Remarks on multilevel bases for divergence-free finite elements

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Abstract

The connection between the multilevel factorization method recently proposed by Sarin and Sameh for solving mixed discretizations of the Stokes equation using a divergence-free finite element formulation, and hierarchical basis preconditioners is established. For the 2D triangular Taylor-Hood element, a preconditioning method is proposed that could be useful in fractional step methods.

Key words: Generalized Stokes problem, mixed finite elements, divergence-free finite elements, preconditioners, hierarchical bases.

AMS(MOS) *classification*: 65N30, 65F10, 65N55, 35Q30

1 Introduction

Recently, Sarin and Sameh [15, 13] have proposed an iterative solver for finite element and finite difference discretizations of Stokes type problems. The idea is an algebraic multilevel factorization of the matrix B corresponding to the discretization of the gradient operator. Implicitly, the factorization yields a multilevel, non-local basis in the space of discretely divergence-free (ddf) velocities for which a basis exchange can be computed efficiently. The method allows to eliminate the pressure variables and leads to solving a reduced system in the space of ddf velocities. Thus, it can be viewed as a modification of both the projection method and of methods that use local bases of ddf velocities to obtain the reduced system directly. A parallel version of this factorization method in the spirit of domain decomposition, and applications to certain flow problems have been demonstrated in [6, 14, 10]. The theoretical analysis [15, 13] of this method seems to be incomplete. In a special case, some non-optimal upper estimates of the L_2-condition numbers of the resulting ddf multilevel basis have been given. The numerical evidence suggests that the method performs reasonably well for some model problems [15, 13] but also in particulate flow calculations [6, 10, 14]. However, it is sensitive to the problem under consideration, and does not scale optimally with respect to problem size (even though the growth is more moderate than in some other solution methods based on representations using ddf velocities). In [15, Section 2], the authors also state that their approach has no obvious similarities with other well-known multilevel methods.

In this note we correct this statement by considering another special case, and exhibit some relationship of the factorization method by Sarin and Sameh with previous research on multilevel methods. We treat a mixed finite element discretization of the Poisson equation using lowest-order Raviart-Thomas elements resp. the nonconforming P1 - P0 discretization of the generalized Stokes problem in a simply-connected two-dimensional domain assuming that the associated triangulations stem from a multilevel refinement procedure. It is shown in Section 2 that if the coarsening strategy underlying the factorization technique is consistent with the refinement pattern and includes a proper scaling then the resulting multilevel basis of ddf fluxes can be mapped one-to-one to the hierarchical basis [17] for standard linear finite elements on the same triangulation. This allows us to derive condition number estimates for the Sarin-Sameh algorithm which exactly capture the asymptotic behavior of the iteration count. In the case of the Stokes problem considered in Section 3, the method leads to an algorithm previously analyzed by Dörfler [4] and Verfürth. The condition number growth in the Stokes case is roughly proportional to h^{-2}, where h is the typical stepsize of the final triangulation, and, thus, far from optimal.

The derivation of the above result implicitly uses the fact that the ddf velocity space admits a local basis which leads to a discrete norm splitting of the L_2-norm of an arbitrary ddf velocity function into L_2 and H_0^1-parts (here a stream function argument is used). The hierarchical basis transformation associated with the Sarin-Sameh algorithm essentially reduces the condition number of the H_0^1-part. This suggests a slight improvement: instead of the linear finite element hierarchical basis, we could have used any other multilevel system suitable for preconditioning a Laplace problem, e.g., the BPX nodal basis function system [2, 9], and relate it to the mixed problem under consideration. In the last section we apply this idea to mixed discretizations for the 2D triangular Taylor-Hood element. We first repeat the construction of a local basis of ddf velocities for this element originally introduced by Hecht (see [11, Section 4.3.3]), and show that it can be made well-conditioned in L_2 by applying an optimal symmetric preconditioner for an associated discrete H_0^1-problem.

2 Poisson problem

We will deal with the mixed discretization of the homogeneous Neumann problem for the Poisson equation

$$\begin{cases} -\Delta p = f & \text{in} \quad \Omega \\ \frac{\partial p}{\partial n} = 0 & \text{on} \quad \partial\Omega \end{cases} ,$$

in weak form (i.e., with respect to the spaces $H_0(\mathrm{div})$ and $L_2\backslash\{0\}$) using lowest order Raviart-Thomas RT0 elements for the auxiliary flux variables $\vec{u}$ $(= \nabla p)$ and piecewise constant P0 approximations to p. Ω is a simply connected polygonal domain in the plane equipped with a regular and (at least, locally) quasi-uniform triangulation $\mathcal{T}$. For all details on mixed finite element methods, we refer to [3] or [12, Chapters 7 and 9]. Recall that RT0 elements are piecewise linear vector functions of the form $\vec{u}(x,y) = (a + cx, b + cy)$ in each triangle, with the degrees of freedom conveniently given by the values of $\vec{u} \cdot \vec{n}_e$ at the midpoints P_e of the edges e in the triangulation (at boundary edges these values are set to 0, which mimics the requirement that p has zero Neumann boundary conditions). The vector $\vec{n}_e$ is the unit vector in normal direction to e, its orientation is arbitrarily fixed. The space of the discrete RT0 fluxes will be denoted by $\vec{V}$, the space of piecewise constants by M. Bases are the usual box functions χ_Δ (i.e., the characteristic functions associated with the triangles Δ) for M, and the definition of the nodal basis functions $\vec{v}_e \in \vec{V}$ associated with the interior edges includes an appropriate scaling:

$$\vec{v}_e(P_e) \cdot \vec{n}_e = |e|^{-1} , \quad \vec{v}_e(P_{e'}) \cdot \vec{n}_{e'} = 0 , \ e' \neq e . \tag{1}$$

The bilinear forms involved are

$$a(\vec{u},\vec{v}) = (\vec{u},\vec{v})_{L_2(\Omega)} , \quad b(p,\vec{v}) = (\mathrm{div}(\vec{v}), p)_{L_2(\Omega)} , \quad \vec{u},\vec{v} \in \vec{V} , \ p \in M .$$

If $x = (x_e)$ and $y = (y_\Delta)$ denote the coefficient vectors of the solutions $\vec{u} \in \vec{V}$ and $p \in M$ of the mixed formulation

$$\begin{array}{lcll} a(\vec{u},\vec{v}) + b(p,\vec{v}) & = & 0 & \forall\, \vec{v} \in \vec{V} \\ b(q,\vec{u}) & = & (f,q)_{L_2} & \forall\, q \in M \end{array} \tag{2}$$

in the above bases then the linear system to be solved takes the form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} , \tag{3}$$

where the entries of A are $A_{e,e'} = a(\vec{v}_{e'}, \vec{v}_e)$ and the entries of B by $B_{e,\Delta} = b(\chi_\Delta, \vec{v}_e)$. For the right-hand side vector r with entries $r_\Delta = (f, \chi_\Delta)_{L_2}$, we assume that it satisfies the compatibility condition. Obviously, since

$$b(\chi_\Delta, \vec{v}_e) = \int_\Delta \operatorname{div}(\vec{v}_e) = \int_{\partial\Delta} \vec{v}_e \cdot \vec{n}_e \ ,$$

and due to the scaling assumed in (1), we have $B_{e,\Delta} = 0$ if Δ does not contain e, and $B_{e,\Delta} = \pm 1$ if $e \subset \Delta$, with the sign $+$ if $\vec{n}_e$ is outer normal with respect to Δ and $-$ otherwise. Thus, each row of B has exactly one $+1$ and one -1 entry, and $\ker B$ is one-dimensional and spanned by the vector $y_0 = (1, 1, \dots, 1)$. Since y is determined by (3) only up to constant multiple of y_0, to achieve unique solvability in (3) we additionally require

$$(y, y_0) = 0 \tag{4}$$

which implies $p \in L_2(\Omega) \backslash \{0\}$ as desired.

