Extremal Properties of Trigonometric Polynomials With Applications to Signal Design

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Abstract. The paper deals with the mathematical aspects of a peak power management problem of radio-frequency signals which is related to minimizing the L_∞ norm of a trigonometric polynomial with given absolute values of the coefficients, and arises, in various forms, in the design of communication systems. We discuss theoretical bounds and numerical results, some generalizations, and open problems.

§1. Introduction

In this note, we discuss the following extremal problem for complex trigonometric polynomials: *Given a set of N non-negative numbers $r_k \in \mathbb{R}$, find real phases $\phi_k \in \mathbb{R}$ such that*

$$\|P\|_\infty = \max_{t\in\mathbb{R}} |P(t)| \,, \qquad P(t) = \sum_{k=0}^{N-1} r_k e^{i(kt+\phi_k)} \,, \tag{1}$$

is minimized. Variations of this problem often arise in the design of communication systems, in particular, in connection with reducing the peak power of the radio-frequency signals. Since the power spectrum of the signal $P(t)$, i.e., the sequence of Fourier coefficients of its auto-correlation function, and the L_2-norm of $P(t)$ do not depend on the phases ϕ_k, the above problem is often cast in the form of minimizing the *peak-to-average-power ratio*, in short *par*-value, of $P(t)$. The problem and its number-theoretical aspect has been discussed in Chapter IX.28 of [12]. In our exposition, we conveniently define the *par*-value of $P(t) \not\equiv 0$ and the minimal *par*-value for a given nonzero sequence $r = \{r_k\}_0^{N-1}$ by

$$par(P) = \frac{\|P\|_\infty}{\|P\|_2} \,, \qquad par(r) = \min_{\phi_k\in\mathbb{R}} \; par(P) \,. \tag{2}$$

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Here, and in the following, we use the notation

$$\|f\|_p = (\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\,dt)^{1/p}\ , \quad \|c\|_p = (\sum_{k=0}^{N-1} |c_k|^p)^{1/p}\ , \qquad 1 \le p < \infty\ ,$$

for L_p- and ℓ_p-norms (with obvious modifications for $p = \infty$).

This equivalent form leads to generalizations of independent interest. E.g., *worst-case estimates* for classes R_N of coefficient sequences r described by the quantities

$$\delta_{p,q}(R_N) = \max_{r=\{r_k\}_0^{N-1}\in R_N}\ \min_{\phi_k\in\mathbb{R}} \frac{\|P\|_p}{\|r\|_q}\ , \qquad 1 \le p, q \le \infty\ , \tag{3}$$

are worth considering. The asymptotic behavior of $\delta_{p,q}(R_N)$ for $N \to \infty$, when R_N is the set of all nonnegative sequences of length N, is studied in Section 2, Theorem 2. As a corollary we obtain

Corollary 1. *For any (non-zero) sequence $r_k \ge 0$, $k = 0, \ldots, N-1$, we have*

$$1 \le par(\{r_k\}_0^{N-1}) \le 1 + C(\log N)^{1/2}\ , \qquad N \ge 1\ , \tag{4}$$

for some absolute constant C. The nontrivial upper bound in (4) cannot be improved asymptotically.

It should be mentioned that the above stated problems, and especially their infinite-dimensional ($N = \infty$) counterparts, have been studied in the context of random Fourier series for many years, see [6], [9], [8] for more information. Most of the statements in Section 2 are relatively straightforward consequences of known results in this area. Another variation, which will not be discussed in this paper, arises if one puts restrictions on the set of phases $\{\phi_k\}$ for the minimization problem. For instance, allowing ϕ_k to take only the values 0 and π leads to the problem of optimal sign distribution for a given set of real Fourier coefficients.

