^T$ and a randomly

- Multiplication A_J times $\mathbf{R}^{m_J}$-vector (in each iteration step).
- Multiplication C_J times $\mathbf{R}^{m_J}$-vector (in each iteration step) using the recursion (32).

We do not discuss the set-up procedure, we just mention that

$$A_J = \tilde{T}_J^T \tilde{A}_J \tilde{T}_J \ , \quad f_J = \tilde{T}_J^T \tilde{f}_J \ , \tag{33}$$

where $\tilde{A}_J$ and $\tilde{f}_J$ are the stiffness matrix of the bilinear form $a(\cdot,\cdot)$ and the right-hand side vector of the first equation (6) discretized with respect to the basis in V_J. These formula (or an evaluation of all scalar products involving the basis functions of Z_J directly) would result in $\mathrm{O}(n_J)$ operations for the set-up.

However, we are much more concerned about the higher density of A_J compared to $\tilde{A}_J$ (which asymptotically amounts to a factor of about 4/3 and leads to more storage and time for the matrix-vector multiplication in each iteration step), and about performing the recursion (32) which involves, according to Lemma 4 , multiplications by I_j represented in either of the factorizations (24). A precomputation and storage of all I_j, though possible, seems disadvantegeous to us, at least, if we think about extending to an adaptive environment. Also, to program the geometrical information hidden in T_j (see the basic exchange described before Lemma 1 which uses certain path systems) for all $j \geq 1$ or, if we wish to use the second fatorization in (24), the imbeddings $\hat{I}_j$ for the X_j-spaces, seems to be not very favourable.

We propose an alternative which simplifies these tasks. Let us describe the modified algorithm for a preconditioned Richardson iteration (the changes for pcg are minor and left to the reader). The usual preconditioned Richardson iteration has the form

$$c_J^{(m+1)} = c_J^{(m)} + \omega C_J (f_J - A_J c_J^{(m)}) \ , \quad m \geq 0 \tag{34}$$

with an arbitrarily given starting vector $c_J^{(0)} \in \mathbf{R}^{m_J}$ and a fixed relaxation parameter $\omega > 0$. Multiply this equation by $\tilde{T}_J$, and introduce the vectors $\tilde{c}_J^{(m)} = \tilde{T}_J c_J^{(m)} \in \mathcal{R}\tilde{T}_J \subset \mathbf{R}^{n_J}$ which are nothing else but the coefficient vectors of $\mathbf{u}_J^{(m)} \in Z_J$ associated with $c_J^{(m)}$ with respect to the (simpler) nodal basis in V_j. What we wish to prove is the following

Theorem 2 *The preconditioned Richardson iteration (34) for (26) with starting vector $c_J^{(0)}$ in $\mathbf{R}^{m_J}$ is equivalent to the Richardson iteration*

$$\tilde{c}_J^{(m+1)} = \tilde{c}_J^{(m)} + \omega \tilde{C}_J (\tilde{f}_J - \tilde{A}_J \tilde{c}_J^{(m)}) \ , \quad m \geq 0 \tag{35}$$

with starting vector $\tilde{c}_J^{(0)} = \tilde{T}_J c_J^{(0)}$ for solving the linear system

$$\tilde{A}_J \tilde{c}_J = \tilde{f}_J \ , \tag{36}$$

in the subspace $\mathcal{R}\tilde{T}_J$ of $\mathbf{R}^{n_J}$, in the sense that $\tilde{c}_J^{(m)} = \tilde{T}_J c_J^{(m)}$. The preconditioner $\tilde{C}_J$ is defined by the recursion

$$\tilde{C}_j = \tilde{I}_j \tilde{C}_{j-1} \tilde{I}_j^T + \tilde{S}_j \ , \quad j = J, \ldots, 1 \ , \qquad \tilde{C}_0 = \tilde{S}_0 \ , \tag{37}$$

By definition of R_j^J, we immediately get

$$\bar{g}_J = \sum_{j=0}^{J} \sum_i g_{j,i} = \sum_{j=0}^{J} g_j \ .$$

Vice versa, starting from an arbitrary decomposition of $\bar{g}_J \in X_J$ as used for defining the triple-bar norm (18), we find a corresponding decomposition of $\mathbf{u}_J = \mathbf{curl}_J \bar{g}_J \in Z_J$ with respect to $Z_{j,i}$-functions $\mathbf{v}_{j,i} = \mathbf{curl}_j g_{j,i}$ involving in addition R_j^J. All this follows from (21).

By these preparations, again using the norm equivalence (23), by Lemma 3, (29), and (18), we obtain

$$a(\mathbf{u}_J, \mathbf{u}_J) = \|\mathbf{u}_J\|_V^2 \asymp \|\bar{g}_J\|_{H^2}^2 \ ,$$

and

$$\begin{aligned} |||\mathbf{u}_J|||_J^2 &= \inf_{\mathbf{v}_{j,i} \in Z_{j,i} : \mathbf{u}_J = \sum_{j=0}^{J} \sum_i R_j^J \mathbf{v}_{j,i}} a(\mathbf{v}_{j,i}, \mathbf{v}_{j,i}) \\ &\asymp \inf_{g_{j,i} : \bar{g}_J = \sum_{j=0}^{J} \sum_i g_{j,i}} \|g_{j,i}\|_{H^2}^2 \\ &\asymp \inf_{g_j : \bar{g}_J = \sum_{j=0}^{J} g_j} 2^{4j} \|g_j\|_{L_2}^2 \ . \end{aligned}$$

We conclude the proof of (31) by using Lemma 2 and (30).

To see the matrix representation, we start from (28). Let $\bar{c}_J$ be the coefficient vector corresponding to $\mathbf{u}_J$. To compute $\mathcal{P}_J \mathbf{u}_J$, we need first the vectors c_j with the entries $a(\mathbf{u}_J, R_j^J \mathbf{N}_{j,i}) / a(\mathbf{N}_{j,i}, \mathbf{N}_{j,i})$. By definition of A_J, $R_j^J = R_{J-1}^J \ldots R_j^{j+1}$ and its corresponding matrix representation $I_J \ldots I_{j+1}$, it is not difficult to see that $a(\mathbf{u}_J, R_j^J \mathbf{N}_{j,i})$ coincides with the i-th entry (corresponding to $\mathbf{N}_{j,i}$) of the $\mathbf{R}^{m_j}$-vector $I_{j+1}^T \ldots I_J^T A_J \bar{c}_J$. Thus, by definition of S_j,

$$c_j = S_j I_{j+1}^T \ldots I_J^T A_J \bar{c}_J \ , \quad j = 0, 1, \ldots, J \ .$$

Now, writing the summation in (28) in a recursive way: $\mathcal{P}_J \mathbf{u}_J = \mathbf{v}_J$ where

$$\mathbf{v}_j = R_{j-1}^j \mathbf{v}_{j-1} + \sum_i c_{j,i} \mathbf{N}_{j,i} \ , \quad j = 0, \ldots, J \quad (R_{-1}^0 \mathbf{v}_{-1} \equiv \mathbf{0}) \ ,$$

and switching to the matrix representation, (32) comes out.

2.3 Discussion of the algorithm

A straightforward implementation of the pcg-method for the solution of (26) using the preconditioner C_J of Theorem 1 contains the following essential steps (the scalar products and linear combinations of vectors necessary in each iteration step will be neglected):

- Set-up of (26), i.e., computation of A_J and f_J.

then the extremal eigenvalues of $\mathcal{P}_J$ are given by

$$\lambda_{\min}(\mathcal{P}_J) = \inf_{0\neq \mathbf{u}_J\in Z_J} \frac{a(\mathbf{u}_J,\mathbf{u}_J)}{|||\mathbf{u}_J|||_J^2}\ , \qquad \lambda_{\max}(\mathcal{P}_J) = \sup_{0\neq \mathbf{u}_J\in Z_J} \frac{a(\mathbf{u}_J,\mathbf{u}_J)}{|||\mathbf{u}_J|||_J^2}\ . \tag{30}$$

The next assertion is the main theoretical result of this paper, it states a condition number estimate for $\mathcal{P}_J$ which is uniform in J, and provides a matrix representation of $\mathcal{P}_J$.