The method proposed in [15, 13] aims at a multilevel factorization

$$P^T B Z = \begin{pmatrix} D \\ 0 \end{pmatrix} \tag{5}$$

of B, where

$$P = P^{(1)} \dots P^{(J)} \ , \quad Z = Z^{(1)} \dots Z^{(J)}$$

are invertible matrices (in addition, Z is orthogonal). Under some straightforward conditions, the matrix-vector multiplications with P, Z, and their transposes can be performed in $\mathrm{O}(\dim \vec{V})$ operations. D is a rectangular matrix with $\dim M - 1$ rows and $\dim M$ columns, it is diagonal, with positive elements on the diagonal. Let

$$P = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$$

be the block decomposition of P that is associated with the decomposition in the right-hand side of (5). Then

$$Z^T B^T P = \begin{pmatrix} Z^T B^T P_1 & Z^T B^T P_2 \end{pmatrix} = \begin{pmatrix} D^T & 0 \end{pmatrix} , \tag{6}$$

Thus, $B^T P_2 = 0$, in other words, the columns in P_2 represent the coefficients of a basis in

$$\vec{Z} = \{\vec{u} \in \vec{V} \;:\; b(q, \vec{u}) = 0 \;\; \forall\, q \in M\} \;, \tag{7}$$

i.e., in the subspace of ddf fluxes in $\vec{V}$.

Setting

$$\hat{x} = P^{-1} x \equiv \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \;, \quad \hat{y} = Z^T y \;,$$

where $x = P\hat{x} \equiv P_1\hat{x}_1 + P_2\hat{x}_2$, gives the following transformation of (3):

$$\begin{pmatrix} P_1^T A P_1 & P_1^T A P_2 & D \\ P_2^T A P_1 & P_2^T A P_2 & 0 \\ D^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Z^T r \end{pmatrix} \tag{8}$$

The last equation in (8) is trivial (it reflects the compatibility assumption on r), and setting the last component of $\hat{y}$ to 0 satisfies (4). The remaining system is then uniquely solvable, and it essentially reduces to solving a subsystem for $\hat{x}_2$ with matrix $\tilde{A} \equiv P_2^T A P_2$. Note that $\tilde{A}$ the stiffness matrix for the form $a(\cdot, \cdot)$ in the above basis of ddf fluxes induced by P_2.

For a detailed description of the factorization (5), we refer to [15, 13]. We will give the construction in a special case, maintaining the following assumptions:

- The underlying triangulation $\mathcal{T}$ is obtained by quasi-uniform refinement of an initial partition of Ω (for simplicity, the initial partition will consist of a few triangles of size ≈ 1).

- To obtain the first step of the factorization,

$$(P^{(1)})^T B Z^{(1)} = \begin{pmatrix} D^{(1)} & 0 \\ 0 & B^{(1)} \end{pmatrix} \;, \tag{9}$$

  the entries y_Δ of the vector y will be partitioned into groups of 4 entries each by correspondence of the associated χ_Δ to triangles of the next-coarser triangulation $\mathcal{T}'$. This rule will also be applied in the subsequent factorization steps as it turns out that the nontrivial part of $B^{(1)}$ is (up to scaling) exactly the B-matrix in the mixed discretization (2) resp. (3) on $\mathcal{T}'$. Note that in [15, 13] much more general partitionings of y are allowed which adds greater flexibility in handling less structured $\mathcal{T}$ and geometries.

- To simplify the exposition, in addition to what is described in [13] a scaling is introduced at each step which can be viewed as a symmetric diagonal scaling of $\tilde{A}$.

We next give the derivation of (9).

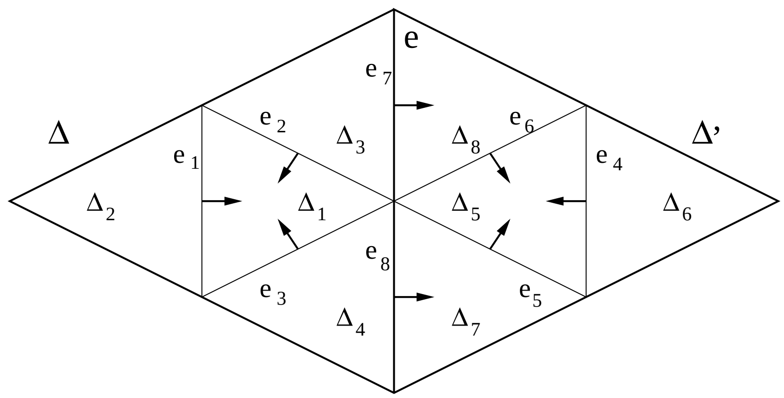

Figure 1: Notation for the local factorization problem

a) Each group of 4 subtriangles $\Delta_1, \ldots, \Delta_4$ of a triangle Δ of $\mathcal{T}'$ has 3 so-called *inner equations* associated with it (these are the equations for the edges e_1, e_2, e_3 in the interior of Δ). Assume that the enumeration of flux basis functions (and therefore of the associated equations) is such that first come, group by group, the inner equations and then the remaining *boundary equations* which correspond to edges e along the boundaries of triangles Δ from $\mathcal{T}'$. For the M-variables (triangles), we go by groups preserving the same ordering of triangles Δ as for the flux variables. This induces a block-structure

$$B = \begin{pmatrix} B_{int} \\ B_{bnd} \end{pmatrix}, \quad B_{int} = \begin{pmatrix} \hat{B}_1 & & & \\ & \hat{B}_2 & & \\ & & \ddots & \\ & & & \hat{B}_m \end{pmatrix},$$

If the enumeration of subtriangles and edges inside any Δ is consistent with Figure 1, then all $\hat{B}_k$ are the same: $\hat{B}_k = \hat{B}$. This 3×4 matrix and its singular value decomposition (SVD) $\hat{B} = \hat{U}\hat{S}\hat{V}^T$ are given by

$$\hat{B} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{U} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 1 & -1 \\ -1 & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -1 & \frac{-\sqrt{3}-1}{2} & \frac{-\sqrt{3}+1}{2} \end{pmatrix}$$

and

$$\hat{S} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2} \\ \frac{-1}{2\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{\sqrt{3}+1}{2\sqrt{3}} & \frac{1}{2} \\ \frac{-1}{2\sqrt{3}} & \frac{-\sqrt{3}-1}{2\sqrt{3}} & \frac{-\sqrt{3}+1}{2\sqrt{3}} & \frac{1}{2} \end{pmatrix}.$$

Note that $\hat{V}$ and $\hat{U}$ are not defined uniquely (they can be formed from different choices of complete orthonormal sets of eigenvectors of $\hat{B}^T\hat{B}$ and $\hat{B}\hat{B}^T$, respectively). However, it is clear that the last column of $\hat{V}$ is unique up to sign changes, and we find it convenient to keep a positive sign in this column. Set

$$Z^{(1)} = V \equiv \begin{pmatrix} \hat{V} & & & \\ & \hat{V} & & \\ & & \ddots & \\ & & & \hat{V} \end{pmatrix}$$

which represents a local (inside each Δ) orthogonal transformation of the basis in M. In particular, one of the new basis functions is the characteristic function of Δ scaled by a factor $1/2$.

