Semiorthogonal linear prewavelets on irregular meshes

Peter Oswald
International University Bremen
e-mail: p.oswald@iu-bremen,de

Dedicated to Professor Zbigniew Chiesielski

Abstract

We extend results on constructing semiorthogonal linear spline prewavelet systems in one and two dimensions to the case of irregular dyadic refinement. In the one-dimensional case, we obtain a necessary and sufficient condition for their L_p stability, $1 < p < \infty$.

1 Introduction

This note is devoted to the L_p-stability of semiorthogonal linear spline prewavelet systems in one and two dimensions. On regular simplicial partitions resp. semiregular refinement of arbitrary partitions, such prewavelet systems have been studied to great extent, see [2, 8, 3, 14, 9] resp. [15, 4, 5]. Our interest in the case of *irregular dyadic refinement* is triggered by recent attempts to theoretically investigate nonlinear approximation processes and multiresolution analyses where the underlying spatial grid structures are less regular. Although the restriction to the linear spline case allows for some simplifications, its separate treatment provides useful insights, and is also justified by a number of concrete applications to irregular sampling, surface discretization, image analysis, empirical density function estimation, and to the numerical solution of partial differential equations by adaptive finite element methods.

Similar studies have recently been undertaken for orthogonal spline systems (in particular, for the Franklin system) the systematic investigation of which was pioneered in the early 1970ies by Zbigniew Ciesielski. E.g., Ciesielski, Gevorkjan, and Kamont [1, 6, 7] have considered Franklin systems on arbitrary 1D partition sequences, and obtained a number of general results implying that often no restrictions need to be imposed on the grid refinement to extend results known for the classical systems to irregular refinement. For generalizations to higher dimensions, see [13] and the papers cited therein. Lyche, Morken, and Quak [10] have independently introduced prewavelet constructions on arbitrary 1D partition sequences.

In this paper, semiorthogonal prewavelet systems $\mathbf{\Psi}$ are constructed following the standard lifting scheme used in [15, 4] for the semiregular refinement case.

For the 1D case, we present necessary and sufficient conditions for L_p-stability of $\mathbf{\Psi}$ in Proposition 5 (some weaker results have independently been obtained by Mikkelsen, Oja, and Quak [11, 12]). In 2D, where the results are still incomplete, two quantities seem to matter: the maximal valence of vertices in the coarse partition (this quantity depends only on the initially given triangulation since all new vertices have valence 6), and an upper bound for the ratios $|\mathrm{supp}\,\tilde{\phi}_Q|/|\mathrm{supp}\,\tilde{\phi}_P|$ of the support areas of fine grid nodal basis functions associated with new vertices Q in a small neighborhood of an old vertex P. The above quantities enter the estimates for the stability constants to the power $p-1$, thus, for $p=1$ we have unconditional stability.

2 Notation and Preliminaries

Let us introduce (and consistently use) the following notation:

- Throughout this paper, L_p-spaces are defined on $\mathbb{R}^d$, most of the time we assume $d \le 2$ (exceptions will be made explicit). The L_p-norm of a function $f \in L_p$ is denoted by $\|f\|_p$. We also need scalar products

$$(f,g) = \int_{\mathbb{R}^d} f(x)g(x)\,dx\ ,$$

  which are well-defined in the appearances below.

- A set $F = \{f_i\} \subset L_p$ is called L_p-stable if there exist constants $0 < C_1 < C_2 < \infty$ such that for all sequences $\{x_i\}$ the following two-sided inequality holds:

$$C_1 \sum_i |x_i|^p \|f_i\|_p^p \le \|\sum_i x_i f_i\|_p^p \le C_2 \sum_i |x_i|^p \|f_i\|_p^p\ . \qquad (1)$$

  This assumes that the summation $\sum_i x_i f_i$ makes sense for arbitrary sequences which is always the case for the systems under consideration. The optimal constants C_1, C_2 in (1) will be called *lower and upper L_p-stability constants of F*. Note that the weights $\|f_i\|_p^p$ are a natural choice since if (1) holds with $\|f_i\|_p^p$ replaced by arbitray weights μ_i then taking the coordinate sequences as $\{x_i\}$ shows that $C_1\mu_i \le \|f_i\|_p^p \le C_2\mu_i$ for all i. This shows that if L_p-stability holds with any weighted ℓ_p coefficient norm then it also holds with the weights given in (1). Consequently, unless we are interested in the best possible stability constants resp. in the smallest possible value of the constant $\kappa(F) = \inf C_2/C_1$ (called *L_p-condition of F*), there is no point in talking about a better choice of weights. For the remainder of this paper, we will stick to the definition (1).

- $\mathcal{T}$ and $\tilde{\mathcal{T}}$ stand for coarse and dyadically refined fine simplicial partitions of $\mathbb{R}^d$ (the case of bounded polyhedral domains requires some boundary treatment which can be handled by an extension procedure, we will avoid these technicalities), $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are the corresponding sets of vertices. We will consistently use the letter P to indicate that P is an old vertex (i.e., $P \in \mathcal{V}$), the letter Q is used for new vertices ($Q \in \tilde{\mathcal{V}} \backslash \mathcal{V}$), and R is used to denote a generic vertex (old or new) in $\tilde{\mathcal{V}}$. Note that *dyadic refinement* means that each edge in $\mathcal{T}$ carries exactly one Q, the new $\tilde{\mathcal{T}}$ is obtained by inserting new edges connecting those Q in a standard way. If the position of Q is restricted to the edge midpoint, this dyadic refinement is called *semi-regular*, otherwise, if there are no restrictions on the placement of Q on the edges, we call the dyadic refinement *irregular*. Throughout this paper, we always have in mind irregular dyadic refinement.

- V and $\tilde{V}$ are the linear spline spaces on these partitions, their standard bases (consisting of hat functions associated with the vertices) are denoted by $\mathbf{\Phi} = \{\phi_P\}$ and $\tilde{\mathbf{\Phi}} := \{\tilde{\phi}_R\}$. By $\tilde{\Delta}_R$ we denote the area of the support of $\tilde{\phi}_R$ (the 1-ring of simplices around R in the fine partition). Recall that nodal bases are unconditionally L_p-stable, i.e.,

$$\| \sum_R c_R \tilde{\phi}_R \|_p^p \asymp \sum_R |c_R|^p \| \tilde{\phi}_R \|_p^p \asymp \sum_R |c_R|^p \tilde{\Delta}_R \, , \quad 1 \le p < \infty \, , \tag{2}$$

  where $\asymp$ stands for a two-sided inequality, with constants that depend on p, at most, but are independent of $\mathcal{T}$, $\tilde{\mathcal{T}}$ (in the following, the letters c, C are used for generic (positive) constants which may change from formula to formula, and generally depend on p but not on any other quantities involved). There is a counterpart of (2) for $p = \infty$ but we will not go into this case.

