Multilevel Solvers for Elliptic Problems on Domains

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Abstract. We study to which extent the geometric multilevel approach based on dyadic scales of shift-invariant subspaces on $\mathbb{R}^d$ can be used to produce accurate discrete solutions of elliptic boundary value problems of positive order on non-rectangular domains. We also deal with the construction of optimal preconditioners including the case of nested refinement. Sufficient geometric conditions on a domain are given such that a robust and asymptotically optimal algorithm can be expected. In contrast to other approaches which emphasize biorthogonal wavelet decompositions we are satisfied with a simpler frame concept which incorporates recent experience with finite element multilevel solvers.

§1 Introduction

For the numerical solution of elliptic boundary value problems for partial differential equations, multilevel methods have gained popularity over the last decade. This is mainly due to their nearly optimal complexity for a number of model problems. In many practical cases, they are based on multiresolution scale of nested finite-dimensional subspaces

$$V_0 \subset V_1 \subset \ldots \subset V_j \subset \ldots \tag{1.1}$$

of a Hilbert space V serving as the energy space for the given variational problem. The scale (1.1) is used to produce stable subspace splittings

$$V_J = V_0 + V_1 + \ldots + V_J \tag{1.2}$$

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and to design fast iterative solvers related to such splittings for the discretized variational problem associated with a computational discretization space V_J (or a properly defined subspace $V_J^* \subset V_J$). For $2m$-th order elliptic boundary value problems in Sobolev spaces, considerable progress has been achieved in the theoretical understanding of multilevel and multigrid methods as well as of other subspace correction methods for finite element discretizations (cf. [65, 67, 11, 60]). We survey some of these results in Section 2.

The underlying theory also applies to various wavelet discretizations, see, *e.g.* , [26, 44, 45, 29] for some papers that deal with wavelet solvers for elliptic problems and are related to our approach. Roughly speaking, in these algorithms suitable "detail spaces" $W_j \subset V_j$ are constructed, together with their algebraic bases, such that $V_j = V_{j-1} \dot{+} W_j$ provides a stable splitting of V_j into "low frequency" (V_{j-1}) and "high frequency" (W_j) parts. Using this two-level decomposition recursively, (1.2) is replaced by

$$V_J = V_0 \dot{+} W_1 \dot{+} \ldots \dot{+} W_J \; . \tag{1.3}$$

One can use this splitting indirectly (*i.e.* , the original problem is discretized with respect to a standard basis in V_J, and the wavelet decomposition behind (1.3) implicitly defines the structure of the multilevel preconditioner) or directly. In the latter case, the discretization is performed with respect to the wavelet basis and is automatically well-conditioned. To achieve asymptotically optimal work estimates one has to use compression arguments. For elliptic problems of order $2m$, and V_j with locally supported basis functions, the first approach is often preferred since the V_J-discretization matrix is automatically sparse and available from standard, engineering codes. The direct use of the wavelet decomposition (1.3) is promising for those situations in which the V_J-discretization is not a priori sparse, *e.g.* , for integral equations.

In both cases, the explicit introduction of detail spaces W_j is the crucial step, and may add some theoretical and practical difficulties. For example, most of the popular examples of wavelet spaces (see [30, 16]) are derived in a one-dimensional, shift-invariant setting on $\mathbb{R}$. Multivariate examples on $\mathbb{R}^d$ are mostly obtained by tensor-product techniques. Adaptions to bounded intervals and domains have been studied in, *e.g.* , [2, 17, 19, 18]. However, up to now there is no comprehensive study of the practical potential of discretizations using multilevel structures based on shift-invariance and dyadic dilation (modulo boundary modifications) in the case of general, non-rectangular geometries. It is not completely clear to the author what will be left from the powerful wavelet machinery if the basic algebraic assumptions (invariance with respect to integer shifts and (dyadic) dilation) are significantly relaxed.

In our opinion, the departure from these assumptions is unavoidable for many engineering problems. A simple example are $2m$-th order elliptic PDE's with rapidly varying coefficients that exhibit a large ratio of ellipticity constants, and are therefore far from a generic H^m-problem. Principal difficulties are to be expected if non-symmetric equations, with dominant low order hyperbolic parts, such as convection-diffusion equations, or nonlinear problems are to be studied. In our opinion, this robustness aspect is one of the target problems for future investigation, *i.e.* , the adaption of the multilevel concept to a class of operator equations (or even to an individual equation) still remains a decisive issue for practical implementations and engineering applications. An indicator of this tendency are analogous efforts within the FEM community and the renewed interest in the algebraic multigrid method and related algorithms.

There is another observation which damps the expectations concerning the practical use of wavelet solvers even for standard symmetric elliptic boundary value problems. Numerical experiments [39, 49] show that for generic H^1-problems, *i.e.* , for second order elliptic equations, wavelet and prewavelet discretizations perform slightly worse than traditional finite element preconditioners associated with (1.2). While condition numbers of L_2-problems (and sometimes also of H^s-problems with negative s) are usually improved and become uniformly bounded if $J \to \infty$, the preconditioning effect in the H^1-norm is reduced by a constant factor. This poses the problem of determining more carefully problem classes in which the use of wavelet preconditioners based on (1.3) is justified compared to simpler methods based on the use of scaling functions only and related to (1.2). Also, one may argue whether new wavelet families (Daubechies orthogonal wavelets [30], AFIF elements [51] etc.) are generally useful, and are able to compete with traditional finite element and spline constructions from this more practical viewpoint.

The reader should not expect an answer to these more philosophical questions. In Sections 3-5, we concentrate on studying the influence of the domain geometry on the optimality of multilevel preconditioners resulting from a standard multiresolution analysis (these sections are sometimes rather technical and represent the original part of this paper). The concept is based on sequences $\{\tilde{V}_j\}$ of subspaces of a fixed sequence $\{V_j\}$ of subspaces in some $H^s(\mathbb{R}^d)$ which "live" on a uniform structure generated by shift-invariance principles and dyadic dilation as usual. These auxiliary subspaces $\tilde{V}_j$ are spans of scaling functions (or, in the finite element terminology, nodal basis functions) of levels $\leq j$ such that a canonical $H^s(\mathbb{R}^d)$-elliptic Galerkin discretization can be solved efficiently, e.g., by preconditioned iterative methods, with the preconditioner inherited from the generating system, or frame, consisting of scaling functions (see Section 4). Thus, we essentially stay with splittings of the type (1.2).

The connection of the auxiliary problems in $\tilde{V}_j$ with the originally given $H^s(\Omega)$-elliptic problem, with natural boundary conditions, on a generic bounded domain $\Omega \subset \mathbb{R}^d$ is established in Section 3 where sufficiently rich subspaces $V_{j,\Omega} \subset V_j|_\Omega$ will be constructed. The construction consists of a local boundary modification which is similar to constructions outlined in other papers on wavelets on intervals and domains, too, but is relatively simple. A drawback is that, in contrast to $\{\tilde{V}_j\}$, the sequence $V_{j,\Omega}$ is not monotone which requires additional considerations when adaptivity is an issue. Information between $V_{j,\Omega}$ and $\tilde{V}_j$ is exchanged by appropriate restriction (R_j) and extension (E_j) operators. To obtain uniform condition number and work estimates, certain geometric conditions on Ω arise in a natural way. Asymptotically, they hold for domains with boundaries satisfying a uniform Lipschitz condition. For details, see Section 3 and 4.

In Section 5 several extensions will be considered in less depth. We have decided to detail the exposition in Sections 3 and 4 to tensor-product spline spaces V_j. In Subsection 5.1, we discuss the conditions on generating functions $\phi^1, \ldots, \phi^l$ and the necessary modifications such that the conclusions of the theory of the previous sections still hold.

In Subsection 5.2 modifications for problems with essential boundary conditions will be considered. This case is somewhat more difficult to handle (compare [44]), and we do not have a satisfactory proposal for $d \geq 3$ at present. For $d = 2$, the basic idea is to enforce additional refinement near the boundary which leads to a modified construction of $V_{j,\Omega}$ resp. $\tilde{V}_j$.

Adaptivity by nested refinement (or, in our terminology, nested basis function selection) is dealt with in Subsection 5.3. We share the more naive viewpoint of most of the adaptive finite element codes, and use local a posteriori estimators based on a sort of local higher regularity or superconvergence assumption (which is hard to justify theoretically but leads to reasonable results in practical computations).

The author admits that there is a lot of closely related work, and that some of the ideas are straightforward and have appeared, in one or another form, in other papers, too. E.g., the construction of Cohen, Dahmen, DeVore [18] of a biorthogonal wavelet system for Sobolev spaces, though complicated in the technical details, looks theoretically much more powerful and is based on a clever boundary modification as well. There is a lot of activity on solving large linear systems arising from finite element discretizations on engineering (so-called unstructured) grids or of obstacle problems, where an embedding into a regular structure has been one of the options. See the recent papers [54, 55, 66, 5, 42] as well as [43, 35, 48]. Some of these investigations originate from the domain embedding or fictitious domain methods for finite difference discretizations where boundary modifications, extension and restriction operators have been used for a long time, also in connection with multigrid techniques. We refer to [40, 52, 36, 8, 54].

Finally, we wish to mention one more time that this is a paper on *geometric multilevel methods*, i.e., the approximating subspaces as well as their multilevel splittings are constructed for a generic linear, symmetric, uniformly elliptic H^s-problem ($s > 0$). No anisotropies, behavior of coefficient functions, physical background, or other specifics of the boundary value problem have been used. It is tempting but rather difficult to further extend this work to a more operator adapted setting (see, *e.g.* , [46, 22, 23]), and to attack the robustness aspect at large.

§2 Stable subspace splittings and iterative methods

This is a short introduction to a class of approximation and solution methods for operator equations in Hilbert spaces which is based on the concepts of multilevel scales and stable subspace splittings. The solvers for the approximate problems fall into the class of iterative subspace correction methods. Particular examples are domain decomposition and multigrid algorithms, orthogonal, Riesz basis, and frame decomposition and reconstruction techniques, and others. The framework typically incorporates adaptivity (with respect to individual features of the solution of the operator equation) in a natural way. We refer to the books and surveys by Bramble [11], Chan, Matthew [15], Dahmen [25], Hackbusch [41], Oswald [60], Xu [65], Yserentant [67], where different aspects and the history of the subject have been illuminated.

We start with the notion of *stable subspace splittings* for Hilbert spaces. Assume that V is a (finite-dimensional or separable) Hilbert space, with scalar product $(\cdot,\cdot)_V$ and norm $||\cdot||_V$. Let $\{V_j\}$ be an at most countable collection of closed subspaces of V such that $V = \sum_j V_j$, in the sense that for each $u \in V$ there is at least one V-converging representation

$$u = \sum_j u_j \ , \quad u_j \in V_j \ .$$

Let us further assume that the symmetric bilinear forms $a(\cdot,\cdot)$ resp. $b_j(\cdot,\cdot)$ are scalar products on V resp. V_j (for all j), such that the spaces are Hilbert when equipped with these alternative scalar products. We use the notation $\{V;a\}$ resp. $\{V_j;b_j\}$ to indicate this assumption. In particular,

$$c||u||_V^2 \leq ||u||_a \leq C||u||_V^2 \qquad \forall\ u \in V \ , \tag{2.1}$$

where $||u||_a = \sqrt{a(u,u)}$ denotes the energy norm in $\{V;a\}$. The two positive constants $0 < c, C < \infty$ in (2.1) are sometimes called ellipticity constants of $a(\cdot,\cdot)$ with respect to V, and do not depend on $u \in V$. In the following, we use the symbol $\asymp$ for two-sided inequalities such as (2.1) while $A \preceq B$ resp. $A \succeq B$ stand for a one-sided inequality $A \leq C \cdot B$

resp. $A \geq c \cdot B$. Thus, $A \asymp B$ is the same as $A \preceq B \preceq A$. The constants $0 < c, C < \infty$ are assumed to be generic, if not stated otherwise, they do not depend on the arguments of the expressions A and B.