Theorem 1 *The additive Schwarz operator $\mathcal{P}_J$ associated with the splitting (27) and defined by (28) possesses uniform lower and upper bounds for its spectrum, i.e.*

$$0 < c \le \lambda_{\min}(\mathcal{P}_J) \le \lambda_{\max}(\mathcal{P}_J) \le C < \infty\ , \tag{31}$$

with constants independent of J. The matrix representation of $\mathcal{P}_J$ with respect to the nodal basis in Z_J has the form C_JA_J where the symmetric positive definite preconditioner C_J is given by the recursion

$$C_j = I_jC_{j-1}I_j^T + S_j\ , \quad j = J, J-1, \ldots, 1\ , \qquad C_0 = S_0\ . \tag{32}$$

Thus, the use of C_J in a pcg-method for solving the linear system $A_Jc_J = f_J$ associated with (9) on the subspace Z_J leads to iteration numbers depending only on the desired accuracy, and to an optimal $\mathrm{O}(n_J)$ operation count per iteration step.

Proof. The last assertion follows from Lemma 4 and formula (32), recall that S_j is diagonal and that the dimensions n_j grow geometrically. The iteration number bound follows from the uniform boundedness of the condition number of $\mathcal{P}_J$ which coincides with that of the matrix $A_J^{1/2}C_JA_J^{1/2}$, and the properties of the pcg-method. What remains to show are the validity of the matrix representation and (31).

We start with the latter. According to (30), we have to show that $a(\mathbf{u}_J,\mathbf{u}_J) \asymp |||\mathbf{u}_J|||_J^2$ on Z_J, with J-independent constants. Consider any representation

$$\mathbf{u}_J = \sum_{j=0}^{J}\sum_i R_j^J\mathbf{v}_{j,i}\ , \quad \mathbf{v}_{j,i} \in Z_{j,i}\ .$$

Denote $\bar{g}_J = (\mathbf{curl}_J)^{-1}\mathbf{u}_J$ and $g_{j,i} = (\mathbf{curl}_j)^{-1}\mathbf{v}_{j,i}$. Furthermore, set

$$g_j = \sum_i g_{j,i} \in X_j\ , \quad j = 0,\ldots,J\ .$$

Note that by $\mathbf{v} \in Z_{j,i}$ and (21) each $g_{j,i}$ is a multiple of a nodal basis function from X_j. By standard properties of finite element basis functions (for our X_j, compare [19]), we have

$$\sum_i \|g_{j,i}\|_{H^2}^2 \asymp \sum_i 2^{4j}\|g_{j,i}\|_{L_2}^2 \asymp 2^{4j}\|g_j\|_{L_2}^2\ , \quad j \ge 0\ .$$

Proof. Take a $g_{j-1} \in X_{j-1}$ such that $\mathbf{curl}_{j-1} g_{j-1}|_{\mathcal{V}_j} = \mathbf{curl}\, g_{j-1}|_{\mathcal{V}_j}$ coincides with $\tilde{c}_{j-1} = \tilde{T}_{j-1} c_{j-1}$ (such a function exists for any $\mathbf{u}_j \in Z_j$ with coefficient vector c_{j-1} and is completely characterized by the nodal vectors on $\mathcal{V}_j$). By definition of I_j, the vector $\tilde{T}_j I_j c_{j-1}$ contains exactly the nodal vectors of $\mathbf{curl}\, g_{j-1}$ on $\mathcal{V}_{j+1}$. Now observe that $\mathbf{curl}\, g_{j-1}$ is a piecewise linear, divergence-free vector function on the triangulation $\tilde{\mathcal{T}}_{j-1}$ and compare with the above definition of $\tilde{I}_j$. This gives $\tilde{T}_j I_j = \tilde{I}_j \tilde{T}_{j-1}$. Now multiply with T_j and use (17). Lemma 4 is established.

2.2 Condition number estimate

The derivation of the preconditioner for solving (9) on the discretely divergence-free subspace $Z_h = Z_J$ (with J standing for the level number of the finest grid) and its properties is based on the abstract theory for additive Schwarz subspace correction methods, see [30, 29, 17, 20] for overviews. Let $A_j \equiv A_{Z_j}$ stand for the stiffness matrix of (7) with respect to the above described basis in Z_j. For later use, set

$$S_j = (\mathrm{diag}(A_j))^{-1}\ , \quad j \geq 0\ . \tag{25}$$

We construct an iterative solver for the linear system

$$A_J c_J = f_J\ , \tag{26}$$

which represents the discretization of (7) with respect to the basis in Z_J (accordingly, the vector $f_J \in \mathbf{R}^{m_J}$ contains the $(\cdot,\cdot)_0$-scalar products of $\mathbf{f}$ with the basis functions of Z_J).

The subspace splitting for Z_J that we use for this purpose is inherited from the BPX-splitting of X_J (cf. [19, Remark 3.4]) and has the form

$$Z_J = \sum_{j=0}^{J} \sum_i R_j^J Z_{j,i} \tag{27}$$

where $R_j^J = R_{J-1}^J \dots R_j^{j+1}$, $j < J$, $R_J = Id$, the inner summation is with respect to all basis functions of Z_j (indexed by $i = 1, \dots, m_j$), and $Z_{j,i}$ stands for the one-dimensional subspace spanned by $\mathbf{N}_{j,i}$. All spaces are considered as subspaces of V, thus, $a(\cdot,\cdot)$ defines the scalar product in Z_J and all $Z_{j,i}$. According to the abstract theory (see, e.g., [20, Section 4.1, Theorem 17] or [17]), the additive Schwarz operator $\mathcal{P}_J : Z_J \to Z_J$ associated with the splitting (27)

$$\mathcal{P}_J = \sum_{j=0}^{J} \sum_i \mathcal{P}_{j,i}\ , \qquad \mathcal{P}_{j,i}\mathbf{u}_J = \frac{a(\mathbf{u}_J, R_j^J \mathbf{N}_{j,i})}{a(\mathbf{N}_{j,i}, \mathbf{N}_{j,i})} R_j^J \mathbf{N}_{j,i} \quad \forall\, (j,i)\ , \tag{28}$$

is symmetric positive definite with respect to $a(\cdot,\cdot)$. Moreover, if we define

$$|||\mathbf{u}_J|||_J^2 = \inf_{\mathbf{v}_{j,i} \in Z_{j,i}\,:\, \mathbf{u}_J = \sum_{j=0}^{J} \sum_i R_j^J \mathbf{v}_{j,i}} a(\mathbf{v}_{j,i}, \mathbf{v}_{j,i}) \qquad \forall\ \mathbf{u}_J \in Z_J\ . \tag{29}$$

(in the previously described steps, the superscript $*$ has to be replaced by either x or y).