- In our considerations, the 1-ring in $\mathcal{T}$ of an arbitrary $P \in \mathcal{V}$ is of special interest. Figure 1 a) shows the notation for $d = 2$. The number $k \equiv k_P$ of

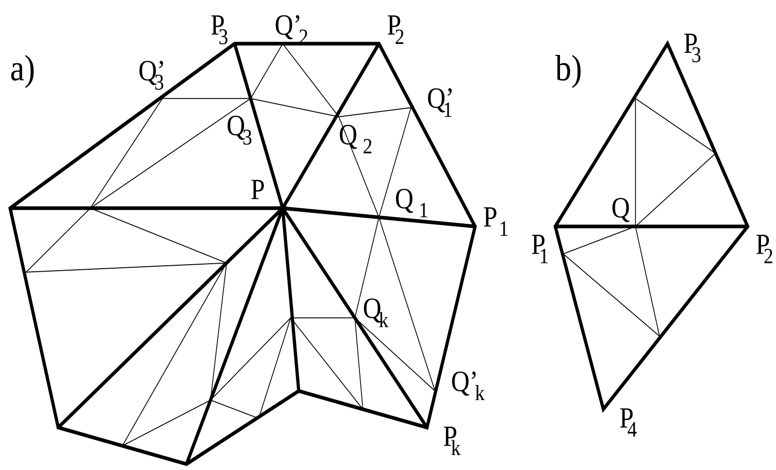

Figure 1: Notation for 1-ring around P, and P-neighborhood of Q

  simplices attached to P is called valence of P. We define the Q-neighborhood $\tilde{\mathcal{V}}_P$ of $P \in \mathcal{V}$ and its subset $\tilde{\mathcal{V}}_P^*$ as

$$\tilde{\mathcal{V}}_P = \tilde{\mathcal{V}}_P^* \cup \{Q_l' \, : \, l = 1, \dots, k\} \, , \quad \tilde{\mathcal{V}}_P^* = \{Q_l \, : \, l = 1, \dots, k\} \, .$$

  Finally, we define a P-neighborhood of $Q \in \tilde{\mathcal{V}} \backslash \mathcal{V}$ by setting

$$\mathcal{V}_Q = \{P \in \mathcal{V} \, : \, Q \in \tilde{\mathcal{V}}_P\} \, .$$

Forf $d = 2$, this neighborhood always consists of 4 vertices (see Figure 1 b)) while $\#\tilde{\mathcal{V}}_P = 2k$, and $\#\tilde{\mathcal{V}}_P^* = k$ depend on the valence. For $d = 1$, we have always $k_P = 2$, $\tilde{\mathcal{V}}_P = \tilde{\mathcal{V}}_P^*$ consists of the two new vertices $Q^\pm$ inserted into the two intervals attached to the left and right of P, and $\mathcal{V}_Q$ consists of two old vertices $P^\pm$ (the left and right endpoints of the interval in $\mathcal{T}$ containing Q).

3 Properties of the dual system

We call $\mathbf{\Theta} = \{\theta_P \,:\, P \in \mathcal{V}\} \subset \tilde{V}$ a dual system for the nodal basis $\mathbf{\Phi}$ if

$$(\theta_P, \phi_{P'}) = 0 \quad \forall\, P \neq P' \,, \qquad (\theta_P, \phi_P) = 1 \quad \forall\, P \,. \tag{3}$$

We call $\mathbf{\Theta}$ a pre-dual system for $\mathbf{\Phi}$ if only the first property is satisfied. A pre-dual system can be turned into a dual system by scaling provided that one can show that

$$(\theta_P, \phi_P) \neq 0 \qquad \forall\, P \,. \tag{4}$$

We make these distinctions only because we want to avoid the appearance of weird scaling factors, and prefer to work with θ_P that are normalized by

$$\theta_P(P) = 1 \qquad \forall\, P \,. \tag{5}$$

Following [15, 4], we restrict our attention to finding a dual system of the form

$$\theta_P(P) = \tilde{\phi}_P - \sum_{m=1}^{k} \alpha_{PQ_m} \tilde{\phi}_{Q_m} \qquad \forall\, P \,, \tag{6}$$

where $\alpha_P = \{\alpha_{PQ_m}\}$ needs to be determined (the notation is as in Figure 1 a)). Obviously, such θ_P automatically satisfy (5).

The following proposition is known for semi-regular dyadic refinement [15] (d=1,2,3), and we claim that it also holds for the irregular case and $d = 1, 2$. This claim will be fully established for $d = 1$ while for $d = 2$ a proof of the second statement (7) is still missing.

Proposition 1 *There is always a unique choice of vectors α_P such that the system $\mathbf{\Theta}$ of the form (6) is pre-dual. Moreover, we have*

$$(\theta_P, \phi_P) = (\theta_P, 1) \asymp \|\theta_P\|_p^p \asymp \tilde{\Delta}_P (\asymp \|\tilde{\phi}_P\|_p^p \,. \tag{7}$$

The constants in (7) depend on p, at most.

Proof. For $d = 1$ we have $k = k_P = 2$ for all P in $\mathcal{V}$, and a simple calculation gives

$$\begin{aligned} \theta_P &= \phi_P - \tfrac{(\phi_P,\phi_{P-})_{L_2}}{(\tilde{\phi}_{Q^-},\phi_{P-})_{L_2}} \tilde{\phi}_{Q^-} - \tfrac{(\phi_P,\phi_{P+})_{L_2}}{(\tilde{\phi}_{Q^+},\phi_{P+})_{L_2}} \tilde{\phi}_{Q^+} \\ &= \tilde{\phi}_P - \alpha(t^-)\tilde{\phi}_{Q^-} - \alpha(t^+)\tilde{\phi}_{Q^+}, \end{aligned} \tag{8}$$

where the auxiliary function $\alpha(t)$ is given by

$$0 \le \alpha(t) := \frac{t^2}{(1+t)} \le \frac{1}{2}\,, \qquad 0 \le t \le 1\,.$$

Here, $Q^\pm$ denote the two new vertices inserted into the intervals to the left $(-)$ and right $(+)$ of P at distance $t^\pm d^\pm$, where $d^\pm = \tilde{\Delta}_{Q^\pm}$ is the length of the corresponding intervals in $\mathcal{T}$. Note that $t^\pm = 1/2$ and thus $\theta_P(Q^\pm) = -1/6$ for semi-regular refinement.