The subspace splitting

$$\{V; a\} = \sum_j \{V_j; b_j\} \tag{2.2}$$

is called stable if

$$|||u|||^2_{\{b_j\}} \equiv \inf_{u_j \in V_j \,:\, u=\sum_j u_j} \sum_j b_j(u_j, u_j) \asymp a(u,u) \,. \tag{2.3}$$

One of the simple consequences of this definition is the fact that the operator equation

$$\mathcal{P}u = \phi \;, \qquad \mathcal{P} = \sum_j R_j T_j \;, \quad \phi = \sum_j \phi_j \;, \tag{2.4}$$

where $R_j : V_j \to V$ denotes the natural injection, and

$$\left\{ \begin{array}{lcl} T_j \,:\, V \to V_j & : & b_j(T_j u, v_j) = a(u, v_j) \\ \phi_j \in V_j & : & b_j(\phi_j, v_j) = \Phi(v_j) \end{array} \right. \qquad \forall\, v_j \in V_j \tag{2.5}$$

for all j, is an equivalent formulation of the variational problem of determining $u \in V$ such that

$$a(u,v) = \Phi(v) \qquad \forall\, v \in V \quad (\Phi \in V^*) \,. \tag{2.6}$$

The problem (2.4) is called additive Schwarz formulation of the variational problem (2.6) associated with the subspace splitting (2.2). Moreover, the additive Schwarz operator $\mathcal{P}$ acting in V is symmetric, positive definite with respect to $a(\cdot,\cdot)$, with spectral condition number defined by

$$\kappa(\mathcal{P}) = \frac{\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})} \;, \tag{2.7}$$

where

$$\lambda_{\max}(\mathcal{P}) \equiv \sup_{u \in V \,:\, a(u,u)=1} a(\mathcal{P}u, u) = \sup_{0 \neq u \in V} \frac{a(u,u)}{|||u|||^2_{\{b_j\}}}$$

and

$$\lambda_{\min}(\mathcal{P}) \equiv \inf_{u \in V \,:\, a(u,u)=1} a(\mathcal{P}u, u) = \inf_{0 \neq u \in V} \frac{a(u,u)}{|||u|||^2_{\{b_j\}}} \,.$$

As a consequence, the operator equation (2.4) is well-conditioned if the norm equivalence (2.3) holds with tight constants. This is the key to preconditioning using subspace splittings.

Before we discuss the typical algorithms associated with finite-dimensional versions of (2.2) we wish to add a few comments. First of all, the inclusion of infinite-dimensional spaces, and countable splittings is very appropriate if we come to our multiscale applications. As a rule, if

$$\{V; a\} = \sum_{j=0}^{\infty} \{V_j; b_j\}$$

is stable then the finite splittings

$$\{V_J; a\} = \sum_{j=0}^{J} \{V_j; b_j\}$$

are *uniformly* stable for $J \to \infty$ vice versa. This is a desirable feature for algorithmical reasons: the convergence rates of the solvers do not degenerate if better resolution is needed. In addition, this "asymptotical" viewpoint links us with the rich mathematical theory of Fourier analysis and function space decompositions. For instance, any complete orthogonal system but also any Riesz basis or frame in V leads to stable subspace splittings. On the other hand, the practical performance of the approach depends on a "non-asymptotical" range of small to moderate J, and is heavily influenced by the constants in (2.3). Thus, constructions which do not lead to stable splittings in the infinite-dimensional setting may well be successful in some applications. The hierarchical basis method of Yserentant designed for H^1-elliptic problems on two-dimensional domains may serve as an illustration. However, especially for methods working in an adaptive refinement environment where larger J are much more likely, the stability of the multiscale splitting in V becomes crucial.

A second remark is on the typical method of proof of the stability condition (2.3) which has been put into its present abstract form by J. Xu in his thesis (see [65]). The lower bound, *i.e.* , $|||u|||_{\{b_j\}} \preceq a(u,u)$, is usually formulated as separate condition, proofs combine (depending on the context) methods of approximation theory and elliptic regularity arguments. The upper bound is often replaced by assuming so-called *strengthened Cauchy-Schwarz inequalities* controlling the interaction between the different subspaces in the splitting. One formulation is to require

$$a(u_j, u_k)^2 \le \gamma_{j,k}^2 b_j(u_j, u_j) b_k(u_k, u_k) \qquad \forall\ u_j \in V_j\ ,\ u_k \in V_k\ . \tag{2.8}$$

Then the finiteness of the largest eigenvalue $\lambda_{\max}(\Gamma)$ of the matrix $\Gamma = ((\gamma_{j,k}))$ is a sufficient condition for the upper estimate. It is convenient to assume that

$$\gamma_{j,j} = 1 \tag{2.9}$$

in (2.8) for $j = k$ which amounts to an appropriate scaling. The inequalities (2.8) and the condition on Γ are also important for the more sophisticated multiplicative algorithms for which stability alone will not give the desired optimal result, see below. With slight modifications, this concept is also surveyed in [67, 38]. A refined theory serving the needs of multigrid applications, and including a discussion of all kinds of perturbations typical for practical implementations, is given in [11]. See also [41, Section 10-11] for an introduction and the link to multigrid. Finally, the approach of [60, 25] emphasizes the close connection of the stability assumption for multiscale splittings with results on scales of approximation spaces. For a survey on applications to domain decomposition methods (not necessarily of multilevel type), see [15].

Thirdly, it is possible to further generalize the concept by removing the assumption $V_j \subset V$. This is important for several reasons. In the multilevel context, we have in mind situations in which the monotonicity condition $V_j \subset V_{j+1}$ is violated. The formal cure is the introduction of a suitable set of mappings $R_j : V_j \to V$ (replacing the natural injections) such that $R = \sum_j R_j : \otimes V_j \to V$ is onto, and

$$|||u|||^2_{\{b_j,R_j\}} \equiv \inf_{u_j \in V_j \,:\, u=\sum_j R_j u_j} \sum_j b_j(u_j,u_j) \asymp a(u,u) \qquad \forall\, u \in V \ .$$

The latter condition replaces (2.3), and guarantees that $\mathcal{P}' = \sum_j R_j T'_j$ preserves the above-mentioned properties of $\mathcal{P}$. (2.5) is replaced by

$$b_j(T'_j u, v_j) = a(u, R_j v_j) \qquad \forall\ v_j \in V_j \ ,$$

where $T'_j : V \to V_j$. All these modifications can be subsumed in a simple Hilbert space lemma, called *fictitious space lemma*, which was first used in connection with fictitious domain and domain decomposition methods by Nepomnyaschikh [54], see [60, Theorem 17] or [66].

As a last remark we note that the abstract formulation of the stability concept in form of the two-sided inequality (2.3) allows to better understand simple transformations of one splitting into another which are useful especially for practical purposes, and add flexibility. In [60, Section 4.1] and [38], *refinement*, *clustering*, and *selection* have been discussed. Just to give an example, selection is typical for adaptivity applications and can be characterized as follows: For each j we select a subspace $V_j^* \subset V_j$ (both extremes $V_j^* = V_j$ and $V_j^* = \{0\}$ are allowed!), and form the new, selected space $V^* = \sum_j V_j^*$. Due to the definition of the triple bar norm, establishing the stability of the new splitting

$$\{V^*; a\} = \sum_j \{V_j^*; b_j\} \ ,$$

requires only the verification of the lower estimate, while the upper estimate is preserved from (2.3), with the same or a better constant. A further simplification comes for the case of splittings into *direct sums* of subspaces where the infimum in the definition of the triple-bar norm can be removed: in this case any selection leads again to a stable splitting, with the same or better condition.

Splittings into *one-dimensional* subspaces are of particular interest. Let V_j be generated by the (nontrivial) element $f_j \in V$. It turns out that then the stability condition (2.3) is equivalent to the frame property of the normalized system

$$\tilde{f}_j = \frac{1}{\sqrt{b_j(f_j, f_j)}} f_j$$

in $\{V; a\}$ (or, equivalently, in V equipped with the original scalar product). Recall that a system $\{g_j\}$ in a Hilbert space V is called *frame* if

$$\|u\|_V^2 \asymp \sum_j |(u, g_j)_V|^2 \qquad \forall\, u \in V$$

(see [30, Section 3.2] for the definition and properties of frames). Indeed, one easily computes that

$$\mathcal{P}u = \sum_j \frac{a(u, f_j)}{b_j(f_j, f_j)} f_j \tag{2.10}$$

which together with the stability of (2.2) implies

$$a(u, u) \asymp a(\mathcal{P}u, u) = \sum_j |a(u, \tilde{f}_j)|^2 \ .$$

Alternatively, look at [30, Proposition 3.2.4] and compare with (2.3). Riesz bases which are, by definition, *minimal* frames are particularly attractive since they lead to stable direct sum splittings of V. Another reason for the interest in frames or Riesz bases is that, based on their stability property, one can derive many other computationally interesting stable splittings. Examples are provided in [60, Section 4] and [58] for finite element applications. The explicit formulas for $\mathcal{P}$ (2.10) resp. for the subspace mappings T_j and the ϕ_j associated with the given functional Φ in the right-hand side of

$$T_j \ : \ u \in V \longmapsto T_j u = \frac{a(u, f_j)}{b_j(f_j, f_j)} f_j \in V_j \qquad (\ \phi_j = \frac{\Phi(f_j)}{b_j(f_j, f_j)} f_j\ ) \tag{2.11}$$

are useful to derive matrix representations suitable for the implementation of the algorithms explained next.

Let us now briefly outline the basic algorithms associated with a subspace splitting (2.2). For this purpose we make the natural assumption that all spaces involved are finite-dimensional, and that their number is finite:

$$\{V;a\} = \sum_{j=0}^{J} \{V_j; b_j\} \ . \tag{2.12}$$

Note that stability itself is obviously guaranteed, the question is the size of $\kappa(\mathcal{P})$.

