approximately divergence-free Q1-rotated vector fields. Clearly, this claim needs verification since the results presented here are asymptotic and only serve the case of uniform dyadic refinement. In practice, one deals with the range $J \leq 15$, spaces are defined on bounded domains and have to include essential boundary conditions. Also, the constants in the asymptotic results become of interest. In this respect, the natural value $t = 3/2$ seems to lead to slightly better performance than $t \approx 0.992$ which corresponds to the optimal Λ.

Table 1

t	0.0	0.5	1.0	1.5	2.0
	4.0000 (10)	4.0000 (3)	4.0000 (3)	4.0000 (3)	4.0000 (3)
	2.0000 (18)	2.6667 (4)	2.0000 (6)	*2.0000* (2)	*2.0000* (2)
λ	**1.0000** (34)	*2.0000* (2)	**1.0076** (1)	1.6000 (4)	1.3333 (4)
		1.7778 (3)	**1.0000** (20)	**1.0362** (1)	**1.0644** (1)
		1.3333 (8)	0.8051 (1)	**1.0000** (9)	**1.0000** (9)
		1.0623 (1)	**0.6289** (1)	0.8000 (8)	0.7486 (1)
Λ	4.0000	1.7778	1.0076	1.0362	1.0644

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or *averaging* as in the scalar case. The latter leads to (compare [3])

$$y^1_{2\alpha} = x^1_\alpha + \theta^1_\alpha/8\,, \quad y^1_{2\alpha+e^1} = x^1_\alpha - \theta^1_\alpha/8, \tag{23}$$

where $\theta^1_\alpha = x^4_\alpha + x^4_{\alpha-e^2} - x^4_{\alpha+e^1} - x^4_{\alpha+e^1-e^2}$, analogously for $l = 2,3,4$. The remaining values y^l_β corresponding to edges in $\mathcal{R}_1$ interior to squares of the coarser partition $\mathcal{R}_0$ have been obtained by satisfying (20) (under the assumption $\boldsymbol{u} \in \boldsymbol{Z}_0$) and minimizing the discrete expression for the $(H^1)^2$-seminorm of $\hat{S}\boldsymbol{u}$, with various values of the parameter t. Note that this reduces locally to an 8-dimensional constrained minimization problem, with a rank-deficient set of constraints (solvability is guaranteed only if (20) is satisfied). In general, the constraints are satisfied in a least-squares sense. Lack of space prevents us from presenting the detailed formula.

As explained in [5], it is easy to determine a relatively low-dimensional space $\hat{\mathcal{K}}$ of Hermitean matrix functions with polynomial entries with given spectra of Fourier coefficients such that it contains the Krylov space $\mathcal{K}$ of interest. Choosing the natural basis in $\hat{\mathcal{K}}$, the matrix representation $\hat{\mathcal{L}}$ for the operator $\mathcal{L}_{S^*,S}|_{\hat{\mathcal{K}}}$ can be computed. Solving the complete eigenvalue problem for $\hat{\mathcal{L}}$ and for $\hat{\mathcal{L}}^T$, one is able to find all λ_r and β_r of interest (see Section 2 for the notation). Our Matlab implementation (for details, see [5]) monitors both conditioning of the eigenvalues and rank of eigenspaces in the case of multiple eigenvalues (such safety measures become necessary, since in the computations below we have $\dim \hat{\mathcal{K}} > 400$). To verify (17), we find a $L \times M$ polynomial matrix function $H(\theta)$ and a scalar polynomial $\Delta(\theta)$ such that $G(\theta)H(\theta) = \Delta(\theta) \cdot Id_M$ and set $R(\theta) = \Delta(\theta) \cdot Id_L - H(\theta)G(\theta)$. Then $R^*(\theta)B_r(\theta)R(\theta) \neq 0$ is sufficient for (17) to hold. In our example, since $M = 1$ we can take $H(\theta) = G^*(\theta)$ and obtain $\Delta(\theta) = 4(\sin^2\theta_1/2 + \sin^2\theta_2/2)$. Again, this is simple linear algebra and easy to implement.