With this, (7) is immediate. Indeed,

$$\begin{aligned}(\theta_P, 1) &= \frac{1}{2}(\tilde{\Delta_P} - \alpha(t^-)\tilde{\Delta_{Q^-}} - \alpha(t^+)\tilde{\Delta_{Q^+}} = \frac{1}{2}((t^- - \alpha(t^-))d^- + (t^+ - \alpha(t^+))d^+) \\ &= \frac{t^-d^-}{2(1+t^-} + \frac{t^+d^+}{2(1+t^+)} \asymp (t^-d^- + t^+d^+) = \tilde{\Delta}_P,\end{aligned}$$

with optimal constants 1/4 and 1/2 in the two-sided inequality.

The L_p norm of θ_P can be estimated in a similar fashion:

$$\begin{aligned}\|\theta_P\|_p^p &\asymp \tilde{\Delta_P} + \alpha(t^-)^p\tilde{\Delta_{Q^-}} + \alpha(t^+)^p\tilde{\Delta_{Q^+}} \\ &= (1 + \frac{(t^-)^{2p-1}}{(1+t^-)^p})t^-d^- + (1 + \frac{(t^+)^{2p-1}}{(1+t^+)^p}t^+d^+ \asymp \tilde{\Delta}_P.\end{aligned}$$

Constants may depend on p, at most.

For $d = 2$, the pre-duality follows from the fact that the $k \times k$ matrix

$$A = ((\, a_{lm} := (\phi_{P_l}, \tilde{\phi}_{Q_m})\,,\ l, m = 1, \ldots, k\,))$$

is non-singular. Indeed, we can then uniquely determine α_P by solving the linear system $A\alpha_P = b$, where b is the column vector with the entries $b_l = (\phi_{P_l}, \tilde{\phi}_P)$, $l = 1, \ldots, k$. This system is equivalent to $(\theta_P, \phi_{P_l}) = 0$, $l = 1, \ldots, k_P$. For the remaining $P' \neq P$ which are not among the immediate neighbors P_l of P we automatically have $(\theta_P, \phi_{P'}) = 0$ since $\theta_P(x)\phi_{P'}(x) \equiv 0$ by their support properties. This implies that (with a unique choice of the α_P) the system $\{\theta_P\}$ from (6) is pre-dual.

The non-singularity of A follows from the fact that A is (columnwise) diagonally dominant. To show this, observe that each column of A contains exactly 3 positive non-zero elements. In the m-th column, these are $(\phi_{P_l}, \tilde{\phi}_{Q_m})$ for $l = m-1, m, m+1$ (with obvious modifications, if $m = 1$ or $m = k$). Since

$$\begin{aligned}\delta_m &:= (\phi_{P_m}, \tilde{\phi}_{Q_m}) - (\phi_{P_{m-1}}, \tilde{\phi}_{Q_m}) - (\phi_{P_{m+1}}, \tilde{\phi}_{Q_m}) \\ &= \int_{\Delta PP_mP_{m+1}} (\phi_{P_m} - \phi_{P_{m+1}})\tilde{\phi}_{Q_m}\,dx + \int_{\Delta PP_mP_{m-1}} (\phi_{P_m} - \phi_{P_{m-1}})\tilde{\phi}_{Q_m}\,dx := \delta_m^+ + \delta_m^-\,,\end{aligned}$$

we establish diagonal dominance if we show that $\delta_m^+ > 0$ for an arbitrary m, independently of the geometry of the subdivided triangle ΔPP_mP_{m+1} (by symmetry arguments this implies that $\delta_m^- > 0$ as well).

Without loss of generality, we will do this for $m = 1$, and use the notation of Figure 2. Note that the parameters x, y, z denote the relative distance (i.e., distance normalized by the length of the corresponding coarse edge) of Q_1 to P, of Q_2 to P, and of Q_1' to P_1, respectively. Relative distances of a Q to its two related coarse grid vertices always sum up to 1. What we are able to show is the following lower bound

$$\delta_1^+ > \frac{\Delta_1}{12} , \tag{9}$$

where the numbers Δ_i, $i = 1, \ldots, 4$, stand for the areas of the 4 subtriangles ($\Delta = \Delta_1 + \ldots + \Delta_4$), compare Figure 2. Note that

$$\Delta_1 = xy\Delta, \quad \Delta_3 = (1-x)z\Delta, \quad \Delta_4 = (1-y)(1-z)\Delta, \quad \Delta_2 = \Delta - \Delta_1 - \Delta_3 - \Delta_4.$$

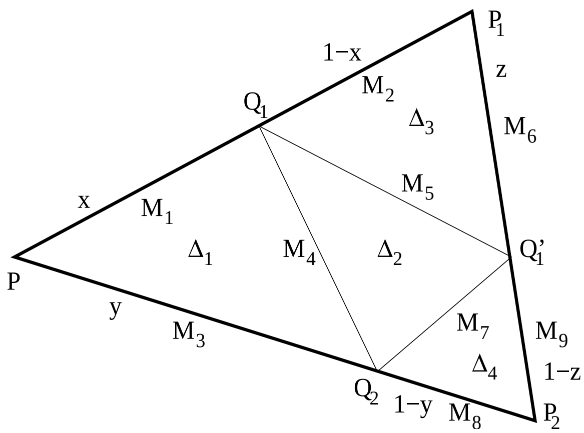

Figure 2: Notation for the proof of (9)

Since the integrand of the integral representing δ_1^+ is a quadratic polynomial on each of the subtriangles, we can apply the midpoint rule on each of them. It is easy to see that the nodal basis function $\tilde{\phi}_{Q_1} \neq 0$ only for 4 of those midpoints (namely, M_1, M_2, M_4, and M_5, where it takes the value 1/2). A straightforward calculation gives the following values of the linear function $g := (\phi_{P_1} - \phi_{P_2})$ at these points:

$$g(M_1) = \frac{x}{2} , \quad g(M_2) = \frac{x+1}{2} , \quad g(M_4) = \frac{x-y}{2} , \quad g(M_5) = \frac{x+1-2z}{2} .$$