If we define U and S in a similar way then a simple exercise (see [15]) gives

$$\underbrace{\begin{pmatrix} \mathrm{Id} & 0 \\ -B_{bnd}VS^{+} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & \mathrm{Id} \end{pmatrix}}_{\equiv(\tilde{P}^{(1)})^T} \begin{pmatrix} B_{int} \\ B_{bnd} \end{pmatrix} V = \begin{pmatrix} S \\ B_{bnd}V(\mathrm{Id} - SS^{+}) \end{pmatrix},$$

where S^+ is the pseudoinverse of S. One easily sees that $B_{bnd}V(\mathrm{Id} - SS^+)$ contains non-zeros only in the columns where S vanishes, these are the columns corresponding to the last column of the diagonal blocks $\hat{V}$ of V. For that reason, these entries are determined uniquely (independently of the specific appearance of $\hat{U}$ and $\hat{V}$) since we have fixed the last row of $\hat{V}$. Thus, after a column permutation in $Z^{(1)}$ which we will not indicate explicitly, we have

$$(\tilde{P}^{(1)})^T B Z^{(1)} = \begin{pmatrix} D^{(1)} & 0 \\ 0 & \tilde{B}^{(1)} \end{pmatrix},$$

where $D^{(1)}$ is a square diagonal matrix with positive diagonal elements.

b) $\tilde{B}^{(1)}$ contains the condensed information from the boundary equations in B_{bnd}. Obviously, each row of $\tilde{B}^{(1)}$ contains exactly two non-zero coefficients $\pm 1/2$. To each edge e in $\mathcal{T}'$, there correspond two such equations representing transformations of the equations associated with e_7 and e_8 in Figure 1. If the normal directions $\vec{n}_{e_7}$, $\vec{n}_{e_8}$ are chosen as indicated, these equations simply coincide. Thus, assuming the appropriate ordering of the flux unknowns, the matrix $\tilde{B}^{(1)}$ is block-diagonal with 2×2 rank-one matrices $\tilde{C}_e$ on the diagonal. Applying SVD to each of these blocks, $\tilde{C}_e = \tilde{U}_e\tilde{S}_e\tilde{V}_e^T$, where $\tilde{S}_e$ has one non-zero and one zero singular value on the diagonal, we can define the block-diagonal matrix $\tilde{U}$ (which is of the same size as $\tilde{B}^{(1)}$) with 2×2 blocks

$$\tilde{U}_e = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

corresponding to the edges e on the diagonal, and set

$$(P^{(1)})^T = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \sqrt{2}\tilde{U}^T \end{pmatrix} (\tilde{P}^{(1)})^T \, .$$

(in the general case of the factorization algorithm, the construction of $\tilde{U}$ is more complicated and involves SVDs applied to small submatrices of linearly dependent rows from $\tilde{B}^{(1)}$, together with a proper scaling). This gives

$$(\tilde{P}^{(1)})^T B Z^{(1)} = \begin{pmatrix} D^{(1)} & 0 \\ 0 & B^{(1)} \end{pmatrix} ,$$

where each second row in $B^{(1)}$ consists of only zero elements. After a final row permutation on $(\tilde{P}^{(1)})^T$ which shifts all those rows towards the end, we arrive at the factorization (9), where

$$B^{(1)} = \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} .$$

Evidently, each row in $\tilde{B}$ corresponds to an interior edge e of $\mathcal{T}'$, and contains nonzero entries 1 and -1 in the columns that are associated with the two triangles Δ and Δ' from $\mathcal{T}'$ attached to e (without loss of generality, we have assumed that the normal directions in the original triangulation are consistent with Figure 1 everywhere). The column of $P^{(1)}$ that corresponds to the zero row in $B^{(1)}$ associated with e will finally become a column of P_2 and therefore contains the coefficients of a ddf basis function associated with e. These coefficients are depicted on the left of Figure 2, they represent the standard divergence-free RT_0 vortex function $\vec{z}_{P_e}$ on $\mathcal{T}$ associated with the midpoint P_e of e. On the right of Figure 2, we indicate the function generated from the column of $P^{(1)}$ which corresponds to the other e-related row in $B^{(1)}$. This function should be considered as a fine-grid replacement for the basis function $\vec{v}_e$ associated with e on $\mathcal{T}'$. For our convenience, we will denote it temporarily by $\vec{v}_{P_e}$.

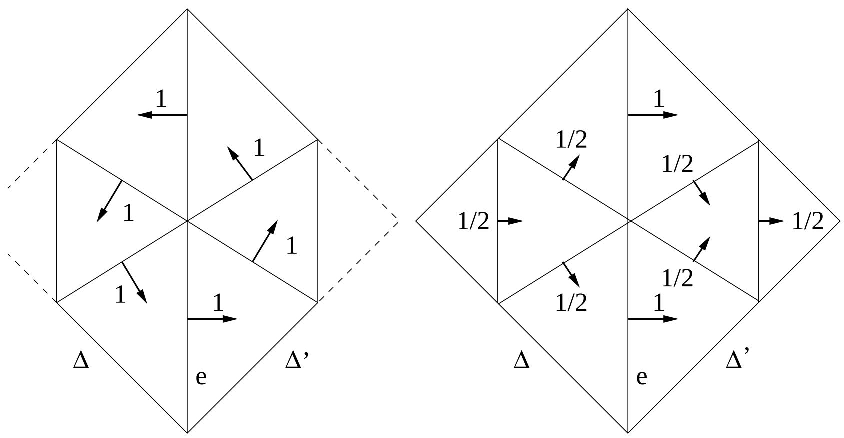

Figure 2: Local functions associated to e: $\vec{z}_{P_e}$ and $\vec{v}_{P_e}$

c) To find the full factorization, we iterate the first step, with $\tilde{B}$ instead of B. This does not pose any difficulties since $\tilde{B}$ is the exact counterpart of B for $\mathcal{T}'$, and we can repeat step a) and b) in complete analogy. This gives new $(P^{(2)})^T$, $D^{(2)}$, $B^{(2)}$, and $Z^{(2)}$, and so on. Note that some columns of $P^{(1)}P^{(2)}$ correspond to the zero rows of $B^{(2)}$, and give the next set of ddf fluxes. To find a convenient expression for them, let us observe that by recursion these new functions are similar to a vortex function (as shown in the left picture in Figure 2), however, now defined with respect to $\mathcal{T}'$, and with the corresponding basis functions $\vec{v}_e$ replaced by their fine-grid analogs $\vec{v}_{P_e}$ shown in the right picture of Figure 2. More precisely, if P is a vertex of $\mathcal{T}'$ which represents an edge midpoint in the next-coarser triangulation to $\mathcal{T}'$, then the associated ddf basis function $\vec{z}'_P$ is given by

$$\vec{z}'_P = \sum_{k=1}^{6} \vec{v}_{P_{e_k}} \ ,$$

where $\{e_k,\, k = 1, \ldots, 6\}$ is the set of edges in $\mathcal{T}'$ emanating from P. The coefficient set and the nodal vector orientations associated with $\vec{z}'_P$ on $\mathcal{T}$ are shown in Figure 3.

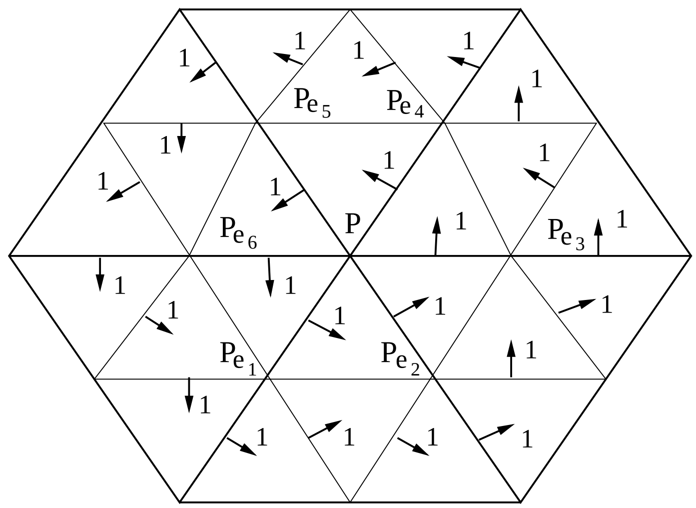

Figure 3: Basis representation of $\vec{z}'_P$ on $\mathcal{T}$

The key is that the ddf flux $\vec{z}'_P$ can be rewritten directly as a linear combination of 7 ddf vortex functions corresponding to $\mathcal{T}$:

$$\vec{z}'_P = 2\vec{z}_P + \sum_{k=1}^{6} \vec{z}_{P_{e_k}} \ .$$

Up to a constant factor 2, this linear combination is exactly the same as in the hierarchical basis transformation for the linear finite element case (replace the fine-grid vortex functions by hat functions), see [17]. All arguments are recursive, i.e., the basis of ddf fluxes obtained as the result of the factorization procedure in form of the matrix P_2 can be associated with the appropriately scaled hierarchical basis for linear finite elements. The scaling contains powers of 2 if we follow the above procedure. The formal relationship between the two bases is given by a standard stream function argument (see below).