Table 1 shows, for various t, the computed leading eigenvalues of $\hat{\mathcal{L}}$ for the prolongation operators based on (23) and local energy minimization for some avlues of t. Their algebraic multiplicity (which coincided with the dimension of the associated eigenspace in all cases) is shown in parentheses. Bold-faced λ indicates that not all associated β_r have vanished. We found that the sufficient condition for (17) was always satisfied for at least one associated $B_r(\theta)$, except for a few λ set in italics. The last row of Table 1 shows the computed value of Λ (according to the above, Λ equals the largest bold-faced λ). The minimal $\Lambda = 1.0074$ was obtained for $t \approx 0.992$, and not for $t = 3/2$. Analogous testing of the $(L_2)^2$-norm behavior of $\tilde{P}_{j,J}$ lead to the optimal value $\Lambda = 1$ for all t. The replacement of (23) by the simpler formula (22) produced significantly worse results (here, $\Lambda \geq 1.8006$ was observed for the same range of t).

Note that values $\Lambda < 1.05$ are quite acceptable from a more practical point of view. This holds, according to our tests, for all $0.95 < t < 1.7$, and hints at a certain robustness of the construction principle. Thus, a generalization to quadrilateral partitions seems to be promising and could lead to a new well-performing (even though not asymptotically optimal) algorithm for the linear system associated with the $\boldsymbol{Z}_j$-discretization of the Stokes problem by

Fig. 1. Supports and edge average vectors for $\boldsymbol{\phi}^l$ and $\boldsymbol{\psi}^l$.

This defines the constraint operators $\hat{G}$ resp. G (here $M = 1$). We note that $\boldsymbol{Z}_0$ possesses an explicit basis generated from the integer shifts of 3 locally supported functions $\boldsymbol{\psi}^1 = \boldsymbol{\phi}^1$, $\boldsymbol{\psi}^2 = \boldsymbol{\phi}^3$, and a so-called vertex function $\boldsymbol{\psi}^3$ (see Figure 1). Since it is not $L_2(\mathbb{R}^2)^2$-stable, we will not use it further.

The energy norm to control is a discrete $(H^1)^2$-seminorm (for discontinuous $\boldsymbol{u} \in \boldsymbol{V}_0$, $|\boldsymbol{u}|^2_{H^1(\mathbb{R}^2)^2}$ is defined by summing up the element-wise well-defined H^1-seminorms of its components). We have (see [3,5] for all needed formulae)

$$|u_1|^2_{H^1(\mathbb{R}^2)} \approx \sum_{\alpha \in \mathbb{Z}^2} (x^1_\alpha - x^1_{\alpha+e^2})^2 + (x^4_\alpha - x^4_{\alpha+e^1})^2 + t(x^1_\alpha + x^1_{\alpha+e^2} - x^4_\alpha - x^4_{\alpha+e^1})^2,$$

analogously for the second component u_2 of $\boldsymbol{u}$. Here, $\approx$ can be replaced by $=$ if $t = 3/2$. In the computations below, we have used this value to determine the norm operator $\hat{E}$ resp. E (with $N = 3$) and the entries of the matrix function $B(\theta)$. Obviously, in this example we have $s_0 = 1$ while $s_1 = 1$ results from the anticipated inverse inequality (7). Consequently, $s_2 = 0$.