- All remaining edges $e \in \mathcal{E}_j$ are interior to some triangle $\Delta' \in \mathcal{T}_{j-1}$, and there is exactly one triangle $\Delta \in \mathcal{T}_j$ with vertices $P_0 \in \mathcal{E}_{j-1}$, and $P_1, P_2 \in \mathcal{E}_j$ such that $e = [P_1, P_2]$. Thus, the nodal vectors ν_i at P_i are known from the first step ($i = 0, 1, 2$). The nodal vector $\nu = (\tilde{c}^x_{M_e,j}, \tilde{c}^y_{M_e,j})^T$ will be chosen such that the linear interpolant to the data $\{\nu_0, \nu, \nu_1\}$ resp. $\{\nu_0, \nu, \nu_2\}$ are divergence-free on the triangles $\Delta_1 = \Delta_{P_0 M_e P_1}$ resp. $\Delta_2 = \Delta_{P_0 M_e P_2}$. This local, two-dimensional problem reduces by Green's formula and some elementary transformations to the equations

$$\begin{aligned} \mathbf{n}_1 \cdot \nu &= \frac{\mathbf{n}_1 + \mathbf{n}_2}{2} \cdot \nu_0 + \frac{\mathbf{n}_1 - \mathbf{n}_2}{2} \cdot \nu_1 \\ \mathbf{n}_2 \cdot \nu &= \frac{\mathbf{n}_1 + \mathbf{n}_2}{2} \cdot \nu_0 - \frac{\mathbf{n}_1 - \mathbf{n}_2}{2} \cdot \nu_2 \end{aligned}$$

from which the Cartesian coordinates of ν can be computed (see Figure 6 for the used notation, note that $\mathbf{n}_i$ is obtained from rotating e_i by an angle $\pi/2$ counter-clockwise and has the same length as the edge e_i).

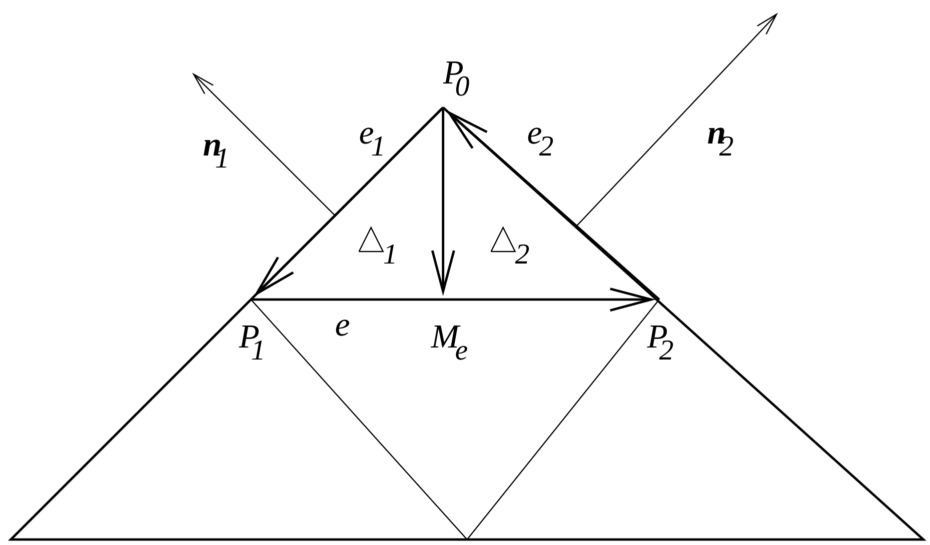

Figure 6: Notation for the definition of I_j

Lemma 4 *The matrix I_j corresponding to the mapping*

$$R^j_{j-1} = \mathbf{curl}_j(\mathbf{curl}_{j-1})^{-1} \,:\, Z_{j-1} \to Z_j$$

possesses the factorizations

$$I_j = T_j \tilde{I}_j \tilde{T}_{j-1} = L_j \hat{I}_j L^{-1}_{j-1} \,, \tag{24}$$

where $\tilde{I}_j$ is defined above. I_j as well as $\tilde{I}_j$ are sparse, with $\mathrm{O}(n_j)$ *non-zero entries.*

Lemma 3 *The discrete curl-operation* $\mathbf{curl}_j : X_j \to Z_j$ *is one-to-one and norm-preserving:*

$$c\|g_j\|_{H^2} \le \|\mathbf{curl}_j\, g_j\|_V^2 \le C\|g_j\|_{H^2}^2 \qquad \forall\, g_j \in X_j\ , \tag{22}$$

uniformly in $j \ge 0$. *Moreover, the matrix representation* L_j *of* $\mathbf{curl}_j$ *with respect to the bases in* X_j *and* Z_j *is diagonal, with the diagonal entries given by (21).*

Proof. What is left to prove is (22). By [16, Theorem 3.1 and 5.5, (5.37)],

$$\|g_j\|_{H^2}^2 \asymp \|\mathbf{curl}\, g_j\|_V^2 \qquad \forall\, g_j \in X_j \subset H_0^2(\Omega)\ ,\ \ j \ge 0\ . \tag{23}$$

From the above considerations, we know that the function $\hat{\mathbf{u}}_j = \mathbf{curl}\, g_j$ belongs to $S_1^0(\tilde{\mathcal{T}}_j)^2 \cap V$ and coincides with $\mathbf{u}_j$ on the edges of $\mathcal{E}_j$. Consider any interior triangle $\Delta \in \mathcal{T}_j$ and map it (by using an affine-linear one-to-one transformation $T_\Delta : \Delta \to \Delta_0$) onto the master triangle Δ_0 with vertices $(0,0),(1,0),(0,1)$ (the slight modifications for boundary triangles are left upon the reader). Obviously, the spaces $X_j|_\Delta, Z_j|_\Delta$, and $\hat{Z}_j|_\Delta$ of all possible restrictions $g_j|_\Delta$, $\mathbf{u}_j|_\Delta$ and $\hat{\mathbf{u}}_j|_\Delta$ to the triangle Δ are mapped one-to-one to similar spaces X_{Δ_0}, Z_{Δ_0}, and $\hat{Z}_{\Delta_0}$ of dimension 12, 11, and again 11, respectively. Natural mappings between these local spaces are provided by the above **curl**-operations (which transform linearly under T_Δ). To see that the $V = H^1 \times H^1$ semi-norms (restricted to Z_{Δ_0} resp. to $\hat{Z}_{\Delta_0}$) are equivalent, let us fix a basis $\{h_i\ :\ i = 0,\dots,11\}$ in the space X_{Δ_0} such that $h_0(x,y) = 1,\ h_1(x,y) = x,\ h_2(x,y) = y$. Then

$$\mathbf{w}_i = \mathbf{curl}_j\, h_i \ \text{ resp. }\ \hat{\mathbf{w}}_i = \mathbf{curl}_j\, h_i\ , \quad i = 1,\dots,11\ ,$$

form bases in Z_{Δ_0} resp. $\hat{Z}_{\Delta_0}$, with the special property that in both cases for $i = 1,2$, the V semi-norm of the basis functions vanishes while for the other $i = 3,\dots,11$ it does not. Since we are on fixed finite-dimensional spaces, this allows us to conclude that

$$\|\mathbf{curl}_j\, g_j(T_\Delta^{-1}\cdot)|_\Delta\|_{V|_{\Delta_0}} \approx \|\mathbf{curl}\, g_j(T_\Delta^{-1}\cdot)|_\Delta\|_{V|_{\Delta_0}} \qquad \forall\, g_j \in X_j|_\Delta$$

with some absolute constants. Using the regularity of $\{\mathcal{T}_j\}$ and summing over all $\Delta \in \mathcal{T}_j$, we get $\|\mathbf{u}_j\|_V^2 \asymp \|\hat{\mathbf{u}}_j\|_V^2$ with some absolute constants. The latter norm equivalence and (23) result in (22). Lemma 3 is proved.