The iteration step of the *additive algorithm* **A** is given by

$$u^{n+1} = u^n + \omega \sum_{j=0}^{J} r_j(u^n) \ , \quad r_j(u) = R_j(\phi_j - T_j u) \ . \tag{2.13}$$

The same amount of work is formally needed to perform one iteration of the *multiplicative algorithm* **M**

$$\left\{ \begin{array}{lcl} v^0 & = & u^n \\ v^{j+1} & = & v^j + \omega r_{J-j}(v^j) \ , \quad j = 0, \ldots, J \\ u^{n+1} & = & v^{J+1} \end{array} \right. \ . \tag{2.14}$$

The role of the relaxation parameter ω is analogous to the classical iterative methods (as a matter of fact, (2.13) corresponds to the extrapolated Jacobi resp. Richardson iteration while (2.14) generalizes SOR). Note that these are stationary linear iterative schemes (in the usual terminology, see [41]), the iteration operators for the two algorithms are

$$M_A = Id - \omega \mathcal{P} \ , \quad M_M = (Id - \omega R_0 T_0)(Id - \omega R_1 T_1) \ldots (Id - \omega T_J) \ .$$

The convergence theory (which is trivial for the additive algorithm) is covered in [65, 67, 11, 38, 60]. We quote the following result from [38], or [60, Theorem 18].

Theorem 1. *Assume that V is finite-dimensional, and that the algorithms* **A** *and* **M** *are defined with respect to (2.12).*

(i) The additive algorithm **A** *converges for $0 < \omega < 2/\lambda_{\max}(\mathcal{P})$. The optimal convergence rate is achieved for $\omega^* = 2/(\lambda_{\max}(\mathcal{P}) + \lambda_{\min}(\mathcal{P}))$, and equals*

$$\rho_A^* = \min_{0<\omega<2/\lambda_{\max}} \|M_A\|_a = 1 - \frac{2}{1+\kappa(\mathcal{P})} \ . \tag{2.15}$$

(ii) Assume (2.8) and (2.9). Then the multiplicative algorithm **M** *converges for* $0 < \omega < 2$. *The (analogously defined) optimal convergence rate can be estimated by*

$$(\rho_M^*)^2 \leq 1 - \frac{\lambda_{\min}(\mathcal{P})}{2\lambda_{\max}(\Gamma) + 1} \ . \tag{2.16}$$

Without the assumption (2.8), one still gets

$$(\rho_M^*)^2 \leq 1 - \frac{1}{\log_2(4(J+1)) \cdot \kappa(\mathcal{P})} \ . \tag{2.17}$$

Intuitively, it might seem that the multiplicative algorithm **M** should perform better than **A** (and this is indeed the case for many standard applications, and parallels the experience with Jacobi- and Gauss-Seidel methods for specific classes of linear systems), however, it was shown for some exotic Toeplitz systems [59] that, in general, the logarithmic factor in (2.17) can not be removed. Note that the choice of ω is only sensitive for the multiplicative algorithm, if **A** is replaced by the conjugate gradient iteration applied to (2.4), one essentially has the same iteration step (2.13), with an automatic choice of $\omega = \omega(u^n)$ (the latter requires some additonal storage and computation of scalar products). Moreover, the cg-iteration results in an even better estimate for the average convergence rate:

$$\rho_{cg}^{aver} \approx 1 - \frac{2}{1 + \sqrt{\kappa(\mathcal{P})}} \ . \tag{2.18}$$

There are numerous modifications of the multiplicative algorithm, and refined theories which serve some applications better, and lead to sharper estimates under special circumstances. We mention a symmetrized version of **M**, the *symmetric multiplicative iteration* **SM**, which is the abstract counterpart of the SSOR-method. It is approximately twice as expensive as **M** and combines two steps of (2.14) (the second performed in the opposite order) into one, see [65]. The iteration operator takes the form

$$M_{SM} = (Id - \omega T_J) \ldots (Id - \omega R_1 T_1)(Id - \omega R_0 T_0)(Id - \omega R_1 T_1) \ldots (Id - \omega T_J) \ .$$

A more general version of **SM**, the *variable symmetric multiplicative algorithm* has been popularized by Bramble et.al. (see [11, Algorithm III] for a general multigrid exposition). In a multiscale environment, the general recommendation is to allow for more subspace correction steps corresponding to small j (which are the low-dimensional subspaces, and, therefore, the unexpensive subproblems in (2.5)). The benefit is that weaker assumptions suffice to state optimal convergence estimates, without increasing the arithmetic complexity of the iteration in the asymptotical range ($J \to \infty$). For details, we refer to [11].

We finish this section by discussing the matrix representations of the above algorithms for the multiscale setting. This also leads us to an understanding of some implementational issues. Assume that the V_j are an increasing multiresolution scale (1.1) of finite-dimensional subspaces, of dimension n_j, and with a designated algebraic basis which we denote by $\mathcal{N}_j = \{\phi_{j,i}\}_{i=1,\ldots,n_j}$. In wavelet resp. finite element discretizations, the $\phi_{j,i}$ are dilates and translates of the scaling functions resp. nodal basis functions. All matrix representations will be with respect to these bases. According to (1.1), there are representations

$$\phi_{j-1,i'} = \sum_{i=1}^{n_j} a_{j;i,i'}\phi_{j,i} \ , \quad i' = 1,\ldots,n_{j-1} \ ,$$

the coefficients of which enter the matrix representations I_j of the natural embeddings $V_{j-1} \to V_j$. More precisely, I_j is the $n_j \times n_{j-1}$ matrix given by

$$I_j = ((a_{j;i,i'}))_{i=1,\ldots,n_j;i'=1,\ldots,n_{j-1}}$$

which transforms the coefficient vector of $u \in V_{j-1}$ with respect to $\mathcal{N}_{j-1}$ into the coefficient vector of the same u with respect to $\mathcal{N}_j$.

We first discuss algorithms associated with (1.2). We denote the stiffness matrices of the bilinear forms $a(\cdot,\cdot)$ (restricted to V_j) and $b_j(\cdot,\cdot)$ with respect to $\mathcal{N}_j$ by A_j and B_j, resp.. Thus, the entries of A_j are $a(\phi_{j,i},\phi_{j,i'})$, $i,i' = 1,\ldots,n_j$, analogously for B_j. It is now obvious that the matrix representations of the operators $T_j : V_J \to V_j$ are given by

$$B_j^{-1} I_{j+1}^T \cdots I_J^T A_J \ , \quad j = 0,\ldots,J \ ,$$

and that the additive operator $\mathcal{P}_J = \sum_{j=0}^J R_j T_j$ acting in V_J leads to the matrix expression

$$\sum_{j=0}^{J} I_J \cdots I_{j+1} B_j^{-1} I_{j+1}^T \cdots I_J^T A_J \equiv C_J^A A_J \ . \qquad (2.19)$$

Note that the preconditioning matrix C_J^A is formally independent of A_J, it depends only indirectly on $a(\cdot,\cdot)$ via the choice of the B_j. The *intergrid transfer operations* behind I_j and I_j^T can be interpreted as *prolongations* and *restrictions*, resp.. Finally we mention that the matrix $Id_J - \omega C_J^A A_J$ corresponds to the iteration operator M_A. Here, and in the following Id_j stands for the identity matrix of dimension n_j.

The *recursive* structure behind (2.19) should be emphasized. The preconditioning matrix C_J^A can be defined recursively as

$$C_0^A = B_0^{-1} \ , \quad C_j^A = I_j C_{j-1}^A I_j^T + B_j^{-1} \ , \ j = 1, \ldots, J \ . \tag{2.20}$$

An analogous structure is exhibited by the multiplicative method. By induction, one verifies that $C_0^M = B_0^{-1}$,

$$C_j^M = I_j C_{j-1}^M I_j^T (Id_j - \omega \tilde{A}_j B_j^{-1}) + B_j^{-1} \ , \ j = 1, \ldots, J \ , \tag{2.21}$$

leading to the matrix expression $Id_J - \omega C_J^M A_J$ for the iteration operator M_M. While (2.20) can be implemented directly for use in a preconditioned cg-iteration, the implementation of the multiplicative algorithm is usually performed in a different fashion, as a multigrid $V^{(1,0)}$-cycle, as explained in [11, 41]. What we wish to show with the formal recursion (2.21) is that the two methods look similar, with the difference that the multiplicative method **M** contains an additional residual evaluation. The latter is therefore slightly more expensive, and needs the so-called *Galerkin coarse grid matrices*

$$\tilde{A}_j = I_{j+1}^T \cdots I_J^T A_J I_J \cdots I_{j+1} \ . \tag{2.22}$$

Provided that all entries of A_J are computed exactly, one easily verifies that $\tilde{A}_j = A_j$. Still these matrices have to be precomputed, and stored. However, in many other cases (when using quadrature rules, for perturbations of the nestedness condition etc.) the matrices $\tilde{A}_j$ are different from A_j, and may change if J is increased. Then additional considerations are required (see, *e.g.* , [11, Sections 4-7]).

In the remainder of this paper we will be concerned only with the properties of additive preconditioners as defined in (2.20). We do not further discuss the multiplicative (and more general multigrid) algorithms which are a valuable and flexible tool in practice. In order to justify this "negligence", let us just recall that Theorem 1, (2.17), guarantees that knowledge about the additive iteration **A**, and thus about the characteristics of the underlying stable splitting, also leads to sufficiently good convergence statements for some simple multiplicative algorithm. Since our examples below are mainly frame-based, *i.e.* we typically consider the refinement

$$\{V_J; a\} = \sum_{j=0}^{J} \sum_{i=1}^{n_j} \{V_{j,i}; a\} \ , \quad V_{j,i} = \mathrm{span}\{\phi_{j,i}\} \ , \tag{2.23}$$

of the splitting (1.2), it is worth mentioning that in this case B_j^{-1} is just a diagonal matrix:

$$B_j = \mathrm{diag}\{a(\phi_{j,i}, \phi_{j,i})\}_{i=1,\ldots,n_j} \ .$$

This follows from the explicit formula for the additive Schwarz operator $\mathcal{P}$ associated with (2.23) which can be derived from (2.10). More involved approximate solvers on the subspaces V_j are possible but will not be considered. The above choice is extremely simple but incorporates at least a minimum of information from A_J into the subspace corrections. Note that the only components of the C_J^A-recursion which changes if (1.2) is replaced by (1.3) are the operations B_j^{-1}. Assume that we have a Riesz basis in V resp. in V_J associated with the splitting (1.3). The basis functions that span W_j are denoted by $\psi_{j,i}$, $i = 1, \ldots, m_j$, where $m_j = n_j - n_{j-1}$, $j \geq 1$. For the preconditioner which results from the Riesz basis

$$\mathcal{N}_0 \cup \{\psi_{1,i}\} \cup \ldots \cup \{\psi_{J,i}\}$$

one would need to put

$$B_j^{-1} = \tilde{I}_j \tilde{B}_j^{-1} \tilde{I}_j^T \ , \quad \tilde{B}_j = \mathrm{diag}\{a(\psi_{j,i}, \psi_{j,i})\}_{i=1,\ldots,m_j} \ , \ j \geq 1 \ ,$$

in the recursion (2.20). Here, the $n_j \times m_j$ matrix $\tilde{I}_j$ describes the natural embedding $W_j \to V_j$, and contains as entries the mask coefficients in the expressions

$$\psi_{j,i'} = \sum_{i=1}^{n_j} \tilde{a}_{j,i}^{i'} \phi_{j,i} \ .$$

This, and the precomputation of the diagonal matrix $\tilde{B}_j^{-1}$, are the places where the choice of the Riesz basis adds to the arithmetical complexity of the preconditioning operation. Other advantages (*e.g.* , better stability estimates, or robustness properties) should compensate for this drawback. Some examples of "cheap" finite element Riesz bases are available, see [50] for a survey.