Finally, we need to describe the prolongations, i.e., $\hat{P}_1$ resp. $\hat{S}$. Prolongations with uniformly bounded H^1-norm growth for the scalar case, i.e., for $\{W_j\}$, are based on natural averaging procedures and described in [3,5]. However, they do not preserve property (20)-(21) when applied to vector fields componentwise. We describe one approach of constructing prolongations suitable for the vector case. According to the abstract theory from [1] applied to the Stokes discretizations under consideration, a desirable property of $\hat{P}_1$ is that it preserves the average of the vector fields on the edges of $\mathcal{R}_0$. I.e., if the coefficients of $\hat{S}\boldsymbol{u}$ w.r.t. $\boldsymbol{\Phi}_0$ are denoted by $\{y^l_\beta\}$ we require that for all $\alpha \in \mathbb{Z}^2$

$$y^l_{2\alpha} + y^l_{2\alpha+e^1} = 2x^l_\alpha,\ l = 1,2\ , \quad y^l_{2\alpha} + y^l_{2\alpha+e^2} = 2x^l_\alpha,\ l = 3,4.$$

There are two natural ways to achieve this: *trivial extension*

$$y^l_{2\alpha} = y^l_{2\alpha+e^1} = x^l_\alpha,\ l = 1,2\ , \quad y^l_{2\alpha} = y^l_{2\alpha+e^2} = x^l_\alpha,\ l = 3,4, \qquad (22)$$

Theorem. *Under the above conditions, we have*

$$\hat{c}_k \approx \Lambda^k, \qquad k \to \infty. \tag{18}$$

If diagonalizability is not assumed, an additional factor k^s may be necessary in (18), with $s > 0$ depending on the Jordan blocks associated with the leading λ_r, on B, and on Z in a more complicated way. Such a case has never appeared in any of our applications. The determination of Λ is now a problem of numerical linear algebra. In our examples, direct eigenvalue solvers can be used. For more details, see [5] and the next section.

§3. Example

Consider the Stokes problem

$$-\Delta \boldsymbol{u} + \nabla p = \mathbf{f}\,, \qquad \nabla \cdot \boldsymbol{u} = \mathbf{0}\,, \tag{19}$$

for the velocity field $\boldsymbol{u} \in \boldsymbol{V} \equiv (H_0^1(\Omega))^2$ and the pressure $p \in L_2(\Omega)\backslash\{const.\}$ in a two-dimensional domain Ω. If Ω can be represented as a union of rectangles (or, more general, quadrilaterals), a low-order discretization by nonconforming rotated Q1 finite elements for the velocity and P0 pressure elements has been introduced in [7]. We describe only the definition of the resulting spaces $\boldsymbol{Z}_j$ corresponding to the shift-dilation invariant setting of the previous section (i.e., the associated spaces of approximately divergence-free rotated Q1 vector fields on nested uniform partitions $\mathcal{R}_j$ of $\mathbb{R}^2$ into squares of side-length 2^{-j}, $j \geq 0$). A vector field $\boldsymbol{u} = (u_1, u_2) \in H \equiv L_2(\mathbb{R}^2)^2$ belongs to $\boldsymbol{Z}_j$ if it is componentwise in the space W_j of scalar rotated Q1 elements and satisfies the discrete divergence-free condition

$$\int_R \nabla u \, dx = 0 \qquad \forall R \in \mathcal{R}_j. \tag{20}$$

A function $u \in L_2(\mathbb{R}^2)$ belongs to W_j if its restriction to any $R \in \mathcal{R}_j$ belongs to $\operatorname{span}\{1, x_1, x_2, x_1^2 - x_2^2\}$, and if the average value of u along any edge e associated with $\mathcal{R}_j$ is the same when taken from either side of that e. The local interpolation problem for W_j is defined by prescribing *edge averages*, compare [7] for alternatives. Accordingly, functions in $\boldsymbol{Z}_j \subset \boldsymbol{V}_j \equiv W_j \times W_j$ are uniquely determined from their *edge average vectors*. Obviously, $\boldsymbol{V}_0$ will then be generated by $L = 4$ functions $\boldsymbol{\phi}^l$. Their supports and defining nonzero edge average vectors (of unit length) are schematically depicted in Figure 1. Thus, an arbitrary $\boldsymbol{u} \in \boldsymbol{V}_0$ is uniquely representable as

$$\boldsymbol{u} = \sum_{l=1}^{4} \sum_{\alpha \in \mathbb{Z}^2} x_\alpha^l T_\alpha \boldsymbol{\phi}^l\,, \quad \|\boldsymbol{u}\|^2_{L_2(\mathbb{R}^2)^2} \approx \sum_{l=1}^{4} \sum_{\alpha \in \mathbb{Z}^2} (x_\alpha^l)^2.$$