As a final auxiliary result, let us derive a suitable matrix representation I_j of the mapping $R_{j-1}^j \equiv \mathbf{curl}_j(\mathbf{curl}_{j-1})^{-1}\ :\ Z_{j-1} \to Z_j$. We do not wish to use the obvious factorization $I_j = L_j \hat{I}_j L_{j-1}^{-1}$ where $\hat{I}_j$ denotes the $m_j \times m_{j-1}$ matrix representation of the natural embedding $X_{j-1} \to X_j$ in the corresponding nodal bases. Define the $n_j \times n_{j-1}$ matrix $\tilde{I}_j$ by the following operations: Given $\tilde{c}_{j-1}$, the coefficient vector of $\mathbf{u}_{j-1} \in V_{j-1}$, find $\tilde{c}_j = \tilde{I}_j \tilde{c}_{j-1}$ according to the rules

- If $P \in \mathcal{V}_j$ set $\quad \tilde{c}_{j,P}^* = \tilde{c}_{j-1,P}^*$.
- If $e \in \mathcal{E}_j$ (with endpoints P_1, P_2) belongs to an edge of $\mathcal{E}_{j-1}$ then

$$\tilde{c}_{M_e,j}^* = \frac{\tilde{c}_{P_1,j-1}^* + \tilde{c}_{P_2,j-1}^*}{2}$$

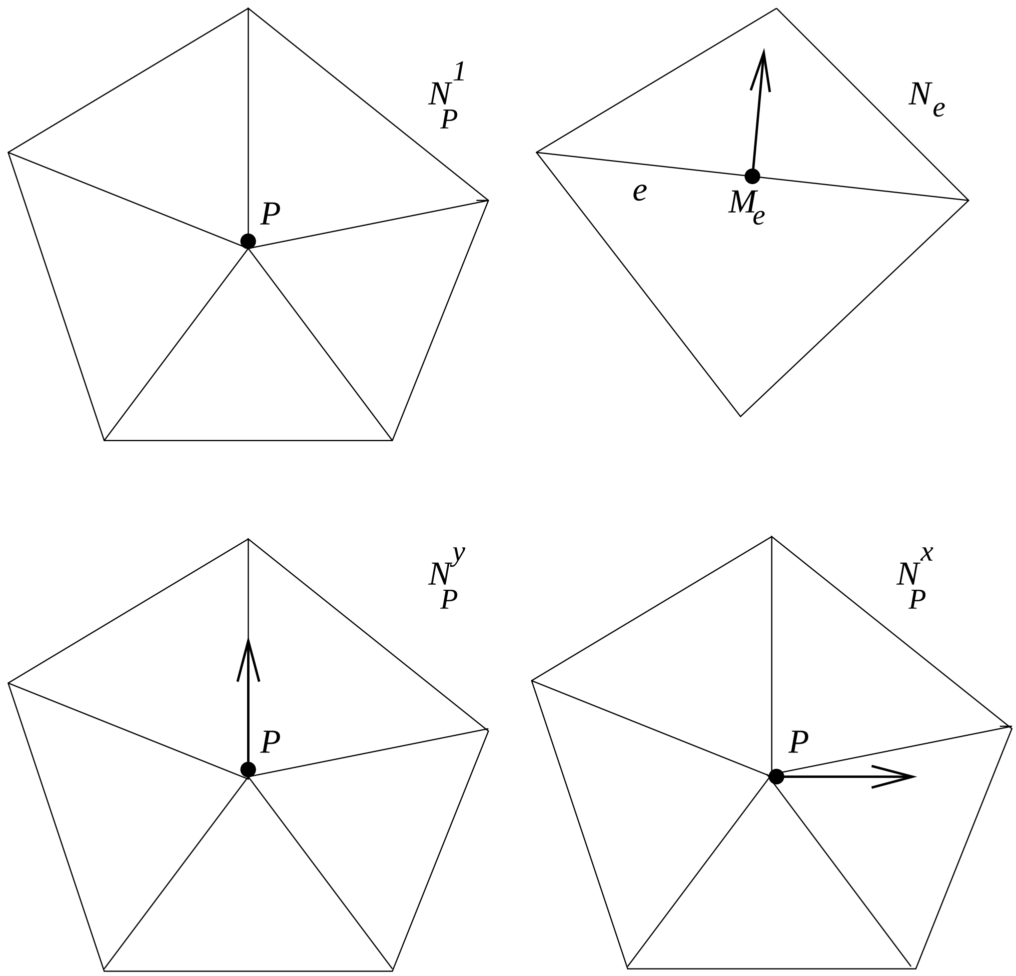

Figure 5: Basis functions in X_j

for all triangles $\Delta \in \mathcal{T}_j$ which is equivalent to $\mathbf{u}_j \in Z_j$.

Moreover, if we denote by $N^1_{P,j}, N^x_{P,j}, N^y_{P,j}$, and $N_{e,j}$ the four possible types of nodal basis functions in X_j associated with $P \in \mathcal{V}_j$ and $e \in \mathcal{E}_j$ (see Figure 5) then from the definition of the basis functions in V_j (see Figure 2) and the curl operations **curl** and $\mathbf{curl}_j$, the following relationships are straightforward:

$$\begin{aligned} \mathbf{N}^1_{P,j} &= -2^{-(j+1)}\mathbf{curl}_j\, N^1_{P,j}\ , & \mathbf{N}^x_{P,j} &= \mathbf{curl}_j\, N^y_{P,j}\ , \\ \mathbf{N}^y_{P,j} &= -\mathbf{curl}_j\, N^x_{P,j} & \mathbf{N}_{e,j} &= \mathbf{curl}_j\, N_{e,j}\ . \end{aligned} \tag{21}$$

The signs in the formulae (21) clearly depend on the choices made for directions of directional derivatives, edge normals, and edge orientations, here they correspond to the situation shown in Figure 2 and 5. The one-to-one relationship between the basis functions of Z_j and X_j expressed by (21) provides a simple isomorphisms between these spaces for all $j \geq 0$.

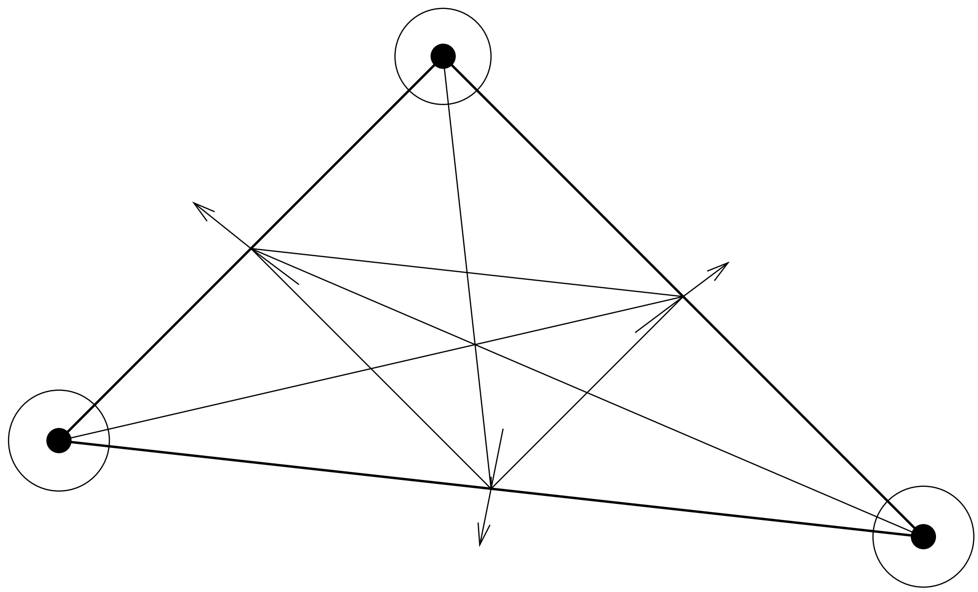

Figure 4: Powell-Sabin macrotriangle

Lemma 2 *Introduce*

$$|||\bar{g}_J|||_J^2 = \inf_{g_j \in X_j : \bar{g}_J = \sum_{j=0}^J g_j} \sum_{j=0}^{J} 2^{4j} \|g_j\|_{L_2(\Omega)}^2 \qquad \forall\, \bar{g}_J \in X_J \,. \tag{18}$$

Then this defines a (multilevel-splitted) norm on X_j which is equivalent to the H^2-norm

$$c|||\bar{g}_J|||_J^2 \le \|\bar{g}_J\|_{H^2}^2 \le C|||\bar{g}_J|||_J^2 \qquad \forall\, \bar{g}_J \in X_J \,, \tag{19}$$

uniformly in $j \ge 0$.