This leads to (we drop the trivial steps of calculating and simplifying the expressions)

$$\begin{aligned}
\delta_1^+ &= \frac{\Delta_1}{12}(2x-y) + \frac{\Delta_2}{12}(2x-y+1-2z) + \frac{\Delta_3}{12}(2x+2-2z)) \\
&= \frac{\Delta}{12}((2x-y)xy + (2x-y+1-2z)(y+xz-xy-zy) + (2x+2-2z)(1-x)z) \\
&= \frac{\Delta}{12}(y + 2z - y^2 - 2z^2 + xy + xz - xyz + zy^2 + 2z^2y - 3zy)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Delta}{12}(xy + (1-y)[y(1-z) + 2z(1-z) + xz]) \\
&> \frac{xy\Delta}{12} = \frac{\Delta_1}{12} > 0\,, \qquad 0 < x, y, z < 1\,.
\end{aligned}
$$

Thus, the strict (columnwise) diagonal dominance of A has been established.

The open question for $d = 2$ is whether the found pre-dual system can be turned into a dual system resp. whether the stronger statement (7) holds. Numerical evidence indicates that this statement holds with moderate constants, probably in the form

$$\frac{\tilde{\Delta}_P}{9} \le (\theta_P, \phi_P) = (\theta_P, 1) \le \frac{\tilde{\Delta}_P}{3}\,.$$

Similar bounds seem to hold for the L_p-norms of the θ_P (we have only collected numerical data for $\|\theta_P\|_2^2$). We also found numerically that α_{PQ_m} can become negative but that $|\alpha_{PQ_m}|$ always is bounded well below 1 (the numerical experiments consisted in generating thousands of randomly shaped and randomly subdivided 1-ring neighborhoods of the origin).

In partial cases, e.g., for semi-regular dyadic refinement, α_P can be found explicitly, which allows to verify (7) directly. As an example (which will be used to construct counterexamples later), we consider a special 1-ring where all new vertices Q_m are inserted at the same relative distance $0 < t < 1$ from P, while the new vertices Q'_m coincide with the midpoints of the respective edges. This corresponds to the parameter choices $x = y = t$ and $z = 1/2$ in our above calculations. The case $t = 1/2$ corresponds to semi-regular refinement. See Figure 3 (left) for a graphical depiction of the situation (only the triangle ΔPP_1P_2 is shown). Under

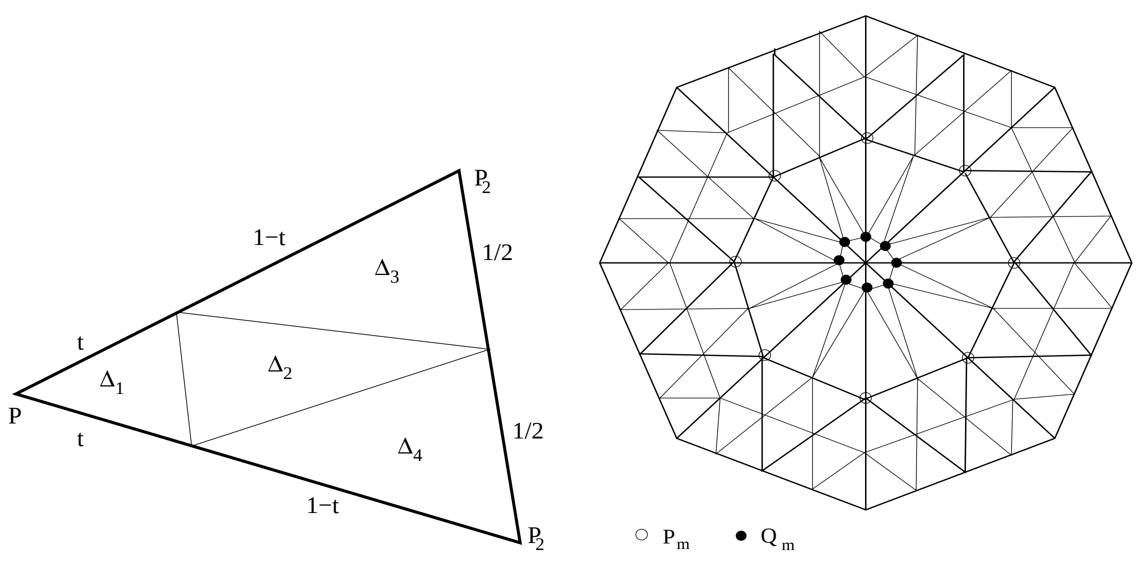

Figure 3: Triangle of special 1-ring refinement (left), and triangulation for counterexamples ($k = 8$, right)

these assumptions, we can claim that α_P coincides with a constant vector, more precisely,

$$\alpha_{PQ_m} = \alpha(t) := \frac{t^3}{1+t+t^2}\ , \qquad m = 1, \ldots, k_P\ . \tag{10}$$

Note that for $t = 1/2$ we get the known value of $\alpha_{PQ_m} = 1/14$.

By symmetry and rotational invariance, it is enough to show that for the above value $\alpha = \alpha(t)$ we have

$$I := \int_{\Delta PP_1P_2} \theta_P \phi_1\, dx = 0\ .$$

As before, we compute the values of θ_P and of ϕ_1 at the edge midpoints for all 4 subtriangles, and apply the midpoint rule. The areas of the subtriangles are $\Delta_1 = t^2\Delta$, $\Delta_3 = \Delta_4 = (1-t)\Delta/2$, and $\Delta_2 = t(1-t)\Delta$. We spare the reader these trivialities.

Using the same midoint rule, we compute

$$\begin{aligned} \int_{\Delta PP_1P_2} \theta_P\, dx &= \frac{\Delta}{3}((1-2\alpha)t^2 - 2\alpha\frac{1-t}{2} - 2\alpha t(1-t)) \\ &= \frac{\Delta}{3}(t^2 - \alpha(1+t)) = \frac{t^2\Delta}{3(1+t+t^2)} \asymp \Delta_1\ , \end{aligned}$$

where the final two-sided inequality holds with constants 1/9 and 1/3, resp., for arbitrary $0 < t < 1$. Summing the similar expressions for all triangles ΔPP_mP_{m+1}, we clearly get

$$\frac{\tilde{\Delta}_P}{9} \le (\theta_P, \phi_P) = (\theta_P, 1) \le \frac{\tilde{\Delta}_P}{3}\ .$$

Similar estimates can be established for the L_p norms of θ_P.