From (20), it follows by Green's Theorem that $\boldsymbol{u} \in \boldsymbol{Z}_0$ is equivalent to

$$x_\alpha^4 + x_\alpha^2 - x_{\alpha+e^1}^4 - x_{\alpha+e^2}^2 = 0 \qquad \forall \alpha \in \mathbb{Z}^2 \quad (e^1 = (1,0), e^2 = (0,1)). \tag{21}$$

Throughout the paper, we assume that the entries of all these matrices are polynomials (this corresponds to situations where norms, constraints, and prolongations are computed by local schemes). For breviety, we call this *compact support assumption.*

We will study the constant $\hat{c}_k$ in

$$\|ES^k y\|^2_{\ell_2(\mathbb{Z}^d)^M} \le \hat{c}_k \|y\|^2_{\ell_2(\mathbb{Z}^d)^L}, \qquad y \in Z \equiv \operatorname{Ker} G \subset \ell_2(\mathbb{Z}^d)^L,\ k>0. \quad (12)$$

One easily detects the connection of (12) with (6) if the following assumptions are accepted. Suppose that two-sided inequalities

$$\begin{aligned} b_j(u_j,u_j) &\approx 2^{2s_1 j}\|u_j\|_H^2 \approx 2^{2s_1 j}\|x_j\|^2_{\ell_2(\mathbb{Z}^d)^L}, \\ a_j(u_j,u_j) &\approx 2^{2s_2 j} a_0(D_{2^{-j}}u_j, D_{2^{-j}}u_j) \approx 2^{2s_2 j}\|\hat{E}D_{2^{-j}}u_j\|^2_{\ell_2(\mathbb{Z}^d)^M}. \end{aligned} \quad (13)$$

hold for some s_1 and $s_2 = s_1 - s_0$, uniformly in $u_j \in V_j$ and $j \ge 0$. Define $\hat{P}_j : V_{j-1} \to V_j$ by $\hat{P}_j u_{j-1} = D_{2^{j-1}}\hat{P}_1 D_{2^{-j+1}} u_{j-1}$, and introduce the subspaces $Z_j = \{u_j \in V_j : \hat{G}D_{2^{-j}}u_j = 0 \ (\iff Gx_j = 0)\}$. We assume that $\hat{S}$ maps Z_0 into Z_0 which implies that $\hat{P}_j$ maps Z_{j-1} into Z_j for all $j>0$. Then, since

$$\tilde{P}_{j,J} = \hat{P}_J \ldots \hat{P}_{j+1} = D_{2^J}(D_{1/2}\hat{P}_1 \ldots D_{1/2}\hat{P}_1)D_{2^{-j}} = D_{2^J}\hat{S}^{J-j}D_{2^{-j}},$$

from (12)-(13) we have for arbitrary $0 \le j < J < \infty$ that

$$a_J(\tilde{P}_{j,J}u_j, \tilde{P}_{j,J}u_j) \le C 2^{2s_2(J-j)}\hat{c}_{J-j} b_j(u_j,u_j) \quad \forall\, u_j \in Z_j. \quad (14)$$