Used on its own, Lemma 2 is the basis for the convergence theory of multilevel preconditioners for X_j- and other discretizations of fourth-order problems (with Dirichlet boundary conditions), see [19, 20]. Here, we use it for deriving an analogous theory for Z_j-discretizations of (6). The relation of X_j and Z_j is relatively straightforward. Introduce a discrete curl operation $\mathbf{curl}_j : X_j \to Z_j$ by defining $\mathbf{u}_j = \mathbf{curl}_j\, g_j$ as the unique function in V_j that interpolates $\hat{\mathbf{u}}_j \equiv \mathbf{curl}\, g_j \in S_1^0(\tilde{\mathcal{T}}_j) \times S_1^0(\tilde{\mathcal{T}}_j)$:

$$\mathbf{u}_j(P) = \hat{\mathbf{u}}_j(P)(= (\mathbf{curl}\, g_j)(P)) \qquad \forall\, P \in \mathcal{V}_{j+1} \,. \tag{20}$$

Following [16], **curl** stands for the differential operator $(\partial_x, \partial_y)^T$. That $\mathbf{u}_j$ belongs to Z_j follows from the definition of the Powell-Sabin macro-element which yields

$$\hat{\mathbf{u}}_j|_{\mathcal{E}} = \mathbf{u}|_{\mathcal{E}} \,,$$

where $\mathcal{E}$ is the union of all interior and boundary edges of $\mathcal{T}_j$. This gives

$$\int_\Delta \nabla \cdot \mathbf{u}_j \, dx = \sum_{i=1}^{3} I_{\Delta,e_i}(\mathbf{u}_j) = \sum_{i=1}^{3} I_{\Delta,e_i}(\hat{\mathbf{u}}_j) = \int_\Delta \nabla \cdot (\mathbf{curl}\, g_j) \, dx = 0$$

the path system). It is easy to check that a basis exchange (i.e., a matrix-vector multiplication with T_j) can be performed in $\mathrm{O}(n_j)$ operations (and be parallelized if necessary). Using (13), the properties of $\mathbf{v}_j$ and of the path system, we get another proof of the basis property since

$$\mathbf{w}_j = \mathbf{v}_j - \sum_{P \in \mathcal{V}_j} c^1_{P,j} \mathbf{N}^1_{P,j} = 0 \ .$$

Indeed, by construction $I_{\Delta,e}(\mathbf{w}_j) = 0$ (and, therefore, $\mathbf{w}_j(M_e) = 0$) for all edges e included into one of the paths. If e is one of the remaining edges then its endpoints are either both in $\mathcal{V}_j$ and belong to different paths or at least one of them is on $\partial\Omega$. In either case, one can determine a new path including e and joining two boundary vertices such that all the $I_{\Delta,e'}$-values along the path vanish with the possible exception for the fixed e. Since Ω is simply connected, we can close the path by adding edges along the boundary and use the fact that by (13)

$$\sum_{e' \in \pi} I_{\Delta,e'}(\mathbf{w}_j) = 0$$

for any closed path π. This shows that $I_{\Delta,e}(\mathbf{w}_j) = 0$ for all edges, thus, $\mathbf{w}_j = 0$ is proved. The argument clearly indicates the place where modifications for multiply-connected domains are necessary, and gives explicit expressions of the $\mathrm{O}(n_j)$ non-zero entries of T_j.

For defining the $n_j \times m_j$ matrix $\tilde{T}_j$ defined by $\tilde{c}_j = \tilde{T}_j c_j$, one just needs to look at the expressions of the basis functions in Z_j by linear combinations of the basis functions of V_j. The coefficient vectors (which we call masks associated with the corresponding basis function of Z_j) form the columns of $\tilde{T}_j$ and contain less than a fixed number of non-zero entries each. To summarize the properties of the two transformation matrices, we have

Lemma 1 *The basis transformation matrices T_j, $\tilde{T}_j$ described above are sparse, each containing $\mathrm{O}(n_j)$ non-zero entries, $j \geq 0$. Moreover,*

$$T_j \tilde{T}_j c_j = c_j \quad \forall\ c_j \in \mathbf{R}^{m_j} \ , \qquad \tilde{T}_j T_j \tilde{c}_j = \tilde{c}_j \quad \forall\ \tilde{c}_j \in \mathcal{R}\tilde{T}_j \subset \mathbf{R}^{n_j} \ . \qquad (17)$$

We introduce now the subspaces X_j of $H^2_0(\Omega)$ mentioned in the introduction. For each $\Delta \in \mathcal{T}_j$, consider a subtriangulation and a local interpolation problem in the local space of piecewise quadratic C^1 splines on the subtriangulation using 12 degrees of freedom as shown in Figure 4. This macrotriangle is used in CAGD for representing globally C^1-smooth surfaces by local triangular patches. If we introduce secondary triangulations $\tilde{\mathcal{T}}_j$ obtained from $\mathcal{T}_j$ by replacing the triangles by the macrotriangles in Figure 4, then it is well-known [14] that the local problems are unisolvent, and that the finite element space composed of these macrotriangles coincides with $S^1_2(\tilde{\mathcal{T}}_j)$. Moreover, $\{\tilde{\mathcal{T}}_j\}$ is itself increasing (this is not the case with other, more ecomically looking, splits of Powell-Sabin type, see [14]). Thus, the spaces $X_j = S^1_2(\tilde{\mathcal{T}}_j) \cap H^2_0(\Omega)$ form an increasing sequence. They have been used in [19] to define a finite element multilevel scale for dealing with fourth order problems. The following lemma is a particular case of results established in sections 2-3 of [19] using methods of approximation theory (see [20] for a general picture).

imposed by (13). The latter equals the number of triangles in $\mathcal{T}_j$ minus one since $b(\mathbf{u}_j, 1) = 0$ for all $\mathbf{u}_j \in V_j$. By Euler's formula, this gives dim $Z_j = m_j$. The linear independence of the m_j functions described by (14), (15) is straightforward, thus, they form a basis. Another proof is contained in the following considerations.