4 Stability estimates for semiorthogonal prewavelets

Given $\mathbf{\Phi}$ and its dual system $\mathbf{\Theta}$, a straightforward definition of a semiorthogonal prewavelet system $\mathbf{\Psi} = \{\psi_Q\}$ in $\tilde{V}$ is to set

$$\psi_Q := \tilde{\phi}_Q - \sum_{P\in\mathcal{V}} \beta_{QP}\theta_P, \qquad \beta_{QP} := \frac{(\tilde{\phi}_Q, \phi_P)}{(\theta_P, \phi_P)} = -\psi_Q(P),\ P \in \mathcal{V}, \tag{11}$$

where $Q \in \mathcal{V}\backslash\tilde{\mathcal{V}}$ (see [15]).

From now on, we restrict our attention to the special $\mathbf{\Theta}$ given by (6), and considered in the previous section. In this case, for each Q only finitely many β_{QP} are non-zero, which implies the desired local support properties of the ψ_Q. For $d = 1$, $\operatorname{supp}\psi_Q$ consists of 3 intervals from $\mathcal{T}$ (the one containing Q and its two neighbors $Q^{\pm}$), which is obviously minimal. Similarily, for $d = 2$ we have

$$\operatorname{supp}\psi_Q \subset \bigcup_{P\in\mathcal{V}_Q} \operatorname{supp}\theta_P$$

(this construction does not lead to the smallest possible support, compare [9, 5]).

The results reported in this section are conditional for $d = 2$, since we will heavily rely on the existence and properties of $\mathbf{\Theta}$ stated in Proposition 1 which have been fully established only for $d = 1$. We start with the upper stability estimate.

Proposition 2 *Let $d = 1, 2$. Assume that the pre-dual system $\mathbf{\Theta}$ satisfies (6) and (4) (the latter property is not yet established if $d = 2$), and that $\mathbf{\Psi}$ is defined as in (11). Then for $1 \le p < \infty$ and arbitrary sequences $\{x_Q : Q \in \mathcal{V}\backslash\tilde{\mathcal{V}}\}$, we have*

$$\| \sum_Q x_Q \psi_Q \|_p^p \le C_p (\sup_P k_P)^{p-1} \sum_Q |x_Q|^p \|\psi_Q\|_p^p \, . \tag{12}$$

The dependency of the bound on the values of k_P^{p-1} is essential for $d = 2$, for $d = 1$ it can be dropped since $k_P = 2$ for all P.

Proof. Let $d = 2$ ($d = 1$ is similar, see also below). By construction, on any triangle from $\mathcal{T}$ (with vertices P_1, P_2, P_3) less than $2(k_{P_1} + k_{P_2} + k_{P_3}) \le 6 \sup_P k_P$ prewavelets ψ_Q do not vanish (more precisely, these are the functions ψ_Q with $Q \in \tilde{\mathcal{V}}_{P_1} \cup \tilde{\mathcal{V}}_{P_2} \cup \tilde{\mathcal{V}}_{P_3}$). Thus, we have a pointwise estimate

$$|\sum_Q x_Q \psi_Q|^p \le (6 \sup_P k_P)^{p-1} \sum_Q |x_Q|^p |\psi_Q|^p \, ,$$

from which (12) follows by integration. The counterexample will be given at the end of this section.

The proof of a lower stability estimate is a bit more involved, and requires the property (7) of $\mathbf{\Theta}$. We need the following lemma.

Lemma 3 *Suppose $\mathbf{\Theta}$ satisfies Proposition 1 including (7).*
a) The coefficients β_{PQ} defined in (11) satisfy

$$\beta_{QP} \asymp \frac{(\tilde{\phi}_Q, \phi_P)}{\tilde{\Delta}_P} \begin{cases} \le \tilde{\Delta}_Q / \tilde{\Delta}_P & , \quad P \in \mathcal{V}_Q \\ = 0 & , \quad P \notin \mathcal{V}_Q \end{cases}$$

b) The prewavelets ψ_Q from (11) satisfy

$$\|\psi_Q\|_p^p \asymp \tilde{\Delta}_Q + \sum_{P \in \mathcal{V}_Q} \beta_{QP}^p \tilde{\Delta}_P \, .$$

Proof. a) is an obvious consequence of the definitions (6), (11), and the property (7) of the dual functions. Concerning b), observe that ψ_Q is the sum of ≤ 5 terms. This, together with (7), gives the upper bound

$$\|\psi_Q\|_p^p \;\le\; 5^{p-1} (\|\tilde{\phi}_Q\|_p^p + \sum_{P \in \mathcal{V}_Q} \beta_{QP}^p \|\theta_P\|_p^p \le C(\tilde{\Delta}_Q + \sum_{P \in \mathcal{V}_Q} \beta_{QP}^p \tilde{\Delta}_P) \, .$$

For the lower bound, we use the L_p-stability (2) of $\tilde{\mathbf{\Phi}}$, and note that for $\psi_Q := \sum_R y_R \tilde{\phi}_R$ we have $y_R = \psi_Q(R)$, $R \in \tilde{\mathcal{V}}$, and, particularly,

$$y_P = \psi_Q(P) = -\beta_{QP} \, , \qquad P \in \mathcal{V}_Q \, ,$$

This gives

$$\sum_{P\in\mathcal{V}_Q} \beta_{QP}^p \tilde{\Delta}_P \le C\|\psi_Q\|_p^p \ .$$

For the remaining term $\tilde{\Delta}_Q$, we proceed as follows. Again using (2) and then (11), we have

$$\tilde{\Delta}_Q \le C\|\tilde{\phi}_Q\|_p^p \le C5^{p-1}(\|\psi_Q\|_p^p + \sum_{P\in\mathcal{V}_Q} \beta_{QP}^p\|\theta_P\|_p^p) \ .$$

According to (7), we can replace $\|\theta_P\|_p^p$ by its upper bound $C\tilde{\Delta}_P$. Together with the previous estimate, this implies the lower bound in b), and concludes the proof of Lemma 3.