§3 Subspaces for bounded domains

3.1 Construction of $V_{j,\Omega}$

Throughout the paper, we use the following notation. Let $\Omega \in \mathbb{R}^d$ be a bounded open d-dimensional domain, and $\partial\Omega$ its boundary. We assume that Ω possesses the extension property for the scale of Sobolev spaces H^s (for it to hold, the uniform cone condition would be sufficient, see [64, 1]). Let the Euclidean space $\mathbb{R}^d$ be partitioned into cubes of sidelength 1 such that the origin is a vertex of one of the cubes. The collection of all these so-called 0-cubes will be denoted by $\mathcal{R}_0$ (*partition of level* 0). The *partitions* $\mathcal{R}_j$ *of level* $j \geq 1$ into j-cubes of sidelength 2^{-j} will be obtained from $\mathcal{R}_0$ by dyadic dilation . Let

$$V_j = S_k^r(\mathcal{R}_j) \cap L_2(\mathbb{R}^d) \qquad (3.1.1)$$

be the L_2-subspaces of tensor-product splines of degree k and smoothness r with respect to $\mathcal{R}_j$ where $0 \le r \le k-1$. Obviously, $\{V_j\}$ is an increasing sequence of subspaces of the Sobolev spaces $H^s(\mathrm{I\!R}^d)$, $0 \le s < r+3/2$. Alternatively, the V_j could be defined by dyadic dilation from V_0: $V_j = \{u(2^j\cdot) \,:\, u \in V_0\}$. Note that V_j locally contains all algebraic polynomials of degree $\le k$.

We fix the local and L_2-stable basis of tensor-product B-splines $\{\phi_{j,i}\}$ in V_j, see [63]. This basis has the remarkable property of *local linear independence*: If $u_j \in V_j$ vanishes on a j-cube $\square$ then $c_{j,i} = 0$ for coefficients in the B-spline representation corresponding to all basis functions $\phi_{j,i}$ which do not vanish identically on $\square$. For our convenience, we introduce the notation $\omega_\square$ for the set of indices i such that $\phi_{j,i}$ does not vanish on the j-cube $\square$. Thus, local linear independence is equivalent to

$$\sum_i c_{j,i}\phi_{j,i}(x) = 0\,, \quad x \in \square \qquad \Longrightarrow \qquad c_{j,i} = 0\,, \quad i \in \omega_\square\,. \tag{3.1.2}$$

A consequence of the local linear independence property is the existence of well-localized biorthogonal functions: For any j-cube $\square \subset \operatorname{supp}\phi_{j,i}$ (or, in other words, for any $i \in \omega_\square$) there is a function $\eta_{j,i} \in L_\infty(\mathrm{I\!R}^d)$ supported on $\square$ and such that

$$\int_\square \eta_{j,i}\phi_{j,i'}\,dx = \delta_{i,i'} \qquad \forall\, i,i'\,. \tag{3.1.3}$$

As is obvious from the translation-dilation invariance of all constructions, the $\eta_{j,i}$ can be obtained as scaled translates of dilates of a finite number of functions associated with the unit cube $\square_0 = [0,1]^d$. The j-cube associated with $\eta_{j,i}$ will be denoted by $\square_{j,i}$. It will be fixed depending on the specific setting. If no explicit choice is made, then any j-cube in $\operatorname{supp}\phi_{j,i}$ will serve.

We introduce some modified basis functions which will be used for the boundary modification below. Consider the finite-dimensional space $X_0 = V_0|_{\square_0}$ (which in this specific case coincides with all polynomials of coordinate-wise degree $\le k$). It contains all monomials x^α, $|\alpha| \le k$, the set of which can be complemented by some other functions to yield a basis in X_0. Let $\{\phi_{\square_0,i}\}$ $(i \in \omega_{\square_0})$ denote this basis, and $\{\eta_{\square_0,i}\}$ the corresponding biorthogonal system in X_0, *i.e.* ,

$$\int_{\square_0} \phi_{\square_0,i}\eta_{\square_0,i'}\,dx = \delta_{i,i'}\,, \quad i,i' \in \omega_{\square_0}\,. \tag{3.1.4}$$

The same notation $\phi_{\square_0,i}$ will be used for the extensions to $S_k^r(\mathrm{I\!R}^d)$ obtained as follows: For the monomials x^α, $|\alpha| \le k$, there is a unique representation

$$x^\alpha = \sum_i c_i^{(\alpha)}\phi_{0,i}\,, \quad x \in \mathrm{I\!R}^d\,,$$

while for the complementing basis functions the minimal extension is used, *i.e.* , the B-spline coefficients of the extension vanish for $\phi_{0,i}$ with $i \notin \omega_{\Box_0}$ (coefficients with $i \in \omega_{\Box_0}$ are uniquely determined by the spline values on $\Box_0$ as follows from (3.1.2)). This construction is illustrated for the bilinear case ($d = 2, k = 1, r = 0$) in Figure 1 a)-d). The upper row shows the nodal values at the integer points near $\Box_0$ of the extended $\phi_{\Box_0,i}$ corresponding to the monomials 1, x_1, x_2, and one arbitrarily fixed complementing basis function, respectively, while in the second row the nodal values of the bilinear functions on $\Box_0$ defining $\eta_{\Box_0,i}$ are depicted. By translation and dyadic dilation we obtain systems $\{\phi_{\Box,i}\}$ and $\{\eta_{\Box,i}\}$ ($i \in \omega_\Box$) for any j-cube $\Box$ and all $j \geq 0$. To be definite, and in order to preserve the biorthogonality relation (3.1.4), we apply scaling (by a factor 2^{jd}) only to the η-functions. In the final construction, suitable restrictions of the extended functions $\phi_{\Box,i}$ will be used, see below.

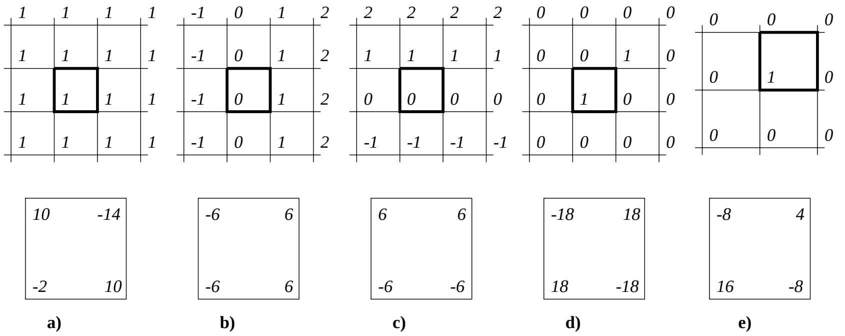


Figure 1. Bilinear elements: Nodal values for ϕ- and η-functions

From now on, we assume that generic constants $C, c, \ldots$ (also those occuring in $\asymp$ and $\preceq$ relations) may depend on k, r, s, and Ω but are independent of other parameters, especially of j, i, l, and the functions involved. For each $j \geq 0$, we define the sets $\Omega_j \subset \Omega \subset \Omega_j^e$ as unions of j-cubes:

$$\Omega_j = \cup\{\Box \in \mathcal{R}_j \,:\, \Box \subset \Omega\} \ , \qquad \Omega_j^e = \cup\{\Box \in \mathcal{R}_j \,:\, \Box \cap \Omega \neq \emptyset\} \ . \quad (3.1.5)$$

We require the following *geometric property* of Ω:

(G1) For each j-cube $\Box \in \Omega_j^e$ and any $l = 0, 1, \ldots, j$, there is at least one l-cube $\Box' \subset \Omega_j$ at a distance $\leq C2^{-l}$ from $\Box$. The constant C is assumed to be independent of $\Box$, $j \geq 0$, and l.

Roughly speaking, this condition means that the domain has a sufficiently "fat" interior and a regular boundary. In particular, **(G1)** implies $\Omega_0 \neq \emptyset$. Though restrictive, **(G1)** seems to be rather natural if robustness of a geometric multilevel method is expected. Asymptotically (*i.e.* , if required only for $j_0 \le l \le j$ and some sufficiently large j_0), the above condition is satisfied for domains with a Lipschitz boundary resp. the uniform cone condition.

By ω_j resp. $\partial\omega_j$ we denote the sets of all indices i such that the support of the tensor-product B-spline $\phi_{j,i}$ intersects Ω_j resp. intersects Ω but not Ω_j. These are the sets of *interior* and *boundary* indices of level j. Let $\mathcal{C}_j$ be a family of j-cubes $\Box \subset \Omega_j$ near the boundary of Ω satisfying the following properties:

- No function $\phi_{j,i}$ contains two different cubes from $\mathcal{C}_j$ in its support.
- There is a constant C such that for each $i \in \partial\omega_j$ there exists a cube $\Box_i \in \mathcal{C}_j$ at a distance $\le C2^{-j}$ from the support of $\phi_{j,i}$.

The second property implies the existence of a partition of the set of boundary indices $\partial\omega_j$ into small sets $\partial\omega_\Box$, $Box \in \mathcal{C}_j\}$, such that for $i \in \partial\omega_\Box$ the distance condition is satisfied with this particular $\Box$ (we will assume that $\partial\omega_\Box$ is nonempty, otherwise the corresponding cube $\Box$ can be excluded from $\mathcal{C}_j$). The existence of the families $\mathcal{C}_j$ easily follows from **(G1)** for $l = j$, the constant C depends on the constant in **(G1)** and k, r. Analogously, the set of interior indices ω_j decomposes into the pairwise disjoint sets $\omega_\Box$ ($\Box \in \mathcal{C}_j$), and a possibly larger "remainder" set

$$\omega_j' = \omega_j \setminus \cup_{\Box\in\mathcal{C}_j} \omega_\Box \ .$$

Figure 2 schematically illustrates the definitions of Ω_j, Ω_j^e, and $\mathcal{C}_j$ for the bilinear case. The j-cubes in $\mathcal{C}_j$ are given by hatching, and the numbers at nodal points associated with $\omega_\Box \cup \partial\omega_\Box$ indicate the number of the corresponding $\Box \in \mathcal{C}_j$, unnumbered points correspond to indices from ω_j') Since for $i \in \omega_j'$ the support of $\phi_{j,i}$ contains at least one j-cube $\Box \subset \Omega_j$ (which is not in $\mathcal{C}_j$!) we can fix $\eta_{j,i}$ such that its support $\Box_{j,i}$ is in $\operatorname{supp}\phi_{j,i} \cap \Omega_j$. Figure 1 e) shows the nodal values of the biorthogonal function for $j = 0$, the general case follows by scaling with a factor $2^{j\,d}$.