Note that without the restriction to Z (i.e., as a problem on $\ell_2(\mathbb{Z}^d)^L$ which then would be directly related to (6)), (12) was studied in [5]. The key observation is that, when going to Fourier series, we have

$$\|ES^k z\|^2_{\ell_2(\mathbb{Z}^d)^M} = ((\mathcal{L}^k_{S^*,S}B)(\theta)y(\theta), y(\theta))_{L_2(\mathbb{T}^d)^L}, \quad y(\theta) \in L_2(\mathbb{T}^d)^L, \quad (15)$$

where $B(\theta) = E^*(\theta)E(\theta)$, and

$$(\mathcal{L}_{S^*,S}X)(\theta) = 2^{-d}\sum_{e\in\{0,\pi\}^d} S^*(\frac{\theta}{2}+e)X(\frac{\theta}{2}+e)S(\frac{\theta}{2}+e) \quad (16)$$

is the matrix transfer operator which was mentioned in section 1. The operator (16) is well-defined on the set $\mathcal{H}$ of $L \times L$ non-negative definite Hermitean matrix functions $X(\theta)$ with polynomial entries. Such transfer operators play a central role in characterizing the regularity of multiwavelets (see [6] for a recent survey). Due to the compact support assumption, the Krylov space $\mathcal{K} \subset \mathcal{H}$ generated from B by repeatedly applying $\mathcal{L}_{S^*,S}$ is finite-dimensional. Let $\{\lambda_r\}$, $\{B_r(\theta)\}$ be a complete set of eigenvalues and eigen'matrices' for $\mathcal{L}_{S^*,S}|_{\mathcal{K}}$ (i.e., we assume diagonalizability). Write $B(\theta) = \sum_r \beta_r B_r(\theta)$ and choose Λ as the maximum of all $|\lambda_r|$ such that $\beta_r \ne 0$ and that

$$\exists z \in Z \qquad (B_r(\theta)z(\theta), z(\theta))_{L_2(\mathbb{T}^d)^L} \ne 0. \quad (17)$$

Condition (17) is the only place where the constraints need to be taken into account. This derivation implies the following result which, in combination with (14), yields the desired sharp estimates for the norm behavior of $\tilde{P}_{j,J}$.

the spaces V_0 associated with non-conforming schemes will necessarily consist of the integer shifts of $L > 1$ *scaling functions* $\phi^1, \ldots, \phi^L$. Representing the operator $D_{1/2}P_1 : V_0 \to V_0$ in this basis (the dilation D_t, $t > 0$ is defined by $D_t u(x) = u(tx)$), it becomes a matrix subdivision operator with compactly supported masks. For details we refer to [5]. In this note, we wish to demonstrate the same approach for investigating multigrid prolongations acting between spaces of approximately divergence-free finite element vector fields. The difference with [5] lies in the incorporation of constraints. In section 2, the necessary notation is introduced, and the connection between norm estimates for iterated prolongations and the above mentioned eigenvalue problem is derived. The theory is applied to the low-order rotated Q1-Stokes element in section 3. We propose prolongation operators that preserve the discrete divergence-free constraints and lead to nearly optimal asymptotic estimates in (6).

§2. Theory

Let H be a shift- and dilation-invariant Hilbert function space on $\mathbb{R}^d$ with shift-invariant norm $\|T_s u\|_H = \|u\|_H$, where $T_s u(x) = u(x-s)$, $s \in \mathbb{R}^d$. Moreover, $\|D_t u\|_H = t^{-s_0}\|u\|_H$ holds for some fixed $s_0 \in \mathbb{R}$. Given $\phi^1, \ldots, \phi^L \in H$, assume that the system

$$\Phi_j = \{\phi^l_{j,\alpha} \equiv 2^{js_0} D_{2^j} T_\alpha \phi^l,\ \alpha \in \mathbb{Z}^d,\ l = 1, \ldots, L\} \tag{9}$$