Proposition 4 *Let the pre-dual system* $\mathbf{\Theta}$ *to* $\mathbf{\Phi}$ *satisfy (6), (7), and assume that* $\mathbf{\Psi}$ *is defined as in (11). Then for* $1 \le p < \infty$ *and arbitrary sequences* $\{x_Q : Q \in \mathcal{V}\backslash\tilde{\mathcal{V}}\}$*, we have*

$$\sum_Q |x_Q|^p\|\psi_Q\|_p^p \le C(1 + \sup_Q(\max_{P\in\mathcal{V}_Q} \beta_{QP}^{p-1}))\|\sum_Q x_Q\psi_Q\|_p^p \ . \tag{13}$$

The dependence of the constant on the values β_{QP}^{p-1} *cannot be neglected.*

Proof. We concentrate on $d = 2$ (a stronger result for $d = 1$ is given below). By Lemma 3 we have

$$\begin{aligned}
\sum_Q |x_Q|^p\|\psi_Q\|_p^p &\le C\left(\sum_Q |x_Q|^p\tilde{\Delta}_Q + \sum_Q |x_Q|^p \sum_{P\in\mathcal{V}_Q} \beta_{QP}^p\tilde{\Delta}_P\right) \\
&\le C\sum_Q |x_Q|^p\tilde{\Delta}_Q(1 + \sum_{P\in\mathcal{V}_Q} \beta_{QP}^{p-1}) \\
&\le C(1 + \sup_Q \max_P \beta_{QP}^{p-1})\|\tilde{g}\|_p^p \ ,
\end{aligned}$$

where in the last step (2) has been applied to $\tilde{g} := \sum_Q x_Q\tilde{\phi}_Q \in \tilde{V}$. The estimation of $\|\tilde{g}\|_p^p$ is similar to what we did for the lower bound in Lemma 3 b): Since $\tilde{g}(P) = 0$, we can write

$$\tilde{g} = g + \sum_P \underbrace{(\sum_{Q\in\tilde{\mathcal{V}}_P} x_Q\beta_{QP}}_{=-g(P)}\theta_P \ ,$$

where $g = \sum_Q x_Q\psi_Q$. In the expression on the right-hand side of this equality at most 4 terms are non-zero on any coarse triangle from $\mathcal{T}$. Therefore, we obtain

$$\|\tilde{g}\|_p^p \le 4^{p-1}(\|g\|_p^p + \sum_P |g(P)|^p\|\theta_P\|_p^p) \le C(\|g\|_p^p + \sum_P |g(P)|^p\tilde{\Delta}_P) \le C\|g\|_p^p \ ,$$

where in the last two steps (7) and (2) were explored. This shows (13).

For $d = 2$, the counterexamples for Proposition 2 and 4 are based on the special, parameter-dependent refinement of a 1-ring around a vertex P discussed in Section 3. Figure 3 (right) shows the essential portion of a special triangulation $\tilde{\mathcal{T}}$, with a vertex P of valence k in the center. The valences of the vertices P_m in the 1-ring around P are 6. How $\tilde{\mathcal{T}}$ looks outside the shown portion is not essential for the result). For simplicity, we assume k-fold rotational symmetry w.r.t. the center P. The new vertices are inserted at the edge midpoints, only the points Q_m on the edges emanating from P are inserted at relative distance $0 < t < 1$ from P. All shown coarse triangles from $\mathcal{T}$ are similar, with two sides of length $k^{1/2}$ and the short side of length $\asymp k^{-1/2}$. Consider the prewavelets ψ_{Q_m}. These functions are "rotational" copies of ψ_{Q_1}. The existence of ψ_{Q_1} can be shown by direct examination. To this end, we have to verify (7) at P (which was already done in Section 3), and at P_m which we leave to the reader.

Note that by construction of the triangulation, all coarse triangles have area $\asymp 1$. Thus, given the described placement of the Q points, we have

$$\tilde{\Delta}_P \asymp kt^2 , \qquad \tilde{\Delta}_R \asymp 1 \quad (R \neq P) .$$

From Lemma 3 we therefore obtain $\beta_{QP} \leq Ck^{-1}t^{-2}$, and, thus,

$$\|\psi_{Q_m}\|_p^p \asymp (1 + (kt^2)^{-(p-1)}) . \tag{14}$$

The lower bound requires that we in addition verify $\beta_{Q_mP} \asymp k^{-1}t^{-2}$ (start from Lemma 3 a)).

Thus, if we choose $t = k^{-1/2}$, and define the (finitely supported) sequence $\{x_Q\}$ by setting $x_Q = 1$ if $Q = Q_m$, $m = 1, \ldots, k$, and $x_Q = 0$ otherwise, we have

$$\sum_Q |x_Q|^p \|\psi_Q\|_p^p \asymp k . \tag{15}$$

On the other hand, because of rotational symmetry $g = \sum_Q x_Q \psi_Q$ has value

$$|g(P)| = k|\psi_{Q_1}(P)| = k\beta_{Q_1P} \asymp k(\tilde{\phi}_{Q_1}, \phi_P) \asymp k .$$

Using (2), this yields

$$\|g\|_p^p \geq c|g(P)|^p \tilde{\Delta}_P \geq ck^p .$$

Thus, comparing this lower bound with (15), we see that the constant in the upper stability estimate (12) stated in Proposition 2 cannot be smaller in order than k^{p-1}. This shows that the valence-dependence of the constant in (12) is essential.

To prove a similar result for the lower bound (13) of Proposition 4, we fix in the above construction k, and consider $g := \psi_{Q_1} - \psi_{Q_2}$ for $t \to 0$. From (14) we have for $0 < t < k^{-1/2}$ that $\beta_{Q_mP} \asymp k^{-1}t^{-2}$. Thus, the left-hand side in (13) is of the order $\beta_{Q_1P}^{p-1} \asymp (kt^2)^{-(p-1)}$. On the other hand, since the term involving θ_P cancels by definition of g, we can represent g as

$$\begin{aligned} g &= \tilde{\phi}_{Q_1} - \tilde{\phi}_{Q_2} - \beta_{Q_1P_k}\theta_{P_k} + \beta_{Q_2P_3}\theta_{P_3} \\ &\quad + (\beta_{Q_2P_1} - \beta_{Q_1P_1})\theta_{P_1} + (\beta_{Q_2P_2} - \beta_{Q_1P_2})\theta_{P_2} . \end{aligned}$$

All coefficients in this representation (and the L_p-norms of the involved functions $\tilde{\phi}_Q$ and $\theta_{P'}$) are bounded by some constant C which gives $\|g\|_p^p \leq C$, independently of t. Comparing these findings with (13), we see that in our particular case the constant there is bounded from below by $c\beta_{Q_mP}^{p-1}$. Thus, the appearance of such terms in the constant for the lower stability estimate (13) is essential.