If we have a discretely divergence-free velocity $\mathbf{u}_j \in Z_j$, given by its coefficient vector $\tilde{c} \in \mathbf{R}^{n_j}$ with respect to the nodal basis in V_j, we can uniquely determine its coefficient vector $c_j \in \mathbf{R}^{m_j}$ with respect to the above basis in Z_j. To find a matrix expression for this basis exchange we consider the following procedure of determining c_j from $\tilde{c}_j$: Obviously, $c^x_{P,j} = \tilde{c}^x_{P,j}$, $c^y_{P,j} = \tilde{c}^y_{P,j}$, for all $P \in \mathcal{V}_j$, and $c_{e,j}$, $e \in \mathcal{E}_j$, coincides with the projection of the nodal vector $(\tilde{c}^x_{M_e,j}, \tilde{c}^y_{M_e,j})^T$ at the midpoint M_e onto the direction of the edge e. By the above definitions,

$$\mathbf{v}_j = \mathbf{u}_j - \sum_{P \in \mathcal{V}_j} (c^x_{P,j} N^x_{P,j} + c^y_{P,j} N^y_{P,j}) + \sum_{e \in \mathcal{E}_j} c_{e,j} N_{e,j}$$

is another velocity in Z_j having zero nodal vectors at all $P \in \mathcal{V}_j$, nodal vectors normal to e at M_e for all $e \in \mathcal{E}_j$, and the same $I_{\Delta,e}$-values as $\mathbf{u}_j$. The latter define the remaining coefficients $c^1_{P,j}$ in a unique way. To see this, fix any set of polygonal paths consisting of $e \in \mathcal{E}_j$ and starting at a boundary point, such that each $P \in \mathcal{V}_j$ belongs to exactly one such path. Sets of this type can be found easily for all j together with the refinement steps, compare Figure 3. For each path which is

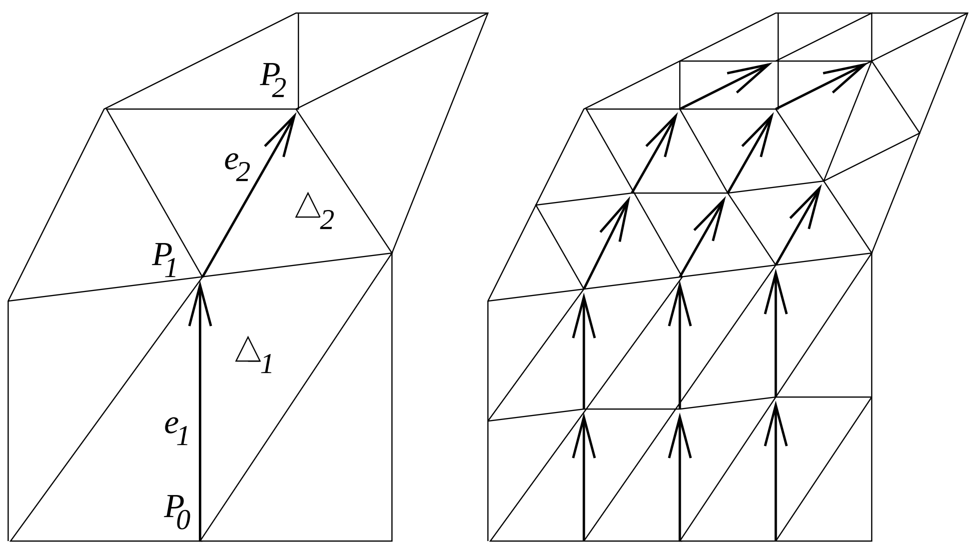

Figure 3: Path sets for $j = 0, 1$

given by a finite sequence of edges $e_k = [P_{k-1}, P_k]$ joining the neighbored vertices P_0 (boundary vertex) and $P_1, \ldots, P_l \in \mathcal{V}_j$ we define now

$$c^1_{P_1,j} = -I_{\Delta_1,e_1}(\mathbf{u}_j)2^j \ , \ \ c^1_{P_k,j} = -I_{\Delta_k,e_k}(\mathbf{u}_j)2^j + c^1_{P_{k-1},j} \ , \ \ k = 2, \ldots, l. \qquad (16)$$

These steps define a matrix representation T_j for the basis exchange (note that this $m_j \times n_j$ matrix T_j is not uniquely determined, it depends on the choice of functions which are described in Figure 2 by indicating their nodal vectors. The three functions associated with an interior vertex P are defined by

$$\mathbf{N}^1_{P,j}(P) = (0,0)^T\ , \quad \mathbf{N}^x_{P,j}(P) = (1,0)^T\ , \quad \mathbf{N}^y_{P,j}(P) = (0,1)^T\ , \qquad (14)$$

complemented by the condition

$$I_{\Delta,e}(\mathbf{N}^1_{P,j}) = 2^{-j} \qquad (15)$$

for each $e \in \mathcal{E}_j$ emanating from P, the corresponding triangle Δ has e as an edge and is taken in clockwise direction. For the function $\mathbf{N}_{e,j}$ associated with an edge $e \in \mathcal{E}_j$, we take the nodal vector at M_e to coincide with the unit vector in direction of e (this induces the orientation on e which will be held fixed throughout the following exposition). All other nodal vectors and $I_{\Delta,e}$-values vanish.