We come to the description of the boundary modification. Roughly speaking, only basis functions $\phi_{j,i}$ with $i \in \omega_\Box$ for j-cubes from $\mathcal{C}_j$ will be changed. For notational convenience, define the restriction operation M_ω associated to an arbitrarily given set ω of indices of level j by

$$u_j = \sum_i c_{j,i}\phi_{j,i} \in S_k^r(\mathcal{R}_j) \longmapsto M_\omega u_j = \sum_{i\in\omega} c_{j,i}\phi_{j,i} \in S_k^r(\mathcal{R}_j) \ . \qquad (3.1.6)$$

If ω is finite, the mapping is clearly into V_j. The restriction of M_ω onto V_j is an L_2-bounded operator, due to the L_2-stability of the B-spline basis. For each $\Box \in \mathcal{C}_j$, we replace the associated set $\{\phi_{j,i} \ : \ i \in \omega_\Box$ by a set of

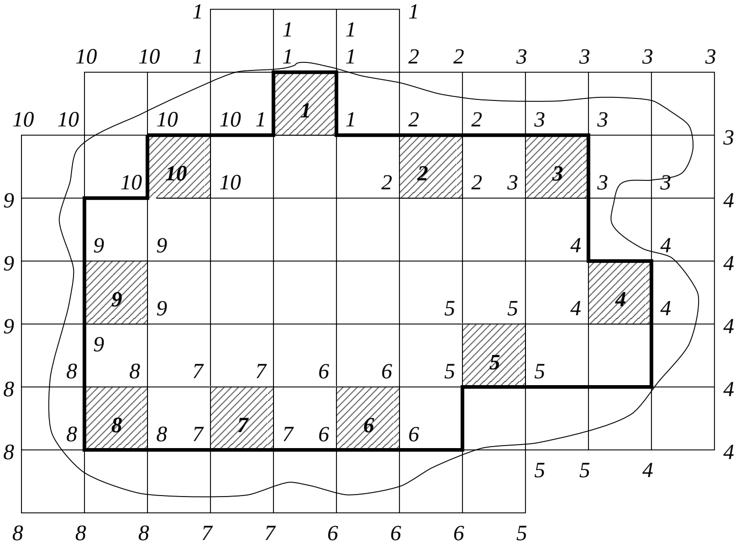


Figure 2. Boundary cubes and nodes for bilinear elements

"new" basis functions $\{\phi^*_{j,i} : i \in \omega_\square\}$ which coincides up to inde ordering with $\{M_{\omega_\square\cup\partial\omega_\square}\phi_{\square,i'}\}$ (*i.e.* , with the restrictions of the dilated and translated extensions $\phi_{\square_0,n}$ defined above to some neighborhood of $\square$, compare Figures 1 and 2). The biorthogonal functions $\eta^*_{j,i}$ are identified with $\eta_{\square,i'}$ accordingly. Thus, our construction ensures that

$$\begin{aligned}
&\|\phi_{j,i}\|_{L_\infty(\mathbb{R}^d)} \preceq 1\,, \quad \|\eta_{j,i}\|_{L_\infty(\square_{j,i})} \preceq 2^{jd}\,, \quad i \in \omega'_j\,,\\
&\|\phi^*_{j,i}\|_{L_\infty(\mathbb{R}^d)} \preceq 1\,, \quad \|\eta^*_{j,i}\|_{L_\infty(\square)} \preceq 2^{jd}\,, \qquad i \in \omega_\square\,,
\end{aligned} \tag{3.1.7}$$

hold for all $\square \in \mathcal{C}_j$, and $j \geq 0$. In addition, the biorthogonality conditions are preserved. The main advantage of our construction is the relatively simple, cube-oriented local basis exchange which still guarantees local reproduction of polynomials of (total) degree $\leq k$ in the spaces $V_{j,\Omega}$ to be defined next.

Let

$$V_{j,\Omega} = \mathrm{span}\{\phi_{j,i,\Omega} : i \in \omega_j\}\,, \tag{3.1.8}$$

where

$$\phi_{j,i,\Omega} = \begin{cases} \phi_{j,i}|_\Omega & \text{if} \quad i \in \omega'_j \\ \phi^*_{j,i}|_\Omega & \text{if} \quad i \in \omega_\square\,,\ \square \in \mathcal{C}_j \end{cases}\,.$$

The space $V_{j,\Omega}$ is a subspace of $V_j|_\Omega$, the restriction of V_j to Ω (as a rule, the inclusion is proper). More precisely, functions from $V_{j,\Omega}$ are uniquely determined by their values on Ω_j, and extended to Ω in a specific way. This can be seen from the definition of the biorthogonal system by

$$\eta_{j,i,\Omega} = \begin{cases} \eta_{j,i} & \text{if} \quad i \in \omega_j' \ , \\ \eta_{j,i}^* & \text{if} \quad i \in \omega_\square \ , \ \square \in \mathcal{C}_j \ , \end{cases}$$

which ensures $\square_{j,i} \equiv \operatorname{supp} \eta_{j,i} \subset \Omega_j$ to hold, and

$$u = Q_j u \equiv \sum_{i \in \omega_j} \lambda_{j,i}(u) \phi_{j,i,\Omega} \ , \quad \lambda_{j,i}(u) = \int_{\square_{j,i}} \eta_{j,i,\Omega} u \, dx \ , \tag{3.1.9}$$

for all $u \in V_{j,\Omega}$ since

$$\int_{\square_{j,i}} \eta_{j,i,\Omega} \phi_{j,i',\Omega} \, dx = \delta_{i,i'} \ , \quad i, i' \in \omega_j \ . \tag{3.1.10}$$

The spaces $V_{j,\Omega}$ still have the same approximation power (with respect to the Sobolev scale) as $V_j|_\Omega$ since the boundary modification allows for local reproduction of polynomials of total degree $\leq k$. Indeed, the quasi-interpolant operator (3.1.9) is well-defined for functions $u \in L_1(\Omega_j)$ and maps into $V_{j,\Omega}$. By construction, Q_j is a projector onto $V_{j,\Omega}$. Also, if u coincides on Ω_j with a spline function from V_j or from $S_k^r(\mathcal{R}_j)$ then its values are preserved on Ω_j.

By a standard argument, we can prove

$$\|Q_j u\|_{L_p(\Omega)} \preceq \|u\|_{L_p(\Omega_j)} \ , \qquad 1 \leq p \leq \infty \ . \tag{3.1.11}$$

To this end, observe that on each $\square \cap \Omega$ at most a fixed number of terms in (3.1.9) does not vanish. This gives

$$\|Q_j u\|^p_{L_p(\square\cap\Omega)} \preceq \sum_{i \in \omega_j \,:\, \square \cap \operatorname{supp} \phi_{j,i,\Omega} \neq \emptyset} \|\lambda_{j,i}(u)\phi_{j,i,\Omega}\|^p_{L_p(\square\cap\Omega)}$$

for all j-cubes $\square$. The generic term in this sum is the L_p-norm of a function of the form $\int_{\square'} \eta u \, dx \cdot \phi$, where ϕ, η satisfy (3.1.7), and $\square' \subset \Omega_j$ is a j-cube at distance $\leq c2^{-j}$ from $\square$. This implies

$$\begin{aligned} \| \int_{\square'} \eta u \, dx \cdot \phi \|_{L_p(\square\cap\Omega)} &\leq |\square|^{\frac{1}{p}} \|\phi\|_{L_\infty(\square)} \|\eta\|_{L_\infty(\square')} |\square'|^{1-\frac{1}{p}} \|u\|_{L_p(\square')} \\ &\preceq \|u\|_{L_p(\square')} \end{aligned}$$

and

$$\|Q_j u\|^p_{L_p(\square\cap\Omega)} \preceq \sum_{\square' \in \Omega' \,:\, \operatorname{dist}(\square,\square') \leq c2^{-j}} \|u\|^p_{L_p(\square')} \ .$$

Now summing over all j-cubes $\Box$ from Ω_j^e and observing that each $L_p(\Box')$-norm term is repeated only finitely many times, we conclude (3.1.11). We gave the proof only for the sake of completeness. On later occasions, analogous considerations are left to the reader.

To show approximation estimates in H^s-norms ($0 \le s \le r+1$) for $u \in H^t$, $s \le t \le k+1$, it suffices to consider integer $s = m = 0, 1, \ldots, r+1$, $t = k+1$, and to prove the norm estimates

$$\|u - Q_j u\|_{H^m(\Omega)} \preceq 2^{(k+1-m)j} \|u\|_{H^{k+1}(\Omega)} \qquad \forall\, u \in H^{k+1}(\Omega)\ . \quad (3.1.12)$$

The general case (as well as estimates in terms of moduli of smoothness) can be concluded from (3.1.11) and (3.1.12) by interpolation methods. To prove (3.1.12), let $E : H^s(\Omega) \to H^s(\mathbb{R}^d)$ denote the bounded extension operator, actually, here we need only $s = k+1$. Note that each

$$u_j = \sum_{i \in \omega_j} c_{j,i} \phi_{j,i,\Omega} \in V_{j,\Omega}$$

has a natural extension to V_j (which is different from Eu_j) given by

$$\tilde{u}_j = \sum_{\Box \in \mathcal{C}_j} \sum_{i \in \omega_\Box} c_{j,i} \phi_{\Box,i} + \sum_{i \in \omega_j'} c_{j,i} \phi_{j,i}\ .$$

Moreover, $Q_j u = Q_j E u$ and, as a by-product of the proof of (3.1.11), we have for the natural extension of $Q_j u$

$$\|\widetilde{Q_j u}\|_{L_2(\mathbb{R}^d)} \preceq \|u\|_{L_2(\Omega_j)}\ .$$

Next, the obvious inequality

$$\|u - Q_j u\|^2_{H^m(\Omega)} \le \sum_{\Box \subset \Omega_j^e} \|Eu - \widetilde{Q_j E u}\|^2_{H^m(\Box)}$$

reduces the global estimation to local estimations on each $\Box \subset \Omega_j^e$. By construction $\omega_\Box \subset \omega_j^e$ for all $\Box \subset \Omega_j^e$. Therefore, to each $\Box \in \Omega_j^e$ and $i \in \omega_\Box$ we can associate a j-cube $\Box_i \subset \Omega_j$ according to the following rules: If $i \notin \omega_j'$ then $i \in \omega_{\Box^*} \cup \partial\omega_{\Box^*}$ for some j-cube $\Box^* \in \mathcal{C}_j$ and $\Box_i = \Box^*$, otherwise, if $i \in \omega_j'$ then set $\Box_i = \Box_{j,i}$ ($\subset \operatorname{supp} \phi_{j,i}$). In all cases, these cubes are at a distance $\le C2^{-j}$ from $\Box$. We choose $U(\Box)$ to be the smallest cube (say, a union of j-cubes) containing $\Box$ and all $\Box_i$, $i \in \omega_\Box$, simultaneously. The diameter of $U(\Box)$ is bounded by $C2^{-j}$, with an absolute constant C which depends on the constant from assumption **(G1)**.