The most heavily studied special case is that of the constant sequence $r_k = 1$, which has a long history and was promoted by conjectures of Littlewood and Erdös (see [3] for some references). What concerns the case of allowing arbitrary phases, a breakthrough result, the existence of *ultraflat polynomials*, was obtained by Kahane [7]. A consequence of Kahane's results is that

$$par(\{1\}_0^{N-1}) \to 1\ , \qquad N \to \infty\ . \tag{5}$$

The more practical issue of actually finding a set of good if not optimal phases for a given sequence r is still widely unsolved. In engineering terms, the quality of a particular $P(t)$ is usually measured by a gain or loss factor in comparison to an appropriate reference value. Here, we define the loss factor

$$l(P) = 10 \cdot \log_{10}(par(P))\ , \tag{6}$$

in units decibel. Signal designs with $l(P) < 1\,\text{dB}$ are usually considered as good. For the case $r_k = 1$, a number of explicit constructions, mostly based on number-theoretical results, with small loss factors is known. For more general sequences r, there is a constructive formula proposed by Schroeder [12], Section 28.2, which often gives reasonable results but is far from a robust and reliable method. Most of the attempts reported in the literature (and suitable for general r and N) to solve the above minimization problems numerically are based on randomized search. Usually, no guarantees on efficiency and robustness can be given. We report on some results and numerical examples in this direction in Section 3.

My interest in this subject was triggered by S. Rangan, who explained to me the connections with practical issues arising in the design of a mobile cellular wireless OFDM system [11]. The author acknowledges numerous contributions by colleagues who helped to significantly improve the initial draft of this note.

§2. Worst-case estimates

In this section, let R_N be the set of all nonnegative sequences of length N. we deal with the asymptotic behavior of the quantities $\delta^N_{p,q} = \delta_{p,q}(R_N)$ from (3) if For this case, the quantities $\delta^N_{p,q} = \delta_{p,q}(R_N)$ have implicitly been studied before in the context of random orthogonal series (see Chapter 14 of [6], [9], and [8]) and for combinatorial problems [13]. Below, the constants c, C are independent of N but may depend on the other parameters, while $A_N \approx B_N$ stands for the two-sided inequality $cA_N \le B_N \le CA_N$ as $N \to \infty$.

Theorem 2. i) *For $1 \le p < \infty$ and $1 \le q \le \infty$ we have the upper estimate*

$$\delta^N_{p,q} \le CN^{\max(0,1/2-1/q)} , \quad N \ge 1 , \tag{7}$$

which is asymptotically correct, i.e., $\delta^N_{p,q} \approx N^{\max(0,1/2-1/q)}$ if either $2 \le p < \infty$ and $1 \le q \le \infty$ or $1 \le p < 2$ and $1 \le q \le 2$.

ii) *For $p = \infty$, we have*

$$\delta^N_{\infty,q} \begin{cases} = 1 & , q = 1 , \\ \approx (\log N)^{1/2} & , \ q = 2 , \\ \approx N^{1/2} & , q = \infty , \end{cases} \qquad N \to \infty . \tag{8}$$

Proof: The proof reveals the non-constructive nature of the upper estimates, and shows that they also hold for the above-mentioned optimal sign distribution case $\phi_k \in \{0, \pi\}$. Let

$$P_\epsilon(t) = \sum_{k=0}^{N-1} \epsilon_k r_k e^{ikt} , \quad \epsilon = \{\epsilon_k\}_0^{N-1} \in \{-1, 1\}^N .$$

If $1 \le p \le 2$ we can use

$$\|P_\epsilon\|_p \le \|P_\epsilon\|_2 = \|r\|_2 \le N^{\max(0,1/2-1/q)} \|r\|_q , \quad 1 \le q \le \infty .$$

which implies the estimate (7) for this parameter range. Looking at the example $r = \{1, 0, ..0\}$, we also see the equality