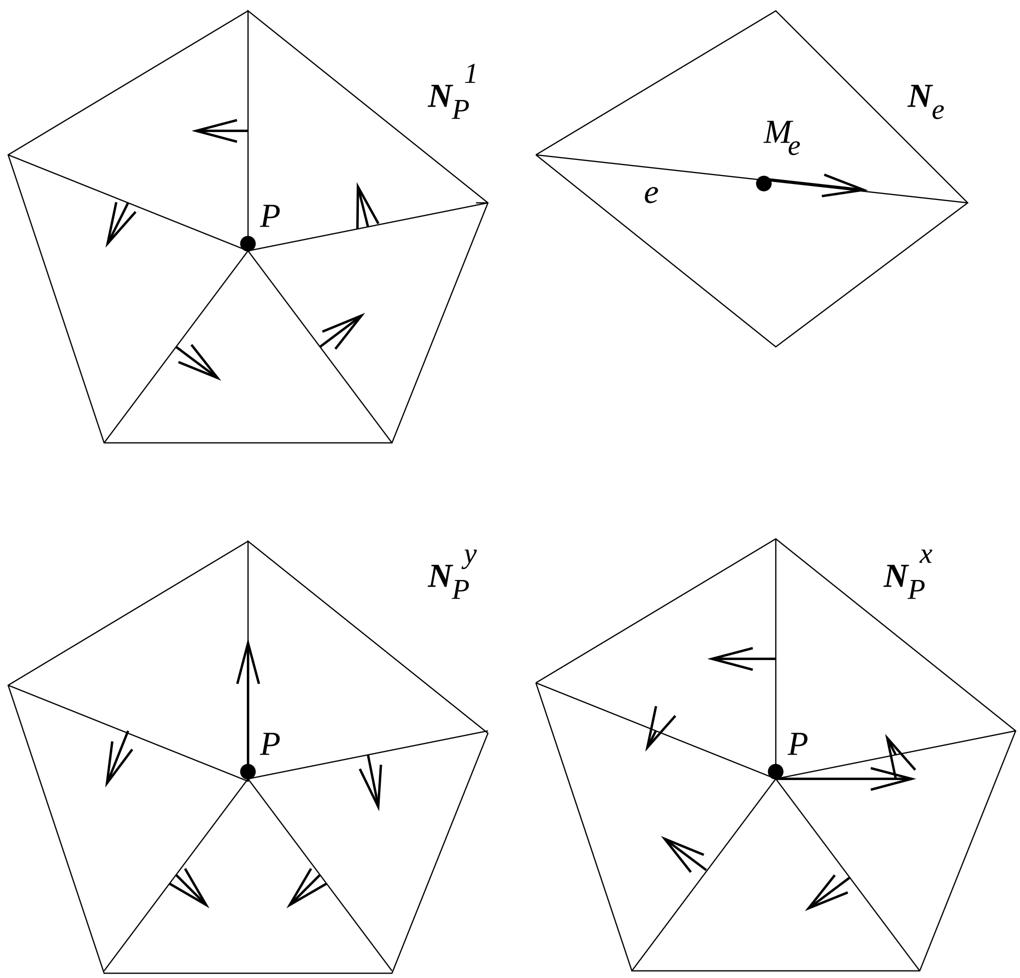

Figure 2: Basis functions in Z_j

To see that this is a basis, we can argue indirectly: The dimension of Z_j should be $n_j = 2|\mathcal{V}_j| + 2|\mathcal{E}_j|$ minus the number of linearly independent conditions trivial subspaces in the following discussion, let $|\mathcal{E}_0| > 0$. With (10) we associate the increasing sequences (11) of conforming subspaces of $V = H_0^1(\Omega)^2$ resp. $M = L_2(\Omega)$. The local interpolation problems (with respect to a triangle $\Delta \in \mathcal{T}_j$) are depicted in Figure 1. The velocity $\mathbf{u}_j \in V_j$ is uniquely determined by prescribing its two components on the set $\mathcal{V}_{j+1}$ (in the sequel, we call them just nodal vectors of $\mathbf{u}_j$), zero values on $\partial\Omega$, and linear interpolation. Thus, a nodal basis in V_j consisting of $n_j = 2|\mathcal{V}_{j+1}| = 2(|\mathcal{E}_j| + |\mathcal{V}_j|)$ functions is obtained from the scalar linear finite element case in an obvious way. To fix the setting, associate with each $P \in \mathcal{V}_{j+1}$ two functions $\tilde{\mathbf{N}}^x_{P,j}$ and $\tilde{\mathbf{N}}^y_{P,j}$ determined by zero nodal vectors except for P where

$$\tilde{\mathbf{N}}^x_{P,j}(P) = (1,0)^T , \qquad \tilde{\mathbf{N}}^y_{P,j}(P) = (0,1)^T .$$

The basis in M_j consists of the characteristic functions of the triangles in $\mathcal{T}_j$.

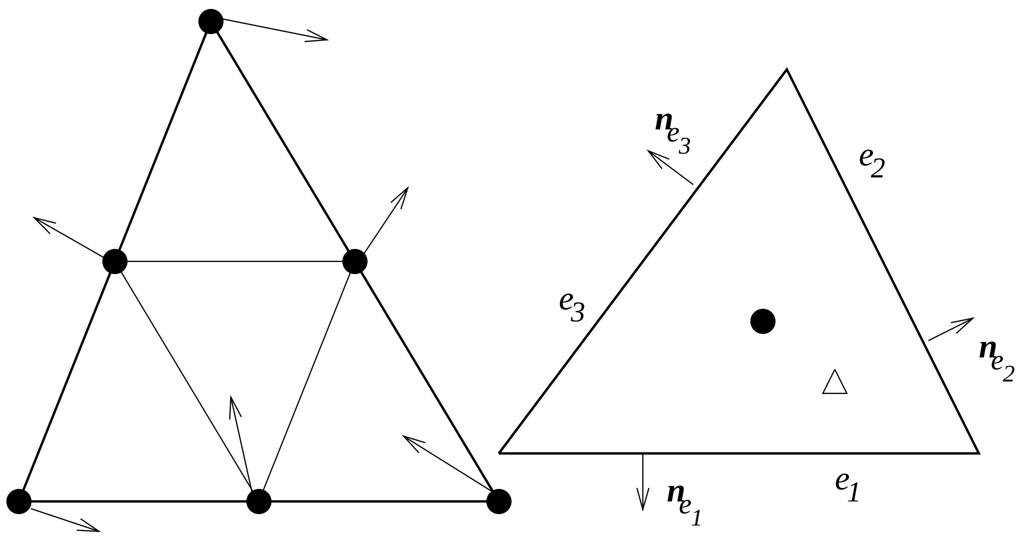

Figure 1: Interpolation problems for V_j and M_j on $\Delta \in \mathcal{T}_j$

By Green's formula, a velocity $\mathbf{u}_j \in V_j$ belongs to

$$Z_j \equiv \{\mathbf{v}_j \in V_j \ : \ b(\mathbf{v}_j, q_j) = 0 \quad \forall\, q_j \in M_j\} \tag{12}$$

iff

$$0 = \int_\Delta \nabla \cdot \mathbf{u}_j \, dx = \sum_{i=1}^{3} \int_{e_i} \mathbf{n}_{e_i} \cdot \mathbf{u}_j \, ds \equiv \sum_{i=1}^{3} I_{\Delta,e_i}(\mathbf{u}_j) \tag{13}$$

for all triangles $\Delta \in \mathcal{T}_j$. By e_i, $i = 1,2,3$, we denote the three edges of Δ and by $\mathbf{n}_e$ the outward normal unit vector to e (the direction will depend on the choice of the triangle the edge is associated with, this choice will be indicated or clear from the context). The integrals $I_{\Delta,e}(\mathbf{u}_j)$ appearing in (13) can obviously be expressed by projections of nodal values of the velocity $\mathbf{u}_j$ onto the directions $\mathbf{n}_e$. In analogy to the P2-P0 element, a basis in Z_j can be given by the following $m_j \equiv 3|\mathcal{V}_j| + |\mathcal{E}_j|$

for the velocity vector, and one interpolation condition for the pressure. One can view (11) as a "mixture" of the widely known P2-P0 resp. the "P1 iso P2" elements, see [4, p.211/212 and p.254/55]. It satisfies the inf-sup condition, the proof is almost identical to that for the P2-P0 element. We do not propose it as an element for discretizing (1) but rather use some nice properties of the associated subspaces $\{Z_j\}$ to develop an efficient solver for other discretizations. The main observation is that there is a norm-preserving, one-to-one mapping between Z_j and a sequence $X_j = S_2^1(\tilde{\mathcal{T}}_j) \cap H_0^2(\Omega)$ of subspaces in $H_0^2(\Omega)$ where the new triangulation $\tilde{\mathcal{T}}_j$ is obtained from $\mathcal{T}_j$ by replacing each $\Delta \in \mathcal{T}_j$ by its Powell-Sabin split. The mapping corresponds to a discrete curl-operation. Details are given in subsection 2.1.

Optimal multilevel solvers for a H_0^2-elliptic, fourth–order problem using the spaces X_j have been described in [19]. As we show in subsections 2.2-2.3, this leads to the components of an optimal additive multilevel preconditioner C_J for the matrix A_{Z_J}. Moreover, we show that one can implement this preconditioner such that one can avoid switching to the basis in Z_j and replace the multiplication by A_{Z_J} by the more sparse A_{V_J}. With this modification, the prolongation operators entering the preconditioner also simplify, and an explicit use of the switch to X_j, especially, to the complicated looking $\tilde{\mathcal{T}}_j$, is not made anymore. The condition numbers of the preconditioned linear system remain bounded if $J \to \infty$, and the overall operation count per iteration is proportional to the number of unknowns in V_J.

In section 3 this preconditioner is combined with a two-level switch for constructing an optimal additive preconditioner for the nonconforming P1-P0 element. The idea of switching to a reference discretization of a certain problem type for which fast solvers of certain type exist, instead of developing a new theory for each new discretization, has meanwhile become very popular, see [21, 7, 3] for applications of this principle. Finally, at the end of section 3 we briefly discuss some other examples.

It is possible to extend the basic preconditioner to the case of triangulations obtained by local nested refinement which is important for incorporating adaptivity concepts. For results on a posteriori error estimators needed in this connection, see [27, 28, 1, 11]. Also, multiply connected domains can be dealt with if an appropriate coarse grid problem is solved exactly. These topics will not be considered in this paper.