Let p denote an arbitrary polynomial of total degree $\le k$. By the above construction it is clear that $\widetilde{Q_j p} = p$ on Ω_j^e (to see the coefficient reproduction in the representation of p with respect to $\{\phi_{j,i}\}$ for indices $i \in \omega_\Box \cup \partial\omega_\Box$, look at the definition of the modified boundary functions $\phi_{\Box,i}$ (part of which coincide on $\Box \in \mathcal{C}_j$ with the monomials of degree $\le k$), and their biorthogonal counterparts). Hence

$$\|Eu - \widetilde{Q_jE}u\|^2_{H^m(\Box)} \le 2(\|Eu-p\|^2_{H^m(\Box)} + \|Q_j(\widetilde{Eu}-p)\|^2_{H^m(\Box)})$$

$$\begin{aligned} &\preceq \|Eu-p\|^2_{H^m(\Box)} + 2^{2jm}\|Q_j(\widetilde{Eu}-p)\|^2_{L_2(\Box)} \\ &\preceq \|Eu-p\|^2_{H^m(\Box)} + \sum_{i\in\omega_\Box} 2^{2jm}\|Eu-p\|^2_{L_2(\Box)} \\ &\preceq \|Eu-p\|^2_{H^m(U(\Box))} + 2^{2jm}\|Eu-p\|^2_{L^2(U(\Box))} \ . \end{aligned}$$

Taking the infimum with respect to all polynomials of degree $\le k$ and taking into account the size of the cube $U(\Box)$, by the Bramble-Hilbert lemma we arrive at

$$\|Eu - \widetilde{Q_jE}u\|^2_{H^m(\Box)} \preceq 2^{-2(k+1-m)j}\|Eu\|^2_{H^{k+1}(U(\Box))} \ .$$

After summing up with respect to $\Box \subset \Omega_j^e$ this yields (3.1.12) (note that by the extension property the $H^{k+1}(\mathbb{R}^d)$-norm of Eu is bounded by the $H^{k+1}(\Omega)$ norm of u. The norm of the extension operator E enters the constants).

The outlined argument establishes

Lemma 1. *Let Ω have the extension property and satisfy the above geometric assumption* **(G1)***. Then the quasi-interpolant operator Q_j defined in (3.1.9) provides good approximation order in Sobolev norms: For all $0 \le s \le r+1$ and $s \le t \le k+1$ we have*

$$\|u - Q_ju\|_{H^s(\Omega)} \le C2^{(t-s)j}\|u\|_{H^t(\Omega)} \qquad \forall\ u \in H^{k+1}(\Omega)\ . \tag{3.1.13}$$

Moreover, the operators Q_j are projectors onto $V_{j,\Omega}$. They also preserve the values on Ω_j of spline functions from V_j resp. $S_k^r(\mathcal{R}_j)$.

By Cea's lemma and the Aubin-Nitsche trick, this lemma ensures optimal approximation rates for solutions of discretized (with respect to $\{V_{j,\Omega}\}$) elliptic variational problems in Sobolev spaces on Ω which are comparable with the rates for traditional finite element schemes. Let $a(\cdot,\cdot)$, denote a generic $H^s(\Omega)$-elliptic bilinear form, in particular, we have

$$\|u\|^2_{H^s(\Omega)} \asymp a(u,u)\ , \qquad \forall\ u \in H^s(\Omega)\ . \tag{3.1.14}$$

We fix s satisfying $0 < s \le r+1$, *i.e.* , we consider only elliptic problems of positive order which is a restriction of our approach. For these s, we have $V_{j,\Omega} \subset V_\Omega \equiv H^s(\Omega)$ resp. $V_j \subset V \equiv H^s(\mathbb{R}^d)$ (there is a possibility to extend the results by additional considerations to the range $r+1 < s < r+3/2$ for which we, however, do not know of any practical need). Cases of practical interest for the bilinear case, namely $s = 1$ (second order elliptic boundary value problems with Neumann or Robin boundary conditions) and $s = 1/2$ (hypersingular integral equations) are still covered.

Let Φ be a bounded linear functional on $H^{-s}(\Omega)$. Then, by the Lax-Milgram result and Cea's lemma, the variational problem

$$\text{Find} \quad u \in V_\Omega \quad \text{such that} \qquad a(u,v) = \Phi(v) \quad \forall\, v \in V_\Omega\ , \tag{3.1.15}$$

as well as the finite-dimensional problems

$$\text{Find } u_j \in V_{j,\Omega} \text{ such that} \quad a(u_j, v_j) = \Phi(v_j) \quad \forall\, v_j \in V_{j,\Omega} \tag{3.1.16}$$

possess unique solutions $u \equiv u_\Phi \in V_\Omega$ resp. $u_j \equiv u_{j,\Phi} \in V_{j,\Omega}$ for which

$$\|u - u_j\|_{V_\Omega} \preceq \inf_{v_j \in V_{j,\Omega}} \|u - v_j\|_{V_\Omega}\ . \tag{3.1.17}$$

This estimate in the energy norm $\|\cdot\|_{V_\Omega}$ leads, in conjunction with Lemma 1, to asymptotical a priori error estimates if additional regularity of u (*e.g.* $u \in H^t(\Omega)$ for some $t > s$) is known. A general treatment of this topic is beyond the scope of this paper.

3.2 Construction of $\tilde{V}_j$

Since usually $V_{j,\Omega} \subset V_{j+1,\Omega}$ does not hold, we have difficulties to use the sequence $\{V_{j,\Omega}\}$ directly for a multilevel scheme (or a multiresolution analysis) in $H^s(\Omega)$. Instead, we construct an auxiliary sequence $\{\tilde{V}_j\}$ of subspaces of V_j, where each $\tilde{V}_j$ is defined as the sum of one-dimensional subspaces $V_{l,i}$ spanned by the B-spline basis function $\phi_{l,i}$:

$$\tilde{V}_j = \sum_{l=0}^{j} \sum_{i \in \tilde{\omega}_{j,l}} V_{l,i}\ . \tag{3.2.1}$$

The choice of the index sets $\tilde{\omega}_{j,l}$, $l \le j$, depends only on Ω_j, and the splitting (3.2.1) defines automatically an additive Schwarz preconditioner for potential variational problems on $\tilde{V}_j$. This preconditioner will be investigated in the next section, together with a brief description of the algorithmical switch between $V_{j,\Omega}$ and $\tilde{V}_j$. Here, besides the construction of $\tilde{V}_j$, we establish the theoretical properties of restriction and extension operators

$$R_j\ :\ \tilde{V}_j \to V_{j,\Omega}\ , \qquad E_j\ :\ V_{j,\Omega} \to \tilde{V}_j\ ,$$

acting between the sequences (3.1.8) and (3.2.1).

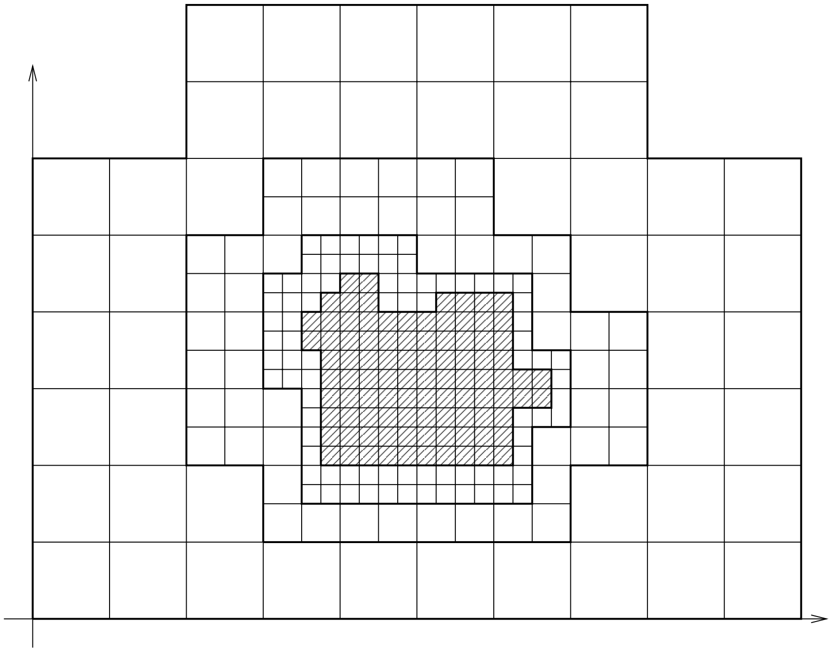


Figure 3. The domains $\{\tilde{\Omega}_l\}$, $l \le j$, and the associated cube partitions

Let $D \ge 1$ be an integer which will be fixed later. We also temporarily fix j and define a monotone sequence

$$\tilde{\Omega}_{j+1} = \Omega_j \subset \tilde{\Omega}_j \subset \cdots \subset \tilde{\Omega}_0 \qquad (3.2.2)$$

of sets according to the following rule: $\tilde{\Omega}_l$ ($l = j, j-1, \ldots, 0$) is the union of all $(l-1)$-cubes with a l_1-distance of $< D \cdot 2^{-l}$ from $\tilde{\Omega}_{l+1}$. Figure 3 shows the construction of $\{\tilde{\Omega}_l\}$ for $j = 2$, $D = 1$, and a fictive $\tilde{\Omega}_3 = \Omega_2$ (which is the hatched region). After this we define

$$\tilde{\omega}_l = \{i \, : \, \operatorname{supp} \phi_{l,i} \subset \tilde{\Omega}_l\} \, , \qquad l = 0, 1, \ldots, j \, , \qquad (3.2.3)$$

and assume that the above D was chosen sufficiently large such that

$$\omega_\Box \subset \tilde{\omega}_l \qquad (3.2.4)$$

for all l-cubes $\Box$ in $\tilde{\Omega}_{l+1}$ and $l = 0, 1, \ldots, j$. In the bilinear case, $D = 1$ would be enough to guarantee (3.2.4), compare Figure 3. The construction of $\tilde{V}_j$ described above on the basis of (3.2.2), (3.2.3) is an example of *nested* *basis function selection* from the infinite splitting $\sum_{j=0}^{\infty}\sum_i V_{j,i}$ as discussed in [60, 4.2.2]. As we will see below, the corresponding systems of basis functions are frames in their spans with respect to H^s-norms, $0 < s < r + 3/2$.

From (3.2.4) for $l = j$, Lemma 1, and the geometric assumption **(G1)** we see that

$$\tilde{V}_j|_{\Omega_j} = \tilde{V}_j|_{\tilde{\Omega}_{j+1}} = V_j|_{\Omega_j} = V_{j,\Omega}|_{\Omega_j} \ ,$$

and $\dim V_{j,\Omega} \le \dim \tilde{V}_j \le C \dim V_{j,\Omega}$ (for sufficiently large D one could also enforce $V_{j,\Omega} \subset \tilde{V}_j|_\Omega$: compare the geometric conditions for the boundary modification required in Subsection 3.1).

We define

$$R_j \ : \ \tilde{u}_j \in \tilde{V}_j \to Q_j \tilde{u}_j \in V_{j,\Omega} \tag{3.2.5}$$

as the restriction of the quasi-interpolant operator (3.1.9) from Subsection 3.1 to $\tilde{V}_j$. Obviously,

$$R_j \tilde{u}_j = \tilde{u}_j \qquad \text{on } \Omega_j \ , \tag{3.2.6}$$

and the boundedness of R_j in the H^s-norm follow from Lemma 1 (set $0 < s = t \le r + 1$).

For the extension operator E_j we need some more preparations. The following lemma has a long history, it follows from the basic direct and inverse inequalities in spline approximation theory in a straightforward way (see, *e.g.* , [34], [56], [60, 3.6]).