Theorem 1 *The Sarin-Sameh factorization for the mixed RT0 - P0 discretization (2) of the homogeneous Neumann problem for the Poisson equation leads to a transformation P_2 which is essentially equivalent to a scaled hierarchical basis transformation of a homogeneous Dirichlet problem with linear finite elements on the same triangulation. As a consequence, with an appropriate diagonal scaling $\tilde{D}$ to be given below, we have*

$$cond(\tilde{D}^{-1/2}\tilde{A}\tilde{D}^{-1/2}) \approx J^2 \ , \qquad \tilde{A} = P_2^T A P_2 \ ,$$

where J is the number of refinement levels.

Proof. The first statement has been proved above. To get the condition number estimate, let us define the discrete stream function Ψ associated with a ddf flux $\vec{u} \in \vec{Z}$: Ψ is a linear finite element function on the same partition, which vanishes at boundary vertices while at interior vertices its value is defined as the path integral

$$\Psi(P) = \int_\gamma \vec{u} \cdot \vec{n}_\gamma$$

where γ is any oriented edge path connecting a boundary vertex with P. The orientation of the path induces the normal direction $\vec{n}_\gamma$ along the path.

Due to the definition of $\vec{Z}$, this definition is correct, the value $\Psi(P)$ does not depend on the choice of the path. Moreover, if x is the coefficient vector of $\vec{u} \in \vec{Z}$ then $|x_e| = |\Psi(P_e^+) - \Psi(P_e^-)|$. Thus,

$$\|\vec{u}\|_{L_2}^2 \approx \sum_e |x_e|^2 \approx \|\nabla\Psi\|_{L_2}^2 \ . \tag{10}$$

Recall that we have used the normalization of the basis functions $\vec{u}_e$ introduced at the beginning. This normalization also ensures that the stream function corresponding to a ddf vortex function (such as indicated in Figure 2 on the left) is nothing but the linear finite element hat function with value 1 at the center vertex P. The hierarchical basis functions for linear finite elements [17] will be denoted by Ψ_P, with P belonging to the set $\mathcal{V}$ of interior vertices of the triangulation ordered in a hierarchical fashion (i.e., $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \ldots$, where $\mathcal{V}_0$ consists of all vertices on the fine grid which are not in the next-coarser grid, etc.).

Let z be the coefficient vector of $\vec{u}$ in the basis in $\vec{Z}$ generated by P_2. Then, by the above remarks, the stream function of $\vec{u}$ has the representation

$$\Psi = \sum_k \sum_{P \in \mathcal{V}_k} z_P 2^k \Psi_P \ .$$

Let $\tilde{D}$ be such that the diagonal entry corresponding to the column in P_2 associated with the function Ψ_P is 2^{2k} if $P \in \mathcal{V}_k$. For $\tilde{D}^{1/2}$ and $\tilde{D}^{-1/2}$ this entry becomes 2^k and 2^{-k}, respectively. Set $E = \tilde{D}^{-1/2} P_2^T A P_2 \tilde{D}^{-1/2}$.

Note that according to [17], we have the sharp two-sided inequality

$$cJ^{-2}\|\tilde{D}^{1/2} z\|^2 \le \|\nabla\Psi\|_{L_2}^2 \le C\|\tilde{D}^{1/2} z\|^2 \ . \tag{11}$$

On the other hand,

$$\|\vec{u}\|_{L_2}^2 = a(\vec{u}, \vec{u}) = (AP_2 z, P_2 z) = (E\tilde{D}^{1/2} z, \tilde{D}^{1/2} z) \ . \tag{12}$$

Together with (10) we immediately arrive at

$$C^{-1} \le (E\tilde{z},\tilde{z})/(\tilde{z},\tilde{z}) \le c^{-1}J^2 \;, \quad \tilde{z} = \tilde{D}^{1/2}z \;.$$

This proves the second statement of the theorem.

We finish this section with a few remarks. In the particular case considered in the present section, the preconditioning method resulting from the factorization technique is straightforward if one would have started directly with the available local basis of vortex functions and the norm equivalence (10). In fact, any multilevel splitting of the linear finite finite element space induces an equivalent multilevel splitting in the above $\vec{Z}$. Thus, optimal multilevel methods (improving upon Theorem 1) could easily be constructed in the above example. E.g., the BPX multilevel system [2], [9, Chapter 4] would lead to uniformly bounded condition number estimates. Moreover, any preconditioner for the linear finite element discretization of the Dirichlet problem would be suitable for solving the variational problem in $\vec{Z}$ when discretized in the basis of ddf vortex functions.

The above arguments can be carried over to other 2D-applications where a scalar stream function is appropriate. However, the factorization technique will not always be equivalent to a *known* hierarchical basis construction in an appropriate space of stream functions although it certainly does generate *some* multilevel basis in this auxiliary space. We do not know about three-dimensional results of this type.

Note also that hierarchical basis constructions with linear finite elements are becoming more difficult if the final triangulation is not the result of a regular refinement process as assumed above (see, e.g., [7, 1]). In particular, this is the case if the domain Ω itself is highly unstructured or multiply-connected as in the applications outlined in [6, 10, 14].

The diagonal scaling of the multilevel basis (introduced in step b) of the recursion procedure, and through $\tilde{D}$) can be replaced by an implicit procedure: Since we are looking for a well-conditioned basis in an L_2-subspace of $\vec{V}$, L_2-normalization of the basis functions (in this case of the columns of P_2) leads to equivalent results.

3 Generalized Stokes problem

We keep the same assumptions on the triangulations and the factorization process as in the previous sections. The generalized Stokes problem we consider is

$$\begin{array}{rcl} -\nu\Delta\vec{u} + \alpha\vec{u} + \nabla p = \vec{f} & \text{in} & \Omega \\ \mathrm{div}(\vec{u}) = 0 & \text{in} & \Omega \\ \vec{u} = \vec{0} & \text{on} & \partial\Omega \end{array} \quad , \qquad (13)$$

where $\vec{u}$ and p are the unknown velocity and pressure functions, respectively. The mathematical theory for (13) requires that the space $H_0(\mathrm{div})$ (which was appropriate in the context of Section 2) is replaced by $(H_0^1)^2$ if $\nu > 0$ which is a major difference. The system (13) includes two parameters $\nu \geq 0$ and $\alpha \geq 0$. The case $\alpha = 1$ and $\nu = 0$ which occurs in operator splitting methods for time-dependent Navier-Stokes problems (see [12, Chapter 13.5]) formally resembles the mixed problem studied in the previous section while for $\nu = 1$, $\alpha = 0$ we get the standard Stokes problem.

To obtain a discrete mixed problem for (13), we will consider the same M for approximating the scalar pressures p but use a nonconforming P1 finite element space for the velocities $\vec{u}$: $\vec{V}$ will now consist of all piecewise linear vector functions which are continuous at the midpoints of interior edges, and vanish at the midpoints of boundary edges in the triangulation. For details on the resulting nonconforming P1 - P0 discretization, see [3, Example 8.1]. To each interior edge e there correspond two velocity basis functions which are conveniently associated with the normal ($\vec{v}_{n,e}$) and tangential directions ($\vec{v}_{t,e}$) along e. We keep the normalization introduced in the previous section. Finally, the respective bilinear forms are

$$\begin{array}{l} a(\vec{u},\vec{v}) = \nu \sum_\Delta (\nabla\vec{u}, \nabla\vec{v})_{L_2(\Delta)} + \alpha(\vec{u},\vec{v})_{L_2(\Omega)} \ , \\ b(p,\vec{v}) = -(\mathrm{div}(\vec{v}), p)_{L_2(\Omega)} \ , \end{array} \qquad \vec{u},\vec{v} \in \vec{V} \ , \ p \in M \ .$$

$\vec{Z}$ is defined as (7), now as a subspace of the new $\vec{V}$.