2 The basic preconditioner

2.1 Definitions and auxiliary results

Throughout this paper we assume that the sequence (10) is obtained by regular, dyadic refinement from an initial triangulation $\mathcal{T}_0$, i.e., $\mathcal{T}_j$ is obtained from $\mathcal{T}_{j-1}$ by subdividing each triangle into four congruent triangles by joining the midpoints. The set of interior edges resp. vertices of $\mathcal{T}_j$ will be denoted by $\mathcal{E}_j$ resp. $\mathcal{V}_j$. Without loss of generality, we assume that $\mathrm{diam}(\Delta) \asymp 2^{-j}$ for $j \geq 0$. To avoid such as the monographs by Brezzi/Fortin [4], Girault/Raviart [16], see also Brenner/Scott [8, ch. 10], Fortin [15].

There are several problems connected with this approach. For many finite element examples a basis in Z_h consisting of locally supported functions is not available which makes a direct discretization of (7) too expensive. An exception are choices where M_h is a discontinuous pressure space. Then one can usually find a basis $\{\mathbf{N}_{h,i}\}$ in Z_h where the $\mathbf{N}_{h,i}$ are expressed by linear combinations of a few basis functions $\{\tilde{\mathbf{N}}_{h,l}\}$ from V_h. However, if one considers the stiffness matrices A_{V_h} and A_{Z_h} corresponding to the form $a(\cdot,\cdot)$ with resp. to the bases in V_h and Z_h, resp., then A_{Z_h}, although of smaller dimension, usually contains more non-zero entries and has a higher condition number than A_{V_h}. Due to these observations, most of the investigations on the numerical solution of the mixed problem (1) concentrate on alternative methods (penalty, augmented Lagrangian, and least-squares methods, reduction to Schur complement problems for the pressure, etc.). We refer to [15] for more information on results in these main-stream directions.

Consequently, not so much is known on the direct use of (7). We mention papers by Dörfler [12] and Verfürth [26] where multilevel preconditioning was proposed for solving (7) for some low-order elements with discontinuous P0 resp. P1 pressure functions. As shown by these authors, condition numbers can be reduced from $\mathrm{O}(h^{-4})$ to about $\mathrm{O}(h^{-2})$ which is still unsatisfactory. If it comes to efficient solvers for the discrete systems by using multigrid techniques then one has to overcome the same difficulties as for nonconforming elements: Even if for a hierarchy of partitions

$$\mathcal{T}_0 \prec \mathcal{T}_1 \prec \cdots \prec \mathcal{T}_J \tag{10}$$

the subspaces $V_j \subset V$, $j = 0, \ldots, J$, are increasing, the corresponding spaces Z_j of divergence-free velocities usually do not have this property, i.e., as a rule $Z_j \not\subset Z_{j+1}$. See [5], [6] for W-cycle multigrid convergence resp. domain decomposition preconditioners for the nonconforming P1-P0 discretization of (4). Another source of motivation are the recent attempts to construct wavelet bases of Z (see, e.g., [24]) which result in multilevel splittings and fast algorithms for the efficient solution of the corresponding problems (7). See [25, 9] for some numerical results which are still restricted to model domains. Finally, note that the spaces Z_h appear sometimes in connection with mixed methods for second order scalar elliptic problems where $V_h \subset H(\mathrm{div})$ represents the space of fluxes, see, e.g., [13]. However, Z_h is now treated as a subspace of $H(\mathrm{div})$ which leads to different problems.

The aim of this note is to show that, to a certain extent, one may overcome the above-mentioned difficulties. We consider first an unusual modified P1-P0 element defined with respect to a sequence of dyadically refined triangulations (10) as follows. Denote by $S_d^r(\mathcal{T}_j)$ the spaces of piecewise polynomial functions of global smoothness C^r coinciding with polynomials of degree $\leq d$ on each triangle $\Delta \in \mathcal{T}_j$. For $r = -1$, there is no smoothness requirement. To define the finite element subspaces under consideration, we set

$$V_j = (S_1^0(\mathcal{T}_{j+1}) \times S_1^0(\mathcal{T}_{j+1})) \cap V \ , \qquad M_j = S_0^{-1}(\mathcal{T}_j) \ , \quad j \geq 0 \ . \tag{11}$$

Thus, on each triangle $\Delta \in \mathcal{T}_j$, we have 6 nodal points (12 interpolation conditions)

1 Introduction

We consider the two-dimensional Stokes problem in a simply-connected bounded polygonal domain $\Omega \subset \mathbf{R}^2$ with homogeneous Dirchlet boundary conditions in its variational formulation: Find the velocity field $\mathbf{u} = (u_1, u_2)^T \in V \equiv H_0^1(\Omega) \times H_0^1(\Omega)$ and the pressure $p \in M \equiv L_2(\Omega)$ (up to an additive constant) satisfying

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v})_0 \qquad \forall\, \mathbf{v} \in V \\ b(\mathbf{u}, q) &= 0 \qquad \forall\, q \in M \end{aligned} \tag{1}$$

The bilinear forms $a(\cdot,\cdot)$ resp. $b(\cdot,\cdot)$ on $V \times V$ resp $V \times M$ are given by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, dx\ , \quad b(\mathbf{u}, p) = -\int_\Omega p \nabla \cdot \mathbf{u}\, dx\ . \tag{2}$$

while $(\cdot,\cdot)_0$ stands for the $L_2(\Omega)$ scalar product of (vector) functions. The second variational equation in (1) expresses the incompressibility condition

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$$

of the velocity $\mathbf{u}$. On a theoretical level, one can use this equation to eliminate the pressure p from the first equation: If we define the subspace

$$Z = \{\nabla \mathbf{v} \in V\ :\ b(\mathbf{v}, q) = 0 \quad \forall\, q \in M\} \subset V \tag{3}$$

of solenoidal or divergence-free velocities then (1) splits into a Z-elliptic problem

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_0 \qquad \forall\, \mathbf{v} \in Z \tag{4}$$

for determining the velocity $u \in Z$, and a subsequent problem for the pressure $p \in M$:

$$b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v})_0 - a(\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in V\ . \tag{5}$$

The same approach would be possible if one discretizes (1) with respect to (conforming) subspaces $V_h \subset V$ and $M_h \subset M$. Then, if the discrete inf-sup-condition is satisfied then

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h)_0 \qquad \forall\, \mathbf{v}_h \in V_h \\ b(\mathbf{u}_h, q_h) &= 0 \qquad \forall\, q_h \in M_h \end{aligned} \tag{6}$$

splits into

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_0 \qquad \forall\, \mathbf{v}_h \in Z_h\ , \tag{7}$$

and

$$b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h)_0 - a(\mathbf{u}_h, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in V_h\ . \tag{8}$$

where

$$Z_h = \{\nabla \mathbf{v}_h \in V_h\ :\ b(\mathbf{v}_h, q_h) = 0 \quad \forall\, q_h \in M_h\} \subset V_h \tag{9}$$

is called subspace of discretely divergence-free velocities in V_h. Note that often Z_h is not a subspace of Z. For a detailed exposition, we refer to standard sources

An optimal multilevel preconditioner for solenoidal approximations of the 2D-Stokes problem

Peter Oswald
Institute of Algorithms and Scientific Computing
GMD German National Research Center for Information Technology
Schloß Birlinghoven
D-53754 Sankt Augustin, Germany
e-mail: peter.oswald@gmd.de

Abstract

We develop an optimal multilevel preconditioner for the divergence-free part of a modified conforming P1-P0 discretization of the two-dimensional Stokes problem which contains a novel prolongation operator preserving the discrete divergence-free property. The proofs utilize the eqivalence to a discretization of the biharmonic Dirichlet problem by Powell-Sabin macrotriangles via a discrete streamfunction. It is shown how the basic preconditioner can be applied to other Stokes discretizations with discontinuous pressure elements.