Lemma 2. *For the above defined spline spaces and* $0 < s < r + 3/2$, *we have the following norm equivalencies:*

$$||u||_{H^s(\mathbb{R}^d)} \asymp |||u|||_s \asymp |||u|||_{\{P_j\},s} \qquad \forall \ u \in H^s(\mathbb{R}^d) \ , \tag{3.2.7}$$

where

$$|||u|||_s^2 = \inf_{u_l \in V_l \,:\, u = \sum_{l=0}^{\infty} u_l} \ \sum_{l=0}^{\infty} 2^{2ls} ||u_l||_{L_2(\mathbb{R}^d)}^2$$

and, for any family $\{P_j\}$ *of uniformly* L_2*-bounded quasi-interpolant operators* $P_j \ : \ L_2(\mathbb{R}^d) \to V_j$ *(preserving locally polynomials of degree* $\le [s]$, *at least),*

$$|||u|||_{\{P_j\},s}^2 = ||P_0 u||_{L_2(\mathbb{R}^d)}^2 + \sum_{l=1}^{\infty} 2^{2ls} ||P_l u - P_{l-1} u||_{L_2(\mathbb{R}^d)}^2 \ .$$

We set

$$E_j \ : \ u_j \in V_{j,\Omega} \to E_j u_j \equiv \sum_{l=0}^{j} M_{\tilde{\omega}_l}(P_l u_j - P_{l-1} u_j) \in \tilde{V}_j \tag{3.2.8}$$

where $\{P_l\}$ ($P_{-1}u \equiv 0$) will be a particular family of quasi-interpolant operators (to be formally correct, in (3.2.8) we apply these operators to the natural extension $\tilde{u}_j$ of u_j to V_j). To define P_l we again make use of **(G1)**. The construction is very similar to the definition of the quasi-interpolant operators Q_j from (3.1.9). To each $\phi_{l,i}$ we associate a l-cube $\square_{l,i}$ *inside* or of *minimal distance* from $\operatorname{supp}\phi_{l,i}$, and such that for $l \le j$

$$\square_{l,i} \subset \Omega_j \qquad \text{if} \quad \operatorname{supp}\phi_{l,i} \cap \tilde{\Omega}_l \neq \emptyset \,.$$

This definition guarantees that $\square_{l,i}$ belongs to the support of the basis function $\phi_{l,i}$ whenever possible. The exceptions are functions (for $l \le j$) whose support overlaps $\tilde{\Omega}_l \backslash \Omega_j$. Due to **(G1)** this set is a thin corridor of "thickness" $\le D(2^{-l} + \ldots + 2^{-j}) \le 2D2^{-l}$ near $\partial\Omega$. This implies that the distance between $\square_{l,i}$ and $\operatorname{supp}\phi_{l,i}$ is either 0 or does not exceed $C2^{-l}$. As above we define functions $\eta_{l,i}$ with support on $\square_{l,i}$ such that the functional

$$\lambda_{l,i}(u) \equiv \int_{\square_{l,i}} \eta_{l,i} u \, dx = c_{l,i} \tag{3.2.9}$$

reproduces the coefficient of $\phi_{l,i}$ in the B-spline representation for any $u = \sum_i c_{l,i}\phi_{l,i} \in S_k^r(\mathcal{R}_l)$ if $\square_{l,i} \subset \operatorname{supp}\phi_{l,i}$, resp. only for all polynomials of total degree $\le k$ if $\square_{l,i} \not\subset \operatorname{supp}\phi_{l,i}$. Since, in both cases, the coefficient is uniquely determined by $u|_{\square_{l,i}}$, the construction of these $\eta_{l,i}$ reduces to a local problem which will be first solved for the unit cube and then transferred to $\square_{l,i}$. Due to the restriction of the distances between $\square_{l,i}$ and $\operatorname{supp}\phi_{l,i}$, we can again assume

$$\|\eta_{l,i}\|_{L_\infty} \preceq 2^{ld} \,.$$

The above properties automatically yield uniform local L_p-boundedness, and local reproduction of polynomials of degree $\le k$ of the quasi-interpolant operators

$$P_l \;:\; u \in L_1(\mathrm{I\!R}^d) \longmapsto P_l u = \sum_i \lambda_{l,i}(u)\phi_{l,i} \,, \qquad l = 0, 1, \ldots \,. \tag{3.2.10}$$

Under these circumstances, Lemma 2 is applicable (the proof is outlined in [60, Chapters 2 and 3]). Moreover, for $l = j$, if u coincides with a spline from $S_k^r(\mathcal{R}_l)$ on Ω_j then $P_j u = u$ on Ω_j. Indeed, according to the above rules for all $\phi_{j,i}$ with support intersecting Ω_j the j-cube $\square_{j,i}$ will be chosen inside $\Omega_j \cap \operatorname{supp}\phi_{\mathrm{j,i}}$ in which case $\lambda_{j,i}$ reproduces the B-spline coefficient corresponding to $\phi_{l,i}$.

These observations, together with the definition of $\{\tilde{\Omega}_l\}$ and $\tilde{\omega}_l$, $l \le j$, ensure that the extension operator E_j from (3.2.8) is well-defined on $V_{j,\Omega}$ (only information from $u_j|_{\Omega_j}$ is needed after the application of the restriction operators M_{ω_l}), and preserves the values of u_j on Ω_j. As a consequence, we have

$$R_j E_j u_j = u_j \ , \qquad \forall\, u_j \in V_{j,\Omega} \ . \tag{3.2.11}$$

Moreover, if $E \ : \ H^s(\Omega) \to H^s(\mathbb{R}^d)$ is again the bounded extension operator then by Lemma 2 and the L_2-stability of the B-spline bases

$$\begin{aligned} ||E_j u_j||^2_{H^s(\mathbb{R}^d)} &= ||E_j E u_j||^2_{H^s(\mathbb{R}^d)} \\ &\preceq \sum_{l=0}^{j} 2^{2js} ||M_{\omega_l}(P_l - P_{l-1}) E u_j||^2_{L_2(\mathbb{R}^d)} \\ &\preceq \sum_{l=0}^{\infty} 2^{2js} ||(P_l - P_{l-1}) E u_j||^2_{L_2(\mathbb{R}^d)} \\ &\preceq ||E u_j||^2_{H^s(\mathbb{R}^d)} \preceq ||u_j||^2_{H^s(\Omega)} \ . \end{aligned}$$

Subsuming these results, we have

Lemma 3. *Under the above assumptions for the construction of* $V_{j,\Omega}$ *and* $\tilde{V}_j$ *(which include* **(G1)***) and the* $H^s(\Omega)$*-elliptic form* $a(\cdot,\cdot)$*, we have for the extension and restriction operators (3.2.8), (3.2.5) the property (3.2.11) and the estimates*

$$a(R_j \tilde{u}_j, R_j \tilde{u}_j) \le C ||\tilde{u}_j||^2_{H^s(\mathbb{R}^d)} \ , \qquad \forall\, \tilde{u}_j \in \tilde{V}_j \ , \tag{3.2.12}$$

and

$$||E_j u_j||^2_{H^s(\mathbb{R}^d)} \le C a(u_j, u_j) \ , \qquad \forall\, u_j \in V_{j,\Omega} \ . \tag{3.2.13}$$

The constants in (3.2.12), (3.2.13) depend on the constants in **(G1)***, on* D, k, r, s*, and the ellipticity constants of* $a(\cdot,\cdot)$*, see (3.1.14).*

§4 Multilevel preconditioners of BPX type

4.1 H^s-preconditioner for $\tilde{V}_j$

In this section we first deal with preconditioning a generic $H^s(\mathbb{R}^d)$-elliptic problem on the finite-dimensional subspaces $\tilde{V}_j$ given by the index sets $\tilde{\omega}_l$ resp. the domains $\tilde{\Omega}_l$, $l \le j$, as indicated in (3.2.1) and (3.2.3). However, we slightly generalize the construction of the subspaces to include some other interesting situations such as discussed in Subsection 5.2 (boundary refinement) and 5.3 (adaptive nested refinement).

Consider an arbitrary but fixed $j > 0$. In contrast with the above construction in the particular setting of Subsection 3.2, we start now with a sequence of pairwise disjoint sets Ω_l^* which should be unions of l-cubes (or empty), $0 \leq l \leq j$, and set

$$\tilde{\Omega}_l = \cup_{l \leq l' \leq j}\, \Omega_{l'}^* \,, \qquad \Omega_l^{**} \equiv \cup_{0 \leq l' \leq l}\, \Omega_{l'}^* \,.$$

After this, define as before $\tilde{\omega}_l$ by (3.2.3), and $\tilde{V}_j$ by (3.2.1). Roughly speaking, $\tilde{\Omega}_0$ is "computational" domain covered by a cube refinement structure, where $\Omega_l^*, \Omega_l^*, \tilde{\Omega}_l$ are the regions covered by cubes of level exactly l, $\leq l$, and $\geq l$, resp. (see Figure 4 a) for an example).

An algebraic basis in $\tilde{V}_j$ is given as follows. Define ω_l^* as the set of all indices $i \in \tilde{\omega}_l$ for which the support of the corresponding B-spline intersects the set of l-cubes Ω_l^*: $\operatorname{supp} \phi_{l,i} \cap \Omega_l^* \neq \emptyset$. we claim that

$$\tilde{\mathcal{B}}_j = \{\phi_{l,i} \;:\; i \in \omega_l^* \;,\; l = 0, \ldots, j\} \tag{4.1.1}$$

is an algebraic basis in $\tilde{V}_j$. This can be shown by induction. Recall that by (3.2.1)

$$\tilde{\mathcal{F}}_j = \{\phi_{l,i} \;:\; i \in \tilde{\omega}_l \;,\; l = 0, \ldots, j\} \tag{4.1.2}$$

is a generating system for $\tilde{V}_j$. Set $l = 0$ and take any $i \in \tilde{\omega}_0 \backslash \omega_0^*$. By definition of the index sets,

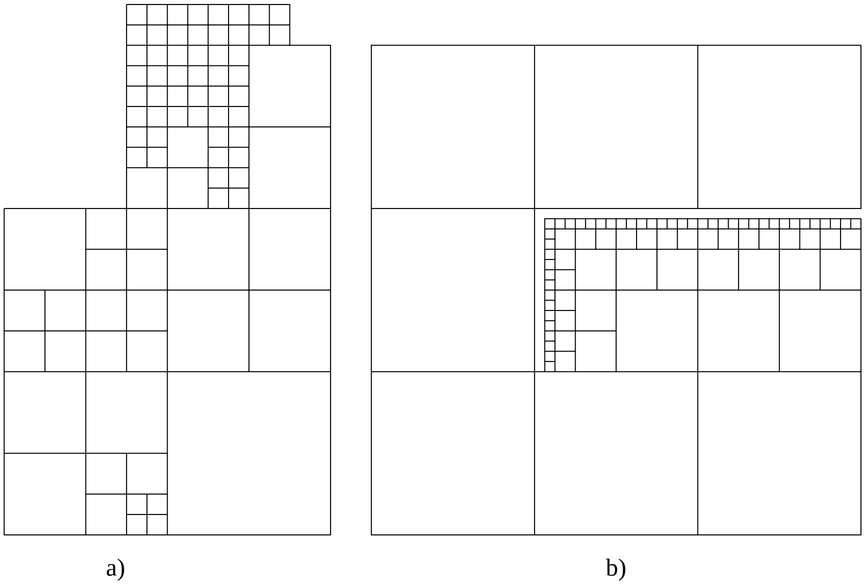
$$\operatorname{supp} \phi_{0,i} \subset \tilde{\Omega}_1 \;,$$

which shows that this $\phi_{0,i}$ can be expressed by a linear combination of basis functions $\phi_{1,i'}$ with $i' \in \tilde{\omega}_1$. Hence all these $\phi_{0,i}$ can be neglected. If $i \in \omega_0^*$ then there is at least one 0-cube in the support of $\phi_{0,i}$ which also belongs to Ω_0^*. Recall that $\tilde{V}_j$ restricted to this cube contains only linear combinations of basis functions from V_0. By the local linear independence property, we conclude that this $\phi_{0,i}$ cannot be dropped from the generating system. After deleting all unnecessary $\phi_{0,i}$, we proceed with $i \in \tilde{\omega}_1 \backslash \omega_1^*$ and so on. This proves that the system $\tilde{\mathcal{B}}_j$ is a basis in $\tilde{V}_j$.