The linear system becomes

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix} \qquad (14)$$

where A, B, r are defined by similar formulas as in Section 2, and the coefficient vector $x = [(x_{n,e}), (x_{t,e})]$ of the discrete solution $\vec{u} \in \vec{V}$ is now twice the size of the old x. A moment's reflection shows that the upper part of the rectangular matrix B in (14) coincides with the B in (3), while the lower part (associated with the tangential degrees of freedom) is a zero matrix (evidently, all $\vec{v}_{t,e}$ are automatically discretely divergence-free).

This means that the factorization method of [15] applied to (14) is essentially equivalent to discretizing $a(\cdot,\cdot)$ with respect to a basis in $\vec{Z}$ which consists of all original tangential basis functions $\vec{v}_{t,e}$ and a hierarchical basis associated with the hierarchical basis $\{\Psi_P\}$ for linear finite element stream functions (the definition of which did not change) associated with the $\vec{v}_{n,e}$ part. Exactly this partial multilevel splitting has been proposed by Dörfler in [4] (and independently by Verfürth). The analysis of Dörfler leads to an condition number estimate for the resulting system which we reestablish in the following

Theorem 2 *Under the above assumptions, the Sarin-Sameh factorization for the mixed nonconforming P1 - P0 discretization of the generalized Stokes problem leads to a partial multilevel basis in the subspace $\vec{Z}$ of ddf velocities which was previously considered in [4]. With an appropriate diagonal scaling $\tilde{D}$, we have*

$$cond(\tilde{D}^{-1/2}\tilde{A}\tilde{D}^{-1/2}) \leq CJ^2\frac{\nu 2^{2J}+\alpha}{\nu+\alpha} , \qquad \tilde{A} = P_2^T A P_2 ,$$

where J is the number of refinement levels.

We will not give the details of the proof but rather mention the few changes in comparison with the argument for Theorem 1. An additional inequality we need is

$$c\|\vec{u}\|_{L_2}^2 \leq \sum_{\Delta}(\nabla\vec{u},\nabla\vec{u})_{L_2(\Delta)} \leq Ch^{-2}\|\vec{u}\|_{L_2}^2 , \quad h \approx 2^{-J} ,$$

which follows from the (discrete) Poincare and inverse inequalities for nonconforming P1 elements. After taking into account the weights ν and α, the comparison of $a(\vec{u},\vec{u})$ with $\|\vec{u}\|_{L_2}^2$ gives the additional factor in the condition number estimate, everything else follows from Theorem 1. Clearly, $\tilde{D}$ in Theorem 2 is obtained from the $\tilde{D}$ of Theorem 1 by extending it by an identity matrix for the positions corresponding to $\vec{v}_{t,e}$ functions.

Numerical evidence in [4] suggests that the condition number estimate cannot be improved upon. It is easy to show that there is a complemeting lower bound of order $2^{2J} \approx h^{-2}$ for the condition numbers in Theorem 2. In other words, with the appropriate scaling implemented, one expects linear growth of the iteration count with respect to problem size if the reduced system with matrix $\tilde{A}$ is solved by the conjugate gradient method.

Note that linear finite elements are not the correct space of stream functions to be naturally associated with this $\vec{Z}$ in the case of a two-dimensional Stokes problem, the correct space would be the Morley element space equipped with a discrete $H_0^2(\Omega)$ norm. For the latter, no truly optimal and simple multilevel basis has been found so far. For an attempt to construct a multilevel preconditioner for a Stokes discretization close to the above one, see [8].

In general, the Sarin-Sameh method is appealing because of its algebraic nature, it can be applied to elements where no (local) basis of divergence-free velocities is known, and to unstructured triangulations and domains. An important modification which has been explored in the parallel implementation for particulate flow problems [6, 14] is that the scheme of forming rank-deficient subsystems of inner equations in step a) of the factorization recursion is not restricted to only small numbers of pressure elements (such as four subtriangles of a coarse triangle, as it was the case in our example). However, we doubt that this flexibility could lead to any significant improvement of the preconditioning effect for the Stokes case, i.e. if $\nu \geq c\alpha$. To support this claim, let us look at an extreme implementation of the Sarin-Sameh algorithm: If all pressure variables are declared one group then all equations are inner to this one group, and the first (and only) step of the factorization algorithm consists of an SVD for B. In this case, P_2 has orthonormal columns which means that the divergence-free basis is L_2-stable. When applied in the context of Theorem 2, this gives an optimal algorithm for the case $\nu = 0$ with respect to preconditioning (but not with respect to operation count). However, for $\nu > 0$ the same strategy would only remove the factor J^2. Thus, it is our belief that while generally leading to a basis in $\vec{Z}$ that is well-conditioned with respect to L_2, the factorization method will not lead to optimal (Navier-)Stokes preconditioning. It would be interesting to give precise statements for this claim which cover the more popular Navier-Stokes elements as well as the 3D case.

4 Triangular Taylor-Hood elements

This section should be viewed as a by-product of our attempts to relate the algebraic factorization method proposed in [15, 13] with other preconditioning methods. Consider the P2 - P1 Taylor-Hood element for the discretization of (13) on a triangulation of a simply-connected polygonal domain $\Omega \in \mathbb{R}^2$ as our choices for $\vec{V}$ and M. Originally, we assumed that the analysis in this case is more complicated, mainly because of the more complicated structure of B but also because main ingredients of the analysis given in the previous sections (such as a simple local basis for the ddf velocity space and an appropriate discrete stream function) seemed to be missing.

However, it turned out that there is a simple local basis in $\vec{V}$ for which B simplifies reasonably, and that allows to explicitly construct a local basis in the associated $\vec{Z}$. Essentially this local basis of ddf velocities has been originally introduced by Hecht in his thesis, a brief description of it can be found in [11]. Since Hecht's paper is widely unknown, we decided to present the derivation of this basis together with the construction of an associated discrete stream function (very much in analogy to the nonconforming P1 - P0 element case in Section 3). For rectangular Q2 - Q1 Taylor-Hood elements, a construction of a local basis of ddf velocities was given in [16], for other research on this topic, see also [5] .

We start with some definitions. Let $\vec{V} \subset (H_0^1(\Omega))^2$ be the space of P2 Lagrange C^0 vector functions, given by their values (which will be conveniently represented by *nodal vectors*) at interior vertices P and midpoints M_e of interior edges, and by zero boundary values. The pressure space M is given by scalar P1 Lagrange C^0 elements on the same partition, given by their values at the vertices, including boundary vertices. Details can be found in [3, Chapter VI] or in [12, Section 9.3.2].