$$\delta_{p,q}^N = 1 \,, \quad N \geq 1 \,, \ \ 1 \leq p, q \leq 2 \,. \tag{9}$$

Let $2 < p < \infty$. It is enough to establish (7) for $q = 2$. Without loss of generality, assume that $\|r\|_2 = 1$. We need the following discrete version of Khinchine's inequality for the Rademacher system: Let $c = \{c_k\}_0^{N-1}$ be an arbitrary finite sequence of complex numbers. Then for any $y > 0$ we have

$$m_y(c) \equiv |\{\epsilon \in \{0,1\}^N \ : \ |\sum_{k=0}^{N-1} \epsilon_k c_k| > y \cdot \|c\|_2\}| \leq 2^{N+2} e^{-y^2/4} \,, \tag{10}$$

where $|E|$ denotes the number of elements of a finite set E. This very useful inequality follows from a more general result proved in Chapter 2, Theorem 5, of [9] by specializing to finite sections of the Rademacher system (since the c_k are complex, the constant in (10) is 2^{N+2} instead of 2^{N+1} for the real case).

We will apply (10) with the sequences $c^j = \{r_k e^{ikt_j^{4N}}\}_0^{N-1}$ of unit ℓ_2-norm for which $P_\epsilon(t_j^{4N}) = \sum_{k=0}^{N-1} \epsilon_k c_k^j$, $j = 0, \ldots, 4N-1$. Thus, using a well-known Marcinkiewicz type theorem [14], Chapter 10, we have

$$\begin{aligned}
\sum_{\epsilon \in \{0,1\}^N} \|P_\epsilon\|^p &\leq \frac{C}{4N} \sum_{j=0}^{4N-1} \sum_{\epsilon \in \{0,1\}^N} |P_\epsilon(t_j^{4N})|^p \\
&\leq \frac{C}{4N} \sum_{j=0}^{4N-1} \sum_{k=0}^{\infty} (k+1)^p |\{\epsilon \in \{0,1\}^N \ : |P_\epsilon(t_j^{4N})| \in [k, k+1]\}| \\
&\leq \frac{C}{4N} \sum_{j=0}^{4N-1} \sum_{k=0}^{\infty} (k+1)^p m_k(c^j) \leq C 2^N \sum_{k=0}^{\infty} (k+1)^p e^{-k^2/4} \,.
\end{aligned}$$

This gives

$$\min_{\epsilon \in \{0,1\}^N} \|P_\epsilon\|_p \leq (2^{-N} \sum_{\epsilon \in \{0,1\}^N} \|P_\epsilon\|_p^p)^{1/p} \leq C \,.$$

On the other hand,

$$\min_{\epsilon \in \{0,1\}^N} \|P_\epsilon\|_p \geq \min_{\epsilon \in \{0,1\}^N} \|P_\epsilon\|_2 = \|r\|_2 = 1$$

for $2 < p < \infty$ (recall that we have assumed $\|r\|_2 = 1$). Together, we arrive at

$$\min_{\epsilon \in \{0,1\}^N} \|P_\epsilon\|_p \approx \|r\|_2 \,, \qquad 2 < p < \infty \,,$$

from which asymptotic equality in (7), i.e., $\delta_{p,q}^N \approx N^{\max(0,1/2-1/q)}$, follows for $2 \leq p < \infty$ and all $1 \leq q \leq \infty$ through standard ℓ_q-norm comparisons.

This and (9) gives the asymptotic sharpness of the upper estimate (7) for the parameter ranges mentioned in the statement of Theorem 2 i).