forms a Riesz basis in its closed linear hull $V_j \subset H$, $j \geq 0$. By definition, $2^{js_0} D_{2^j} : V_0 \to V_j$ is an isometry, and the Riesz basis property of Φ_j yields an isomorphism between V_j and $\ell_2(\mathbb{Z}^d)^L$. Thus, operators acting on V_j induce equivalent operators acting on $\ell_2(\mathbb{Z}^d)^L$ (or, after going to Fourier series, on $L_2(\mathbb{T}^d)^L$). We need to introduce three basic operators: a *constraint operator* $\hat{G} : V_0 \to \ell_2(\mathbb{Z}^d)^M$, a *norm operator* $\hat{E} : V_0 \to \ell_2(\mathbb{Z}^d)^N$, and a *prolongation operator* $\hat{P}_1 : V_0 \to V_1$ which all are assumed to be invariant w.r.t. $\mathbb{Z}^d$-shifts. In particular, $\hat{P}_1 T_\alpha v_0 = T_\alpha \hat{P}_1 v_0$ for all $\alpha \in \mathbb{Z}^d$ and $v_0 \in V_0$. If we use the isomorphism between V_0 and $\ell_2(\mathbb{Z}^d)^L$, then the operators $\hat{G}$, $\hat{E}$, and $\hat{S} = D_{1/2}\hat{P}_1$ turn into bounded linear matrix operators between $\ell_2(\mathbb{Z}^d)^k$ spaces ($k = L, M, N$) denoted by G, E, and S, resp. By the above shift-invariance assumptions, S corresponds to a dyadic matrix subdivision scheme, i.e.,

$$(Sy)^k_\beta = \sum_{l=1}^{L} \sum_\alpha s^{kl}_{\beta-2\alpha} y^l_\alpha,\ \beta \in \mathbb{Z}^d,\ k = 1, \ldots, L,\ y \in \ell_2(\mathbb{Z}^d)^L, \tag{10}$$

while G, E are Toeplitz. In terms of Fourier series,

$$y \in \ell_2(\mathbb{Z}^d)^L \ \longleftrightarrow\ y(\theta) \equiv (y^l(\theta) = \sum_\alpha y^l_\alpha e^{-i\alpha\theta},\ l = 1, \ldots, L)^T \in L_2(\mathbb{T}^d)^L,$$

these operators become multiplication operators characterized by matrix functions $G(\theta) = (g^{ml}(\theta))$, $E(\theta) = (e^{nl}(\theta))$, and $S(\theta) = (s^{kl}(\theta))$:

$$(Gy)(\theta) = G(\theta)y(\theta), \quad (Ey)(\theta) = E(\theta)y(\theta), \quad (Sy)(\theta) = S(\theta)y(2\theta). \tag{11}$$

The matrices P_j resp. R_j represent the prolongation $P_j : V_{j-1} \to V_{j-1}$ and restriction operators $R_j : V_j \to V_{j-1}$. For simplicity, we restrict ourselves to the symmetric case where $A = A^T$, $S_j = S_j^T$, and $R_j = P_j^T$. In finite element applications, the matrix representations of all these operators are extremely sparse due to the local support property of the standard bases in finite element spaces.

The convergence properties of the method (4)-(5) solely depend on the choices of S_j and P_j. A particularly important design problem occurs if $V_{j-1} \not\subset V_j$ (this is called *nonnestedness* in multigrid theory). While under the assumption $V_{j-1} \subset V_j$ one naturally accepts the natural embedding as P_j, for non-nested spaces the choice of the prolongations P_j becomes nontrivial. An application where nonnestedness occurs are discretizations of elliptic boundary problems by nonconforming finite elements. Mixed finite element discretizations of the Stokes system lead to nonnested sequences of approximately divergence-free finite element spaces. Previous work [1] indicates that for nonnested spaces the energy norm behavior of the *iterated prolongations* $\tilde{P}_{j,J} \equiv P_J \dots P_{j+1} : V_j \to V_J$ is crucial for the convergence rate of (4)-(5). More precisely, the constants $c_{j,J}$ in

$$a_J(\tilde{P}_{j,J}u_j, \tilde{P}_{j,J}u_j) \le c_{j,J} b_j(u_j,u_j) \qquad \forall\, u_j \in V_j, \quad j < J, \tag{6}$$