For $d = 1$, we have stronger results. Obviously, in Lemma 3 we can now establish

$$\beta_{QP} \asymp \frac{\tilde{\Delta}_Q}{\tilde{\Delta}_P}, \quad P \in \mathcal{V}_Q, \qquad \|\psi_Q\|_p^p \asymp \tilde{\Delta}_Q(1 + \sum_{P\in\mathcal{V}_Q} \beta_{QP}^{p-1}). \tag{16}$$

Define the quantity

$$\tau := \sup_P \min_{Q\in\tilde{\mathcal{V}}_P} \beta_{QP} \asymp \tau' = \sup_{P\in\mathcal{V}} \frac{\min(\tilde{\Delta}_{Q^-}, \tilde{\Delta}_{Q^+}}{\tilde{\Delta}_P} \tag{17}$$

(as before, $Q^\pm$ are the new vertices left and right from P). Note that τ' is a specific quantitative measure for the irregularity of the refinement from $\mathcal{T}$ to $\tilde{\mathcal{T}}$. Note that $\tau, \tau' \geq c_0 > 0$ for some absolute constant c_0.

Proposition 5 *Let $d = 1$. Suppose that $\mathbf{\Theta}$ and $\mathbf{\Psi}$ are defined by (6) resp. (11). Then the L_p-condition of $\mathbf{\Psi}$ satisfies*

$$\kappa(\mathbf{\Psi}) \asymp \tau^{p-1}, \qquad 1 \leq p < \infty, \tag{18}$$

with constants independent of $\mathcal{T}, \tilde{\mathcal{T}}$.

Proof. For the upper estimate of $\kappa\mathbf{\Psi}$ we need the following improvement of Proposition 4: For $g \in \tilde{V}$ of the form $g = \sum_Q x_Q\psi_Q$, we have

$$\sum_Q |x_Q|^p \|\psi_Q\|_p^p \leq C\tau^{p-1}\|g\|_p^p. \tag{19}$$

The counterpart of Proposition 2 for $d = 1$ is

$$\|g\|_p^p \leq 3^{p-1} \sum_Q |x_Q|^p \|\psi_Q\|_p^p$$

(because in the sum $\sum_Q x_Q\psi_Q$ at most 3 terms are non-zero for any argument). These two inequalities give the upper bound in (18).

The proof of (19) heavily relies on the following recovery formula for the coefficients x_Q of $g = \sum_Q x_Q\psi_Q$:

$$x_Q = \lambda_Q(g) := g(Q) + \alpha_{P^-Q}\, g(P^-) + \alpha_{P^+Q}\, g(P^+), \tag{20}$$

where $\ldots$, Q^-, P^-, Q, P^+, Q^+, $\ldots$ denote the vertices immediately to the left and right from Q. The values $\alpha_{P^\pm Q}$ are defined by (6) and (8), and belong to $[0, 1/2]$. The reader can easily verify this expression for x_Q by using the explicit formulas for the nodal values of the prewavelets which follow from the definitions (11), (6), and (8):

$$\psi_Q(Q) = 1+\beta_{QP^-}\alpha_{P^-Q}+\beta_{QP^+}\alpha_{P^+Q}, \quad \psi_Q(P^\pm) = -\beta_{QP^\pm}, \quad \psi_Q(Q^\pm) = \beta_{QP^\pm}\alpha_{P^\pm Q^\pm}. \tag{21}$$

Using (21), we obtain $\lambda_Q(\psi_Q) = 1$ and $\lambda_Q(\psi_{Q^\pm}) = 0$ (on all other $\psi_{Q'}$ the functional λ_Q vanishes trivially due to the support properties of $\psi_{Q'}$). This implies (20).

As a preparation, we will show that

$$\|g\|_p^p \asymp \sum_Q |x_Q|^p \tilde{\Delta}_Q + \sum_P |g(P)|^p \tilde{\Delta}_P. \tag{22}$$

Indeed, by definition of $\alpha_{P^\pm Q}$ (see (6) and (8) we have

$$\alpha_{P^\pm Q} \le \min(\frac{1}{2}, \left(\frac{\tilde{\Delta}_Q}{\tilde{\Delta}_P^\pm}\right)^2),$$

and in conjunction with (20)

$$\begin{aligned} |x_Q|^p \tilde{\Delta}_Q &\le (1 + \alpha_{P^-Q}^{1-\frac{1}{2p}} + \alpha_{P^+Q}^{1-\frac{1}{2p}})^{p-1} \tilde{\Delta}_Q (|g(Q)|^p + \alpha_{P^-Q}|g(P^-)|^p + \alpha_{P^+Q}|g(P^+)|^p \\ &\le 3^{p-1}(|g(Q)|^p \tilde{\Delta}_Q + |g(P^-)|^p \tilde{\Delta}_{P^-} + |g(P^+)|^p \tilde{\Delta}_{P^+}). \end{aligned}$$

Summation with respect to Q and use of the L_p stability of $\tilde{\mathbf{Phi}}$ gives one direction of (22). The opposite inequality follows in the same way if (20) is rewritten as

$$g(Q) = \lambda_Q(g) := x_Q - \alpha_{P^-Q} g(P^-) - \alpha_{P^+Q} g(P^+),$$

and used to estimate the terms $|g(Q)|^p \tilde{\Delta}_Q$.