The following choice of basis functions in $\vec{V}$ will greatly simplify the considerations: The basis functions associated with interior vertices P are defined as usual by setting

$$\vec{v}_{P,1}(Q) = \begin{cases} h_P^{-1}\vec{e}_1 & \text{, if} \quad Q = P \\ \vec{0} & \text{, if} \quad Q \neq P \end{cases} , \qquad \vec{v}_{P,2}(Q) = \begin{cases} h_P^{-1}\vec{e}_2 & \text{, if} \quad Q = P \\ \vec{0} & \text{, if} \quad Q \neq P \end{cases} ,$$

where Q runs through all nodal points (vertices and edge midpoints), h_P is the characteristic element diameter near P (e.g., we could set $h_P = \mathrm{diam}(\mathrm{supp}\, \vec{v}_{P,1})$), and $\vec{e}_1$, $\vec{e}_2$ are unit vectors in coordinate directions. The definitions for the two basis functions associated with the midpoints of interior edges e are similar:

$$\vec{v}_{e,t}(Q) = \begin{cases} c_e\vec{e} & \text{, if} \quad Q = M_e \\ \vec{0} & \text{, if} \quad Q \neq M_e \end{cases} , \qquad \vec{v}_{e,d}(P') = \begin{cases} c_e\vec{d} & \text{, if} \quad Q = M_e \\ \vec{0} & \text{, if} \quad Q \neq M_e \end{cases} .$$

Here, the vectors $\vec{e}$ and $\vec{d}$ are the two diagonal vectors of the quadrilateral given by the two triangles attached to e, and c_e is a normalization constant to be chosen below. We will call $\vec{v}_{P,1}$, $\vec{v}_{P,2}$ *vertex basis functions*, and $\vec{v}_{e,t}$, $\vec{v}_{e,d}$ *tangential* and *diagonal edge basis functions*, respectively. Figure 4 shows the supports of these basis functions, it also introduces some notation which is used below.

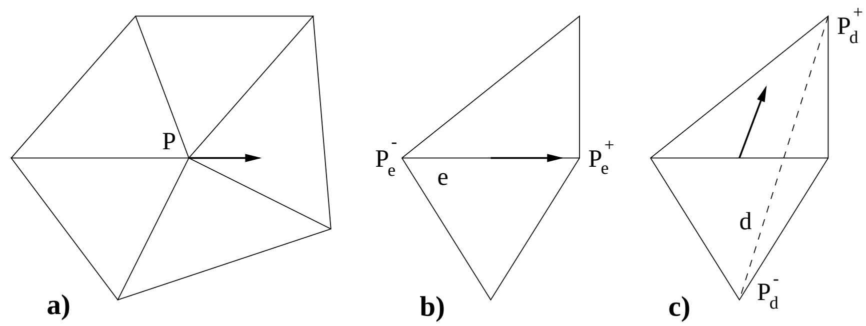

Figure 4: Basis functions in $\vec{V}$: a) $\vec{v}_{P,1}$, b) $\vec{v}_{e,t}$, and c) $\vec{v}_{e,d}$

With these choices, the matrix B has a simple structure. Note that we can use

$$b(p,\vec{u}) = \sum_{\Delta} \int_{\Delta} \vec{u} \cdot \nabla p = \sum_{\Delta} \frac{|\Delta|}{3} \sum_{e \subset \Delta} \vec{u}(M_e) \cdot \nabla p|_{\Delta} \ ,$$

since $\vec{u} \cdot \nabla p$ is a quadratic polynomial on Δ for which the triangular edge-midpoint rule is exact. Thus, all vertex basis functions are automatically in $\vec{Z}$ (the corresponding row of B is a zero row).

On the other hand, each edge basis function contributes a nontrivial row to B. Indeed, if $\phi_P \in M$ denotes the usual hat function at the vertex P, then for any triangle Δ attached to P we have

$$\frac{|\Delta|}{3} \nabla \phi_P = \frac{|\Delta|}{3h_e} \vec{n}_e = \frac{|e|}{6} \vec{n}_e \equiv \frac{\vec{e}^{\perp}}{6} \ ,$$

where e is the side of Δ opposite to P, and h_e the corresponding height. Note that $\vec{e}^{\perp} = |e|\vec{n}_e$ coincides with the edge vector $\vec{e}$ rotated by $\pi/2$ into the direction of $\vec{n}_e$ which we have assumed to point towards P. This immediately gives

$$b(\phi_{P_d^+}, \vec{v}_{e,t}) = 0 \ , \quad b(\phi_{P_d^+}, \vec{v}_{e,d}) = c_e \frac{\vec{d} \cdot \vec{e}^{\perp}}{6} \ .$$

Setting $c_e = 6(\vec{d} \cdot \vec{e}^{\perp})^{-1}$ and dealing with P_d^-, P_e^+, P_e^- in an analogous way we get

$$b(\phi_P, \vec{v}_{e,t}) = \begin{cases} 1 & , \text{ if } P = P_e^+ \\ -1 & , \text{ if } P = P_e^- \\ 0 & , \text{ otherwise} \end{cases} \ , \quad b(\phi_P, \vec{v}_{e,d}) = \begin{cases} 1 & , \text{ if } P = P_d^+ \\ -1 & , \text{ if } P = P_d^- \\ 0 & , \text{ otherwise} \end{cases} \ .$$

In other words, each nontrivial row of B has exactly two non-vanishing entries, 1 and -1, associated with either the endpoints of an interior edge e or the endpoints of the corresponding diagonal d.

This geometric description of B makes it easy to describe the ddf condition, and to construct a local basis in $\vec{Z}$. Indeed, a function from $\vec{V}$ given by its coefficients with respect to the above nodal basis in $\vec{V}$ belongs to $\vec{Z}$ *if and only if*, for any vertex P, the sum of coefficients associated with incoming tangential and diagonal basis functions equals the sum of coefficients associated with outgoing tangential and diagonal basis functions (incoming means that P coincides with P_e^+ resp. P_d^+, outgoing means that P coincides with P_e^- resp. P_d^-). In Figure 5 we schematically depict the coefficients (together with the directions of nodal vectors of the involved basis functions from $\vec{V}$) of some locally supported functions from $\vec{Z}$ for which the above ddf condition can easily be checked.

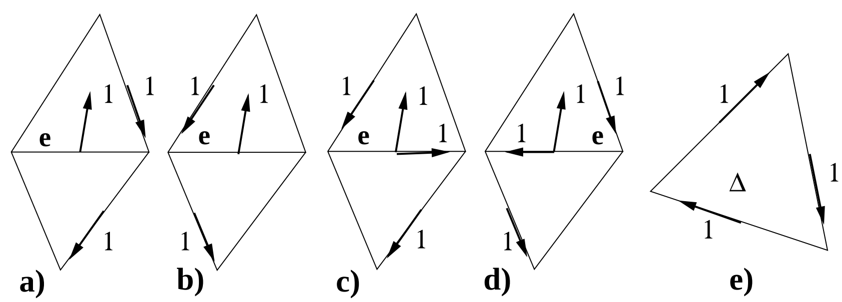

Figure 5: Local ddf velocities: a-d) $\vec{z}_{e,d}$ and e) $\vec{z}_\Delta$

Next we will show that a certain collection of functions $\vec{z}_{e,d}$ (*diagonal edge functions*), and $\vec{z}_\Delta$ (*triangle functions*), together with the vertex basis functions $\vec{z}_{P,k} \equiv \vec{v}_{P,k}$, spans a basis in $\vec{Z}$. To this end, pick an arbitrary $\vec{u} \in \vec{Z}$. There is a unique linear combination $\vec{z}_1$ of vertex basis functions $\vec{z}_{P,k}$, $k = 1, 2$, which after subtraction from $\vec{u}$ leads to another $\vec{v} = \vec{u} - \vec{z}_1 \in \vec{Z}$ with

$$\vec{v}(P) = \vec{0}\ , \quad \vec{v}(M_e) = \vec{u}(M_e)\ ,$$

for all vertices P and edge midpoints M_e in the triangulation. As a result, $\vec{v}$ is uniquely represented by a linear combination of $\vec{v}_{e,t}$ and $\vec{v}_{e,d}$. Take an arbitrary interior edge e, and denote the attached triangles by Δ, Δ'. If in either of the two triangles the two other edges are on the boundary of Ω then necessarily the coefficient in front of $\vec{v}_{e,d}$ is zero (otherwise the ddf property would be violated at the boundary vertex common to the two boundary edges). Thus, there is at least one more interior edge $\neq e$ in both Δ and Δ' which implies that at least one of the ddf functions shown in Figure 5 a)-d) is well-defined. Pick any such function as the basis function $\vec{z}_{e,d}$ associated with e. Obviously, there is a unique linear combination $\vec{z}_2$ of these $\vec{z}_{e,d}$ such that $\vec{w} = \vec{v} - \vec{z}_2 \in \vec{Z}$ satisfies