The constant C in (7) depends on p and deteriorates if $p \to \infty$. Thus, (8) cannot be obtained by a simple limiting argument. For $q = 1$, (8) is trivial (use the triangle inequality for $\|P_\epsilon\|_\infty \leq \sum_{k=0}^{N-1} |r_k| = \|r\|_1$ and again the example $r = \{1, 0, \ldots, 0\}$). The upper bound for $q = 2$ follows from (10) by a similar argument (compare the proof of Theorem 10 in Chapter 4 of [9]). We use the same notation as before. If we choose $y = y_0$ such that $e^{-y_0^2/4} \leq 1/(32N)$ then the sets

$$E^j = \{\epsilon \in \{0,1\}^N \ : \ |P_\epsilon(t_j^{4N})| \leq y_0\} \ , \quad j = 0, \ldots, 4N-1 \ ,$$

have cardinality $\geq 2^N(1 - 1/(8N))$. Consequently, the intersection of all $4N$ sets E^j has cardinality $\geq 2^N(1 - 1/2) = 2^{N-1} \geq 1$. Thus, there exist sign sequences ϵ for which

$$\|P_\epsilon\|_\infty \leq C \max_{j=0,\ldots,4N-1} |P_\epsilon(t_j^{4N})| \leq C y_0 \ .$$

The first estimation step uses again the above-mentioned Marcinkiewizc type inequality. This gives the estimate since we can choose $y_0 \approx (\log N)^{1/2}$ to satisfy the requirements. A weaker version of the upper estimate related to the case of arbitrary ϕ_k is also contained in Chapter IV.14 of [1]. Complementing lower bounds are given below.

Next, the result for $p = q = \infty$ can be derived from a theorem by Spencer (see p.77 in [13]) which reads as follows: Given any collection of vectors $u^k \in \mathbb{R}^n$, $k = 0, \ldots, n-1$, there is a sequence of signs $\epsilon_k = \pm 1$ such that

$$\|\sum_{k=0}^{n-1} \epsilon_k u^k\|_\infty \leq K n^{1/2} \max_{k=0,\ldots,n-1} \|u^k\|_\infty \ , \tag{11}$$

where K is an absolute constant which is independent of $n \geq 1$. We apply this result with $n = 8N$, where the vectors u^k, $k = 0, \ldots, N-1$, are given componentwise by

$$u_j^k = r_k \cos(k t_j^{4N}) \ , \quad u_{4N+j}^k = r_k \sin(k t_j^{4N}) \ , \qquad j = 0, \ldots, 4N-1 \ .$$

For $k = N, \ldots, 8N-1$, we set $u^k = \{0\}$. We have

$$\|P_\epsilon\|_\infty \leq C \max_{j=0,\ldots,4N-1} |P_\epsilon(t_j^{4N})| \leq 2C \|\sum_{k=0}^{N-1} \epsilon_k u^k\|_\infty = 2C \|\sum_{k=0}^{n-1} \epsilon_k u^k\|_\infty \ ,$$

and $\max_{k=0,\ldots,n-1} \|u^k\|_\infty \leq \|r\|_\infty$. Thus, substitution into (11) yields the upper bound in (8) for $q = \infty$, the lower bound is provided by looking at the example $r = \{1, \ldots, 1\}$.

The lower bound in the case $q = 2$ of (8) follows from a different type of examples: lacunary polynomials. Let

$$Q_m(t) = \sum_{l=1}^{m} b_l e^{i2^{l-1}t} , \quad N \geq 2^{m-1} , \ m \geq 2 .$$

By well-known theorems about norms of functions with lacunary Fourier series (see, e.g., Chapter 15 of [6]) we have

$$\|Q_m\|_p \approx (\sum_{l=1}^{m} |b_l|^2)^{1/2} , \quad 1 \leq p < \infty , \qquad \|Q_m\|_\infty \approx \sum_{l=1}^{m} |b_l| .$$

The constants in this two-sided inequality do not depend neither on m nor on the coefficient sequence $\{b_l\}$ but may depend on p. Thus, if we consider any sequence with $|b_l| = 1$, then for any $N \geq 2$ we can choose $m \geq 2$ such that $2^{m-1} \leq N \leq 2^m$ and get

$$\|Q_m\|_p \approx m^{1/2}, \quad 1 \leq p < \infty , \qquad \|Q_m\|_\infty \approx m .$$