We are now in the position to attack (19). Since by (16)

$$\sum_Q |x_Q|^p \|\psi_Q\|_p^p \le C \sum_Q |x_Q|^p \tilde{\Delta}_Q (1 + \beta_{QP^-}^{p-1} + \beta_{QP^+}^{p-1}),$$

it is enough to concentrate on the critical terms with $\beta_{QP^+} > \tau$ resp. $\beta_{QP^-} > \tau$ (the other terms are bounded by $\le C\tau^{p-1}|x_Q|^p \tilde{\Delta}_Q$). E.g., let $\beta_{QP^+} > \tau$ for some Q. By definition of τ we have $\beta_{Q^+P^+} \le \tau$, and we can use the identity $g(P^+) = -(\beta_{QP^+} x_Q + \beta_{Q^+P^+} x_{Q^+})$ together with (16) to estimate as follows:

$$\begin{aligned} \beta_{QP^+}^{p-1} |x_Q|^p \tilde{\Delta}_Q &= |\beta_{QP^+}^{-1/p}(g(P^+) + \beta_{Q^+P^+} x_{Q^+})|^p \tilde{\Delta}_Q \\ &\le C\beta_{QP^+}^{-1}(|g(P^+)|^p + \beta_{Q^+P^+}^p |x_{Q^+}|^p) \tilde{\Delta}_Q \\ &\le C(|g(P^p)|^p \tilde{\Delta}_{P^+} + \tau^{p-1}|x_{Q^+}|^p \tilde{\Delta}_{Q^+}). \end{aligned}$$

With this estimate for the critical terms at hand, the overall result is

$$\sum_Q |x_Q|^p \|\psi\|_p^p \le C(\tau^{p-1} \sum_Q |x_Q|^p \tilde{\Delta}_Q + \sum_P |g(P)|^p \tilde{\Delta}_P \le C\tau^{p-1}\|g\|^p,$$

where (22) was used. This gives (19), and the upper estimate for $\kappa(\mathbf{\Psi})$ in (18).

For the lower estimate, note that the stability constants C_1, C_2 in (1) must always satisfy $C_1 \le 1 \le C_2$ (just take $x_1 = 1$, and $x_i = 0$ for all $i \ne 1$). Thus, the result follows if we establish $C_1 \le C\tau^{1-p}$, or, equivalently, we find $g = \sum_Q x_Q \psi_Q$ such that

$$\sum_Q |x_Q|^p \|\psi_Q\|_p^p \ge c\tau^{p-1}\|g\|_p^p\,. \tag{23}$$

Moreover, it is enough to consider $\tau_0 \le \tau$, where τ_0 is fixed but sufficiently large. By definition of τ, there exists $P \in \mathcal{V}$ such that $\beta_{Q^+P}, \beta_{Q^-P} \ge \tau/2$, where this time

$$\ldots,\ Q_1^-,\ P^-,\ Q^-,\ P,\ Q^+,\ P^+,\ Q_1^+,\ \ldots$$

denote the vertices immediately to the left and right from P (the case $\tau = \infty$ requires some obvious modifications). Set

$$g = \beta_{Q^+P}^{-1}\psi_{Q^+} - \beta_{Q^-P}^{-1}\psi_{Q^-}.$$

The coefficients $x_{Q^\pm}$ have been chosen such that $g(P) = 0$, moreover, by (21) we compute

$$|g(P^\pm)|^p = |\beta_{Q^\pm P}^{-1}\beta_{Q^\pm P^\pm}|^p \le C\tau^{1-p}\frac{\tilde{\Delta}_P}{\tilde{\Delta}_{P^\pm}}.$$

To get the last estimate for P^+, we use that according to (16)

$$\tilde{\Delta}_{P^+} + \tilde{\Delta}_P > \tilde{\Delta}_{Q^+} \ge c_0\beta_{Q^+P}\tilde{\Delta}_P \ge \frac{c_0\tau}{2}\tilde{\Delta}_P,$$

and conclude that for $\tau \ge \tau_0 := 4/c_0$

$$\frac{\beta_{Q^\pm P^\pm}}{\beta_{Q^\pm P}} \le C\frac{\tilde{\Delta}_P}{\tilde{\Delta}_{P^+}} \le \frac{C}{c_0\tau/2 - 1} \le C\tau^{-1}.$$

Similarily for P^-.

Substituting into (22), we obtain

$$\begin{aligned} \|g\|_p^p &\le C(\beta_{Q^+P}^{-p}\tilde{\Delta}_{Q^+} + \beta_{Q^-P}^{-p}\tilde{\Delta}_{Q^-} + |g(P^+)|^p\tilde{\Delta}_{P^+} + |g(P^-)|^p\tilde{\Delta}_{P^-}) \\ &\le C\tau^{1-p}\tilde{\Delta}_P. \end{aligned}$$

On the other hand, by (16) we get

$$|x_Q^+|^p\|\psi_{Q^+}\|_p^p \ge c\beta_{Q^+P}^{-p}\tilde{\Delta}_{Q^+}\beta_{Q^+P}^{p-1} \ge c\tilde{\Delta}_P.$$

This gives the desired lower bound (23), and concludes the proof of Proposition 5.

Proposition 6 *Let $\{\mathcal{T}_j,\ j \ge 0\}$ be a sequence of partitions of $\mathbb{R}^1$ obtained recursively by irregular dyadic refinement from $\mathcal{T}_0$. Denote by τ_j' the quantity defined in (17) with $\mathcal{T} = \mathcal{T}_{j-1}$, $\tilde{\mathcal{T}} = \mathcal{T}_j$, and by $\mathbf{\Psi}_j$ the associated semiorthogonal prewavelet system, $j \ge 1$. Let $\mathbf{\Psi}_0 := \mathbf{\Phi}_0$ be the system of hat functions with respect to $\mathcal{T}_0$. Then the multilevel prewavelet system $\{mathbf\Psi_0, \mathbf{\Psi}_1, \mathbf{\Psi}_2, \ldots\}$, after normalization with respect to the L_2 norm, forms a Riesz basis in $L_2(\mathbb{R})$ if and only if $\sup_{j\ge1}\tau_j < \infty$.*

This result is an obvious consequence of the L_2 stability result of Proposition 5 applied to each $\mathbf{\Psi}_j$, and the mutual orthogonality between $\mathbf{\Psi}_j$ with different level indices j. We conjecture that the above condition also characterizes the L_p-unconditionality ($1 < p < \infty$) of this multilevel prewavelet system. This needs to be contrasted with the L_p unconditionality results for Franklin systems [7] which do not require any conditions on the refinement process. We do not know whether allowing larger support sizes for the ψ_Q in the semiorthogonal case (or replacing semiorthogonality by biorthogonality, see [16]) would lead to the construction of L_p stable locally supported prewavelet systems with no constraints on the irregular dyadic refinement. However, the main open problem is to close the gaps in the 2D case, where a full proof of the claims in Proposition 1 is missing.

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