$$\vec{w}(P) = \vec{0} \ , \quad \vec{n}_e \cdot \vec{w}(M_e) = 0 \ ,$$

for all vertices P and edge midpoints M_e in the triangulation. In other words, $\vec{w}$ is representable by a linear combination of only tangential edge functions

$$\vec{w} = \sum_e a_{e,t} \vec{v}_{e,t} \ . \tag{15}$$

The crucial last step of the basis construction is to show that any $\vec{w} \in \vec{Z}$ of the form (15) is uniquely representable by a linear combination of functions from $\{\vec{z}_\Delta\}$, where Δ runs through the set of *interior triangles* of the triangulation (a triangle is called interior if all its edges are interior, otherwise it is a *boundary triangle*). This can conveniently be done by introducing a discrete stream function mapping $\vec{w} \mapsto \Psi_{\vec{w}}$, where $\Psi_{\vec{w}}$ is a scalar, piecewise constant function on the underlying triangulation. Set $\Psi_{\vec{w}}|_\Delta = 0$ if Δ is a boundary triangle. To define the value of $\Psi_{\vec{w}}|_\Delta$ for an interior triangle Δ, consider any (directed) chain

$$\mathcal{C} = \{\Delta_0 \to \Delta_1 \to \ldots \to \Delta_m = \Delta\} \tag{16}$$

of triangles such that $\Delta_{j-1} \cap \Delta_j$ consists of an (interior) edge e_j, $j = 1, \ldots, m$, and which starts at some boundary triangle Δ_0. Such a chain exists for any interior Δ but is usually not unique. See Figure 6 for an illustration of this situation (the dashed line represents another chain which is shown in reverse direction). Without loss of generality, we can also assume that all Δ_j, $j = 1, \ldots, m$, are interior and are pairwise different. Let $\vec{d}_j$ denote the vector which connects the barycenter Δ_{j-1} with that of Δ_j, and set $\epsilon_j = \operatorname{sign}(\vec{d}_j \cdot \vec{e}_j)$. Then

$$\Psi_{\vec{w}}|_\Delta = S(\mathcal{C}) \equiv \sum_{j=1}^{m} \epsilon_j a_{e_j,t} \ . \tag{17}$$

It is easy to see that this value does not depend on the chain (16). Indeed, consider two different chains $\mathcal{C}_1$ and $\mathcal{C}_2$ connecting to the same Δ. Reversing the second chain and appending it to the first one, we get another chain $\mathcal{C}'$ which now connects two boundary triangles. In the most general case, after some reordering of the triangles, the new chain can be represented as the sum of a chain $\mathcal{C}'_0$ connecting two boundary triangles and, possibly, some more *cycles* $\mathcal{C}'_k$, $k \geq 1$, of interior triangles (a chain (16) is a cycle if

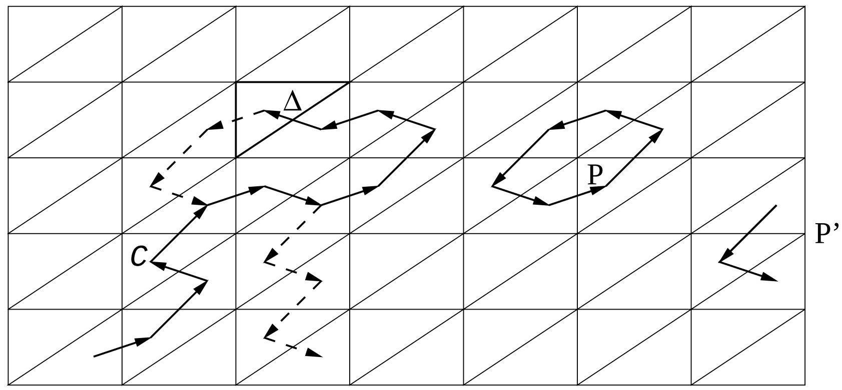

Figure 6: Chains and cycles

$\Delta_0 = \Delta_m$). In the example of the two chains in Figure 6, we have just one such cycle. Accordingly, we can write

$$S(\mathcal{C}_1) - S(\mathcal{C}_2) = S(\mathcal{C}') = S(\mathcal{C}'_0)\ (\ + \sum_{k\geq 1} S(\mathcal{C}'_k)\ )\ . \tag{18}$$

But for any of the cycles $\mathcal{C}'_k$, $k \geq 1$, we have $S(\mathcal{C}'_k) = 0$ since it can be represented as the superposition of elementary cycles $\mathcal{C}_P$ around the vertices P interior to the cycle $\mathcal{C}'_k$ (note that $S(\mathcal{C}_P)$ automatically vanishes by the ddf condition applied to $\vec{w}$). Figure 6 shows one such elementary cycle. The argument is the same for the chain $\mathcal{C}'_0$ if we take into account also some $\mathcal{C}_{P'}$ with P' along the boundary of Ω (in both cases, we need the assumption that Ω is simply connected). Thus, all terms in the right-hand side of (18) are zero which gives the chain-independence of the definition in (17).

With (17) at hand, we can write

$$\vec{w} = \sum_{\Delta} \Psi_{\vec{w}}|_{\Delta} \vec{z}_{\Delta} \tag{19}$$

for any $\vec{w} \in \vec{Z}$ of the form (15), where the summation is over all interior triangles Δ. This assumes that the triangle functions $\vec{z}_\Delta$ are defined with an orientation as shown in Figure 5 e). To see this, consider any interior e with triangles Δ and Δ' attached to it. Append $\Delta_{m+1} = \Delta'$ to the chain (16) defining $\Psi_{\vec{w}}|_\Delta$ according to (17). Since the value of $\Psi_{\vec{w}}|_{\Delta'}$ does not depend on the chain, we have (assuming that the orientation of e is such that $\epsilon_{m+1} = 1$)

$$a_{e,t} = S(\{\mathcal{C} \to \Delta'\}) - S(\mathcal{C}) = \Psi_{\vec{w}}|_{\Delta'} - \Psi_{\vec{w}}|_{\Delta}\ .$$

Thus, according to the definition of $\vec{z}_\Delta$ and the fact that $\Psi_{\vec{w}}|_\Delta = 0$ for all boundary triangles, (19) is established.

Altogether, we arrive at

Theorem 3 *The space $\vec{Z}$ of ddf velocities for the triangular P1 - P2 Taylor-Hood element with respect to a triangulation on a simply-connected polygonal domain Ω possesses a basis*

$$\mathcal{Z} = \mathcal{Z}_P \cup \mathcal{Z}_e \cup \mathcal{Z}_\Delta$$

of locally supported functions (originally introduced by Hecht, see [11]), where $\mathcal{Z}_P$ consists of all vertex basis functions $\vec{z}_{P,k}$, $\mathcal{Z}_e$ of a selection of diagonal edge functions $\vec{z}_{e,d}$ as specified above, and $\mathcal{Z}_\Delta$ of all triangle functions $\vec{z}_\Delta$ associated with interior Δ. Moreover, let

$$x = [(x_{P,k}), (x_{e,d}), (x_\Delta)]$$

denote the coefficient vector of $\vec{z} \in \vec{Z}$ with respect to this basis. Set $\tilde{x}_\Delta = x_\Delta$ if Δ is an interior triangle, and $\tilde{x}_\Delta = 0$ if Δ is a boundary interior triangle. Then

$$\|\vec{z}\|_{L_2}^2 \approx \sum_P \sum_{k=1}^{2} x_{P,k}^2 + \sum_e x_{e,d}^2 + \sum_{\Delta,\Delta' : \exists e = \Delta \cap \Delta'} |\tilde{x}_\Delta - \tilde{x}_{\Delta'}|^2 . \qquad (20)$$

The norm equivalence (20) is an obvious consequence of the locality of all basis functions and the above construction. We leave the details to the reader. (20) indicates that $\mathcal{Z}$ is not uniformly L_2-stable. However, the problems come from the $\mathcal{Z}_\Delta$-part. Note that multiply-connected domains could be considered as well, at the expense of including one non-local basis functions per hole into the construction.