Since $m \approx \log N$, defining the sequence $r = \{r_k\}_0^{N-1}$ by $r_k = 1$, $k = 2^l$, $l = 0, \ldots, m-1$, and $r_k = 0$ otherwise, we obtain $\|r\|_q = m^{1/q} \approx (\log N)^{1/q}$, and $\min_{\phi_k} \|P\|_\infty \approx m \approx \log N$. This gives the nontrivial lower bound

$$\delta^N_{\infty,q} \geq C(\log N)^{1-1/q} , \quad 1 < q \leq 2 . \tag{12}$$

The case $q = 2$ provides the lower bound in (8). In particular, Corollary 1 is established. Note that the possible deterioration of par-values is relatively slow and associated mainly with very non-smooth, lacunary power spectra. □

Let us briefly discuss the cases where, to our knowledge, the exact asymptotic behavior is not yet known. We believe that (12) possesses a matching upper bound for $p = \infty, 1 < q < 2$ but were unable to prove this claim. From (12) and the upper estimate in (8) for $q = 2$ we only have

$$c(\log N)^{1-1/q} \leq \delta^N_{\infty,q} \leq C(\log N)^{1/2} , \quad N \geq 2 , \ 1 < q < 2 . \tag{13}$$

Analogously, we have

$$N^{1/2-1/q} \leq \delta^N_{\infty,q} \leq CN^{1/2-1/q}(\log N)^{1/2} , \quad N \geq 2 , \ 2 < q < \infty , \tag{14}$$

where the lower bound is covered by the example $r = \{1, \ldots, 1\}$ while the upper bound follows again from (8), $q = 2$, in conjunction with simple ℓ_q-estimates. We predict the lower bound to give the correct order.

The last case left open concerns the parameter region $2 < q \leq \infty$, $1 \leq p < 2$. Here, we have

$$c_\epsilon N^{1/2-1/q-\epsilon} \leq \delta^N_{p,q} \leq CN^{1/2-1/q} , \quad N \geq 1 , \tag{15}$$

for arbitrarily small $\epsilon > 0$. The upper bound, which we believe is the correct result for $N \to \infty$, comes from (7) while the lower bound is a consequence of Bourgain's result [4], Theorem 1, on the existence of high-density $\Lambda(r)$-sets for bounded orthonormal systems and $r > 2$, see Section 15.5 in [6] for some basic facts on $\Lambda(r)$-sets. Indeed, picking some $\alpha > 2$, we can conclude that for some $C_\alpha, c_\alpha > 0$ depending on α and all N, there exist sets $\Lambda_N \subset \{0, \ldots, N-1\}$ of cardinality $|\Lambda_N| \geq c_\alpha N^{2/\alpha}$ such that

$$\|Q\|_2 \leq \|Q\|_r \leq C_\alpha \|Q\|_1 \qquad \forall\, Q(t) = \sum_{k \in \Lambda_N} c_k e^{ikt} \, .$$

Set $r_k = 1$ if $k \in \Lambda_N$, and $r_k = 0$, otherwise. Then, $\|r\|_q = |\Lambda_N|^{1/q}$ and

$$\min_{\phi_k} \| \sum_{k \in \Lambda_N} e^{i\phi_k} r_k e^{ikt} \|_p \geq \min_{\phi_k} \| \sum_{k \in \Lambda_N} e^{i\phi_k} r_k e^{ikt} \|_1 \geq C_\alpha^{-1} |\Lambda_N|^{1/2} \, .$$

This gives $\delta_{p,q}^N \geq C_\alpha^{-1} |\Lambda_N|^{1/2-1/q} \geq C_\alpha^{-1} c_\alpha^{1/2-1/q} N^{(1/2-1/q)2/\alpha}$ for all parameters of interest. Finally, set $\alpha = 2\frac{1/2-1/q}{1/2-1/q-\epsilon} > 2$.