need to be controlled. In practice, the forms a_j represent discrete versions of Sobolev norms and the forms $b_j(u_j,u_j) \equiv \omega_j(S_j^{-1}x_j, x_j)$, $u_j \in V_j$, are related to scaled L_2-norms. Choosing appropriate scaling factors ω_j, we may assume that an inverse inequality

$$a_j(u_j,u_j) \le b_j(u_j,u_j) \qquad \forall\, u_j \in V_j, \quad j \ge 0, \tag{7}$$

is satisfied. Although estimates for the prolongations

$$a_j(P_j u_{j-1}, P_j u_{j-1}) \le c_j a_{j-1}(u_{j-1},u_{j-1}) \qquad \forall\, u_{j-1} \in V_{j-1}, \quad j \ge 1, \tag{8}$$

are easy to obtain and imply $c_{j,J} \le c_{j+1} \dots c_J$, $j < J$, this trivial upper estimate is usually a crude overestimation. Proofs of sharper, optimal estimates in (6) have been obtained so far only for two particular cases of nonconforming finite element schemes [2,3]. More details and references on multigrid theory with nonnested spaces and numerical examples for the behavior of the $c_{j,J}$ for several nonconforming finite element schemes can be found in [1,4].

Recently, in [5], we have shown that under simplifying assumptions, for sequences of generic finite element spaces on dyadic sequences of uniform partitions of $\mathbb{R}^d$ and homogeneous norms, with the strong assumption of *invariance w.r.t. dyadic shifts and dilations* for all spaces and operators, asymptotically exact estimates for the constants in (7) can be derived from solving a finite-dimensional eigenvalue problem for an associated matrix transfer operator. Similar reductions have shown their usefulness for estimating the regularity of multiwavelets [6]. The analogy comes from the following observation: The shift-dilation-invariant setting assumed, the standard basis for

Multigrid prolongations and matrix subdivision

Peter Oswald

Abstract. Attention is drawn to the connection between theoretical properties of prolongation operators in multigrid methods, on the one hand, and the regularity problem for solutions of matrix subdivision schemes, on the other. Both problems can be reduced to spectral properties of an associated matrix transfer operator. An application to multilevel preconditioners for the Stokes problem is discussed.

§1. Introduction

We consider multigrid methods for iteratively solving linear systems

$$Ax = f \tag{1}$$

arising from a variational problem

$$find \quad u \in V \ : \quad a(u,v) = F(v) \qquad \forall v \in V, \tag{2}$$

in a finite-dimensional real Hilbert space V. Roughly speaking, multigrid methods are based on embedding (2) into a sequence of analogous problems

$$find \quad u_j \in V_j \ : \quad a_j(u_j,v_j) = F_j(v_j) \qquad \forall v_j \in V_j, \tag{3}$$

on real Hilbert spaces V_j, $j = 0,\ldots,J$ ($V \equiv V_J$, $a \equiv a_J$, etc.), of increasing dimension, and constructing an iterative method for solving (1) by combining approximate solvers (smoothers) S_j for the problems (3) and prolongation/restriction operations between different V_j. In the simplest version (multilevel preconditioning), the resulting iterative method takes the form

$$x^{(n+1)} = x^{(n)} - \omega C(Ax^{(n)} - f), \quad n = 0,1,\ldots, \tag{4}$$

where $x^{(0)}$ is an arbitrary starting vector, $\omega \neq 0$ a relaxation parameter, and the multilevel preconditioner $C = C_J$ is recursively defined as

$$C_0 = S_0, \qquad C_j = P_j C_{j-1} R_j + S_j, \quad j = 1,\ldots,J. \tag{5}$$

Approximation Theory IX
Charles K. Chui and Larry L. Schumaker (eds.), pp. 1–8.

ISBN 1-xxxxx-xxx-x.