The Sarin-Sameh factorization technique would replace $\mathcal{Z}_e \cup \mathcal{Z}_\Delta$ by some multilevel basis $\mathcal{Z}'$ while leaving $\mathcal{Z}_P$ untouched (by the way, it would automatically handle the case of multiply-connected domains). As in the non-conforming P1 - P0 element case considered in section 3, one expects that this basis possesses relatively moderate L_2-conditioning even though there is no proof for this claim (for numerical experiments with a similar element, see [15, 13]). However, using the above norm equivalence (20), one can directly address the L_2-preconditioning problem. Indeed, all what is needed is a preconditioner for the symmetric positive definite (spd) matrix C associated with the quadratic form

$$c((x_\Delta), (x_\Delta)) \equiv \sum_{\Delta,\Delta' : \exists e = \Delta \cap \Delta'} |\tilde{x}_\Delta - \tilde{x}_{\Delta'}|^2 . \qquad (21)$$

But this quadratic form resembles a discretization of the Dirichlet problem for Laplace's equation for which preconditioning has been studied extensively. E.g., if the triangulation is quasi-uniform then we can use the fictitious space lemma (see [9, Theorem 17]) to relate the form in (21) to a linear finite element discretization of Laplace's equation on an auxiliary refined triangulation. In Figure 7, two versions of such refined triangulations are shown. The common feature is that they contain one vertex P_Δ interior to each triangle such that a linear mapping

$$R \;:\; v \in V \;\longmapsto\; x_\Delta = v(P_\Delta) \quad \forall \text{ interior } \Delta$$

can be given which has the properties required in the fictitious space lemma. V denotes the finite linear element space on the refined partition, equipped with zero boundary. Preconditioners for C can now be constructed in the form $M = R\tilde{M}R^T$, where $\tilde{M}$ is a spd preconditioner for the discrete Laplace problem with respect to V. Similar constructions can be given for locally quasi-uniform triangulations.

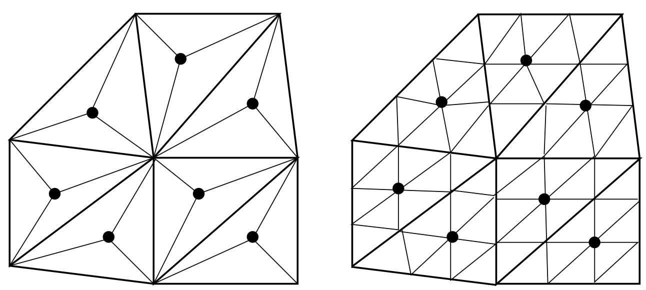

Figure 7: Two possibilities of refined triangulations

We do not go into the details of this reduction step. We rather conclude with a numerical example for which the above switch to a linear finite element discretization for preconditioning purposes can be implemented in a simpler fashion, without introducing auxiliary triangulations. Let $\Omega = [0,1]^2$ be equipped with a uniform partition into triangles $2n^2$ triangles. Then

$$\dim \vec{V} = 2(2n-1)^2 \;, \quad \dim M = (n+1)^2; \;, \quad \dim \vec{\mathcal{Z}} = 7n^2 - 10n + 2 \;.$$

There are $2(n-1)^2$ vertex basis functions, $4n^2 - 2n - 4$ diagonal edge functions, and another $2(n-1)^2$ triangle functions in $\mathcal{Z}$. According to Theorem 3, in order to obtain an optimal spd preconditioner for the L_2-limit of the generalized Stokes problem (13) (i.e., for $\nu = 0,\ \alpha = 1$), we need a spectrally equivalent representation for the quadratic form (21) corresponding to the $\mathcal{Z}_\Delta$-part of the basis. This can be achieved as follows. In the interior squares $\Box$ which consist of two interior triangles Δ, Δ' each we introduce a local change of basis by setting

$$\vec{z}_\Box = \frac{1}{2}(\vec{z}_\Delta + \vec{z}_{\Delta'}) \ , \quad \vec{z}\,'_\Box = \frac{1}{2}(\vec{z}_\Delta - \vec{z}_{\Delta'}) \ .$$

In the boundary squares with one interior triangle Δ, set $\vec{z}\,'_\Box = \vec{z}_\Delta$. Let $(x_\Box)$ and $(x'_\Box)$ denote the coefficients of a $\vec{w} \in \operatorname{span} \mathcal{Z}_\Delta$ with respect to this equivalent basis $\mathcal{Z}_\Box = \{\vec{z}_\Box, \vec{z}\,'_\Box\}$. Finally, denote by $v_{(x_\Box)}$ the bilinear finite element function which is defined on the underlying $n \times n$ square partition by setting its nodal values to $x_\Box$ at $P_\Box$ (the upper-right vertex of an interior square), and to 0 otherwise. With this, it is easy to see that

$$c((x_\Delta),(x_\Delta)) \approx \sum_\Box (x'_\Box)^2 + (\nabla v_{(x_\Box)}, \nabla v_{(x_\Box)})_{L_2} \ . \qquad (22)$$

The last expression can be replaced by a quadratic form in $(x_\Box)$ derived from any standard spd preconditioner for bilinear finite element discretizations on the square partition. In our numerical experiment below, we have used the BPX multilevel preconditioner [2].

Table 1 shows the parameter n, the dimension N of $\vec{Z}$, and the iteration count for the preconditioned conjugate gradient (pcg) method with the above described preconditioner. The L_2 column corresponds to solving the reduced spd problem in $\vec{Z}$ if we have $\nu = 0$ and $\alpha = 1$ in (13). Accordingly, the H^1 column corresponds to solving the Stokes problem ($\nu = 1$, $\alpha = 0$) in the reduced formulation using $\mathcal{Z}$. The iteration was stopped when an error reduction by a factor of 10^{-4} in the preconditioned residual norm was achieved. All cases have been run repeatedly, with randomly generated solutions and zero starting vector for the iteration. The iteration count shown is the average value (rounded to the next integer), however, in all cases the variations were negligible.

The results of the L_2 column confirm our theory which predicts optimal scalability in this case. The iteration count does not grow if N becomes large. For all n, the relative euclidean error of the approximate solution vector obtained from the iteration did not exceed $4 \cdot 10^{-4}$ which indicates reliable preconditioning. We conclude that the construction of a basis of locally supported ddf velocities given by Theorem 1, together with a simple preconditioning step, might provide an alternative for performing the projection step in operator-splitting (or fractional step) methods for instationary Navier-Stokes problems.

n	N	L_2	H^1	n	N	L_2	H^1
6	194	23	59	48	15650	25	86
8	370	24	72	64	28034	24	85
12	890	25	87	96	63554	24	84
16	1634	25	83	128	113410	24	84
24	3794	24	89	192	256136	24	84
32	6850	25	89	256	456194	24	84

Table 1: Iteration count for the ddf formulation with P2 - P1 elements

The results of the H^1 column require some comment. In the Stokes case, using the same argument as in Section 3, we only have an upper bound of $O(n^2)$ for the condition numbers of the preconditioned linear system. Thus, in the worst case, the iteration count of the pcg method could grow linearly in n. What we observed in our experiments was an almost constant (although 4 times higher) iteration count. The explanation is, however, very simple. Note that the preconditioned residual norm which was used as a stopping criteria is reliable only if the preconditioner is close to optimal. Thus, although the preconditioned residual was consistently reduced by a factor of about 10^{-4} after about 80 pcg steps, monitoring of the error in the euclidean norm showed less and less progress with n increasing (for $n = 256$ the euclidean error reduction was only about 0.3 after 80 iterations). Thus, we do not recommend using the above preconditioning method for large-scale Stokes problems.

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