In summary, there is a wealth of known facts about the asymptotic behavior of the quantities $\delta_{p,q}^N$. However, a few details seem to be still unfinished, and we propose to close the gaps in (13), (14), and (15). From the point of view of communications applications, improved estimates for smaller sets R_N are of importance. It is also of interest to ask for one set of phases which reduces the peak power for a whole family R_N, i.e., to reverse the order of max and min in (3).

§3. Practical methods and examples

Let us start with revisiting Schroeder's heuristic formula (see formula (28.1) in [12]), which we write without loss of generality in the form

$$\phi_k = 2\pi \|r\|_2^{-2} \sum_{l=0}^{k-1} (k-l) r_l^2 \, , \quad k = 1, \ldots, N-1 \, , \qquad \phi_0 = 0 \, . \tag{16}$$

For $r_k = 1$, (16) essentially turns into $\phi_k = k^2\pi/N$, which is a special case of phase sets from the one-parameter family

$$\phi_k = a\frac{k^2\pi}{N} \, , \quad k = 1, \ldots, N-1 \, , \qquad \phi_0 = 0 \, , \qquad a \in \mathbb{R} \, , \tag{17}$$

investigated, e.g., in [10], [2], and related to Gauss sums. As was established in [2], by taking $a_N = (1 + N^{-1/4})^{-1})$ one obtains signals $P(t)$ with $par(P) \leq 1.1717...$ or $l(P) \leq 0.6882\,\mathrm{dB}$ as $N \to \infty$. Although Schroeder's formula gives slightly worse results ($l(P) \approx 1.3\,\mathrm{dB}$), it is considered a good ad hoc choice for smoothly varying power spectra. However, if $\{r_k\}_0^{N-1}$ becomes more lacunary, (16) may fail considerably. An example is provided by setting

$N = n^2$, $r_{ln} = 1$ for $l = 0, \ldots, n-1$, and $r_k = 0$ otherwise. Thus, $\|r\|_2 = n^{1/2}$, and from (16) we compute $\phi_{ln} = 2\pi n^{-1} \sum_{m=0}^{l} (ln - mn) = \pi l(l+1)$. This gives $P(t) = \sum_{l=0}^{n-1} e^{ilnt}$ and implies $\|P\|_\infty = n$ as well as

$$par(P) = n^{1/2} = N^{1/4} >> par(\{r_k\}_0^{N-1}) = par(\{1\}_0^{n-1}) \to 1 , \qquad N \to \infty .$$

It is an intriguing question whether a simple formula or deterministic method improving upon (16) can be found at all.

In our opinion, the par-problem might serve as an interesting benchmark for algorithms designed for non-smooth global optimization. It is experimentally known that $\|P\|_\infty$ as a function of $\phi = \{\phi_k\}_0^{N-1} \in [0, 2\pi)^N$ possesses an exponentially growing number of local minima, the majority of which gives values far off the global minimum. As to theory, two results would be desirable to obtain: the existence of an efficient method that gives $par(P) \to 1$ as $N \to \infty$ for the case $r_k = 1$, or one that gives par-values which are provably within a small constant factor of the optimal value for arbitrary $r = \{r_k\}_0^{N-1}$ and N. Although these questions might have simple negative answers in general, we believe that attacking them would lead to improvements for the range of practical applications ($N \leq 512$).

Our own contributions are very modest. To obtain good designs needed for the applications in [11], we evaluated the above described proposals for small N, both for the case $r_k = 1$ and for another sequence r associated with the Dolph-Chebyshev design [5] which is characterized by minimizing the absolute value of the autocorrelation function outside an interval of length 2δ around the origin $t = 0$. We also implemented a simple probabilistic hill-climbing algorithm which was based on the following principles:

- For a polynomial $P(t) \equiv P_\phi(t)$ as defined in (1), an approximation f_ϕ to $\|P_\phi\|_\infty$ is computed by DFT with $M = 2^m \geq 4N$ sampling points.
- Starting with a random ϕ^0, the current ϕ^l is updated as follows. For a random search direction $\beta \in (-1, 1]^N$, we set $\phi^{l+1} = \phi^l + h\beta$ if $f_{\phi^l + h\beta} < f_{\phi^l}$, or we set $\phi^{l+1} = \phi^l - h\beta$ if $f_{\phi^l - h\beta} < f_{\phi^l} \leq f_{\phi^l + h\beta}$, where $0 < h < \pi$ is fixed. If both comparisons fail, the step is repeated with a new β.
- If the number of repeated failures in the above update procedure exceeds a preset limit, the value of h is reduced resp. $M = 2^m$ is increased. The process is terminated if h and M reach certain minimal and maximal values $h_{\min}$ resp. $M_{\max}$.
- The basic iteration described above is repeated many times, with new random ϕ^0, only the best solution is recorded.

This unsophisticated algorithm was used to produce the values in the last column of Table 1 and the phases for the example of Table 2. Its efficiency decreases if N increases, more and more restarts were needed to get to small values for f_ϕ. Statistically, our experiments confirmed the observation made in the literature that only a small portion of the search space leads to P_ϕ with close-to-optimal par-value.

In both tables below, the reported par-value is obtained approximately by DFT with $M = 2^{15}$ sampling points. Table 1 shows par-values for $r_k = 1$

and various N obtained by the different methods. The last column shows the best result obtained from our random search algorithm, all others are based on (17), where $a = 1$ corresponds to Schroeder's formula, $a = a_N$ to the proposal from [2], while $a = a_{opt}$ was found by bisection (a_{opt} is shown first, followed by the par-value).

Table 1

N	$a = 1$	$a = a_N$	$a = a_{opt}$		Random Search
12	1.3846	1.4910	1.0748	1.3336	1.1647
16	1.3519	1.5000	0.9768	1.3298	1.1221
20	1.3608	1.5034	1.0244	1.3404	1.2135
32	1.3545	1.4871	0.9464	1.3339	1.2318
128	1.3487	1.3889	0.9344	1.3115	
512	1.3469	1.3243	0.9344	1.2801	
2048	1.3434	1.2785	0.9446	1.2523	

Table 2 contains more details for $N = 16$. It also shows results for the best sign distribution problem obtained by exhaustive search. The first two columns contain results for $r_k = 1$. It shows the phases ϕ_k normalized by 2π, optimal signs (+ corresponds to $\phi_k = 0$ and − to $\phi_k = \pi$), the corresponding $par(P)$- and $l(P)$-values can be found in the last two rows row. The other columns contain the corresponding information for the Dolph-Chebyshev design for $\delta = \pi/16$. The ϕ_k are again obtained by the random search algorithm. For comparison, Schroeder's formula leads to $par(P) = 1.3865$ and $l(P) = 1.4192$ in the last example.

Table 2

k	$\phi_k/(2\pi)$	signs	r_k	$\phi_k/(2\pi)$	signs
0	0.000000	+	0.262372	0.000000	−
1	0.166458	+	0.198243	0.102677	+
2	0.004228	+	0.219716	0.145050	+
3	0.138135	−	0.238323	0.133352	−
4	0.034767	+	0.253700	0.383789	−
5	0.155074	+	0.265533	0.975175	+
6	0.469273	+	0.273567	0.158599	−
7	0.811796	−	0.277628	0.551148	+
8	0.701856	−	0.277628	0.857623	−
9	0.133372	−	0.273567	0.712594	−
10	0.618992	+	0.265533	0.113037	−
11	0.281630	−	0.253700	0.665970	−
12	0.211549	+	0.238323	0.409246	−
13	0.845512	+	0.219716	0.263537	−
14	0.791936	−	0.198243	0.063477	+
15	0.384692	+	0.262372	0.844039	+
par	1.1221	1.3085	1.1644		1.3031
$l(P)$	0.5003	1.1677	0.6610		1.1498

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