We need the following geometric condition.

(G2) For each $l \leq j$ and each i such that $\operatorname{supp} \phi_{l,i} \not\subset \tilde{\Omega}_0$, there is an l-cube $\square_{l,i}$ outside $\tilde{\Omega}_0$ at a distance $\leq C2^{-l}$ from the support of $\phi_{l,i}$. Again, C is assumed to be independent of l, i, and j.

The subspaces $\tilde{V}_j$ defined in Subsection 3.2 fit these definitions if one takes $\Omega_l^* = \tilde{\Omega}_l \backslash \tilde{\Omega}_{l+1}$. Then **(G2)** is trivially satisfied since $\tilde{\Omega}_0$ is the union of (-1)-cubes (and, therefore, of l-cubes for any $l \geq 0$). A more general example for the bilinear case and $j = 3$ is given in Figure 4 a) while b) shows a domain with a slit where condition **(G2)** would be violated if the construction is continued for $j \to \infty$.



a) b)

Figure 4. Illustrations for **(G2)**

Now we define appropriate l-cubes $\Box_{l,i}$ for all possible l and i which enter the construction of the quasi-interpolant operators $\{P_l\}$ as explained in Subsection 3.2 (see (3.2.10), (3.2.9)). If $l \geq j$, the only condition is

$$\Box_{l,i} \in \operatorname{supp} \phi_{l,i} \ . \tag{4.1.3}$$

If $l < j$ we have three cases:

- if $\operatorname{supp} \phi_{l,i} \subset \tilde{\Omega}_{l+1}$ then only (4.1.3) has to be observed,
- for i with $\operatorname{supp} \phi_{l,i} \cap \Omega_l^{**} \neq \emptyset$ there is at least one l-cube (denoted by $\Box_{l,i}$) in the intersection of $\operatorname{supp} \phi_{l,i}$ and Ω_l^{**},
- finally, for the remaining $i \notin \tilde{\omega}_l$ we fix as $\Box_{l,i}$ an l-cube outside $\tilde{\Omega}_0$ which satisfies (4.1.3) or, if this is impossible, is closest to $\operatorname{supp} \phi_{l,i}$ (according to the first two cases and **(G2)**, this distance cannot exceed $C2^{-l}$).

By Lemma 2, for $0 < s < r + 3/2$, the second norm equivalence in (3.2.7) holds. In particular, it applies to all $\tilde{u}_j \in \tilde{V}_j \subset V_j$. By construction, we have $P_l u_j = u_j$ for all $l \geq j$ and $u_j \in V_j$. Hence,

$$||\tilde{u}_j||^2_{H^s(\mathbb{R}^d)} \asymp ||P_0 \tilde{u}_j||^2_{L_2(\mathbb{R}^d)} + \sum_{l=1}^{j} 2^{2ls} ||(P_l - P_{l-1})\tilde{u}_j||^2_{L_2(\mathbb{R}^d)} \ . \tag{4.1.4}$$

Now we observe that according to our specific choices of the cubes $\Box_{l,i}$, for $l < j$ the functionals $\lambda_{l,i}(\tilde{u}_j)$ given by (3.2.9) vanish if the index i corresponds to the last case. Moreover, on the set Ω_l^{**} we have $P_l \tilde{u}_j = \tilde{u}_j$. Indeed, let us fix l. By definition of the index sets $\tilde{\omega}_{l'}$, each $\tilde{u}_j \in \tilde{V}_j$ can be split into two parts

$$\tilde{u}_j = \underbrace{\sum_{i\,:\,\mathrm{supp}\,\phi_{l,i}\subset\tilde{\Omega}_0} c_{l,i}\phi_{l,i}}_{\equiv v_l} + \underbrace{\sum_{l'=l+1}^{j} \sum_{i\in\tilde{\omega}_{l'}} c_{l',i}\phi_{l',i}}_{\equiv w_{l+1}} \ ,$$

where w_{l+1} has support in $\tilde{\Omega}_{l+1}$. For i corresponding to the second case (where $\Box_{l,i} \subset \mathrm{supp}\,\phi_{l,i} \cap \Omega_l^{**} = \tilde{\Omega}_0 \backslash \tilde{\Omega}_{l+1}$), we get by examining the values of the linear functionals $\lambda_{l,i}(\cdot)$

$$\lambda_{l,i}(v_l) = \begin{cases} c_{l,i} & \text{if} \quad \mathrm{supp}\,\phi_{l,i} \subset \tilde{\Omega}_0 \\ 0 & \text{otherwise} \end{cases} \quad , \qquad \lambda_{l,i}(w_{l+1}) = 0 \ .$$

Since there is no contribution from i corresponding to the third case, we have

$$\tilde{u}_j - P_l\tilde{u}_j = \sum_{i\,:\,\mathrm{supp}\,\phi_{l,i}\subset\tilde{\Omega}_{l+1}} \tilde{c}_{l,i}\phi_{l,i} + w_{l+1} \ .$$

This shows the coincidence of $P_l\tilde{u}_j$ and $\tilde{u}_j$ on Ω_l^{**}.

In turn, this implies that the difference

$$\begin{aligned} P_l\tilde{u}_j - P_{l-1}\tilde{u}_j &= (\tilde{u}_j - P_{l-1}\tilde{u}_j) - (\tilde{u}_j - P_l\tilde{u}_j) \\ &= (w_l - w_{l+1}) + \sum_{i\,:\,\mathrm{supp}\,\phi_{l-1,i}\subset\tilde{\Omega}_l} \tilde{c}_{l-1,i}\phi_{l-1,i} \\ &\qquad - \sum_{i\,:\,\mathrm{supp}\,\phi_{l,i}\subset\tilde{\Omega}_{l+1}} \tilde{c}_{l,i}\phi_{l,i} \\ &= \sum_{i\in\tilde{\omega}_l} \tilde{c}_{l,i}\phi_{l,i} \end{aligned}$$

belongs to the subspace

$$\hat{V}_l = \mathrm{span}\{\phi_{l,i} \ : \ i \in \tilde{\omega}_l\} \ . \tag{4.1.5}$$

The latter produce, according to (3.2.1), a subspace splitting of $\tilde{V}_j$, for which we have just proved the following analog of Lemma 2.

Lemma 4. *Assume that the construction of* $\{\tilde{\Omega}_l\}$, $\{\tilde{\omega}_l\}$ *(see (3.2.3)) from the sequence* Ω_l^* *satisfies the above conditions, especially* **(G2)**. *Let the spaces* $\tilde{V}_j$ *and* $\hat{V}_l$ *be defined by (3.2.1) and (4.1.5), resp.. Then, for* $0 < s < r + 3/2$, *we have the norm equivalence*

$$\|\tilde{u}_j\|^2_{H^s(\mathbb{R}^d)} \asymp (|||\tilde{u}_j|||\hat{}_s)^2 \equiv \inf_{u_l^* \in \hat{V}_l \,:\, \tilde{u}_j = \sum_{l=0}^j \hat{u}_l} \sum_{l=0}^{j} 2^{2js} \|\hat{u}_l\|^2_{L_2(\mathbb{R}^d)} \qquad (4.1.6)$$

for all $\tilde{u}_j \in \tilde{V}_j$. *The constants in (4.1.6) depend on the constants in the norm equivalencies of Lemma 2 and in* **(G2)**.

One direction of the two-sided estimate follows from

$$\|\tilde{u}_j\|^2_{H^s(\mathbb{R}^d)} \preceq |||\tilde{u}_j|||^2_s$$

(see (3.2.7) in Lemma 2), and

$$|||\tilde{u}_j|||^2_s \le (|||\tilde{u}_j|||\hat{}_s)^2 \ ,$$

which is obvious by the infimum definition of the $|||\cdot|||$-norms.

The other direction is the consequence of our construction of the family $\{P_l\}$ of quasi-interpolants as given above, specifically, of (4.1.4) and $(P_l - P_{l-1})\tilde{u}_j \in V_l^*$:

$$(|||\tilde{u}_j|||\hat{}_s)^2 \le \|P_0\tilde{u}_j\|^2_{L_2(\mathbb{R}^d)} + \sum_{l=1}^{j} 2^{2ls} \|(P_l - P_{l-1})\tilde{u}_j\|^2_{L_2(\mathbb{R}^d)} \preceq \|\tilde{u}_j\|^2_{H^s(\mathbb{R}^d)} \ .$$

The result of Lemma 4 is the final preparation for the construction of preconditioners. On its own, according to the theory of subspace correction methods outlined in Section 2, it yields a frame-based multilevel preconditioner for the $\tilde{V}_j$-discretization for any symmetric $H^s(\mathbb{R}^d)$-elliptic variational problem $(0 < s < r + 3/2)$.

To be precise, let $\tilde{a}(\cdot,\cdot)$ be a symmetric positive definite bilinear form on $H^s(\mathbb{R}^d)$. In particular,

$$\tilde{a}(u,u) \asymp \|u\|^2_{H^s(\mathbb{R}^d)} \qquad \forall\ u \in H^s(\mathbb{R}^d) \ .$$

Consider the variational problem

$$\text{Find} \quad \tilde{u}_j \in \tilde{V}_j \quad \text{such that} \qquad \tilde{a}(\tilde{u}_j, \tilde{v}_j) = \tilde{\Phi}(\tilde{v}_j) \quad \forall\ \tilde{v}_j \in \tilde{V}_j \ , \qquad (4.1.7)$$

where $\tilde{\Phi}$ is a linear functional on $\tilde{V}_j$. Using $\tilde{\mathcal{B}}_j$ defined in (4.1.1) as the standard basis in $\tilde{V}_j$, the problem (4.1.7) turns into a linear system

$$\tilde{A}_j \tilde{x}_j = \tilde{f}_j \ , \qquad (4.1.8)$$

where the vector $\tilde{x}_j = \{\tilde{x}_{l,i} \, : \, i \in \omega_l^*, \; l = 0, \ldots, j\}$ contains the unknown coefficients of the basis representation

$$\tilde{u}_j = \sum_{l=0}^{j} \sum_{i \in \omega_l^*} \tilde{x}_{l,i} \phi_{l,i}$$

of the solution $\tilde{u}_j$ of (4.1.7). The symmetric positive definite matrix $\tilde{A}_j$ resp. the vector $\tilde{f}_j$ are given by their elements

$$\tilde{a}_{j;(l,i),(l',i')} = \tilde{a}(\phi_{l,i}, \phi_{l',i'}) \quad \text{resp.} \quad \tilde{f}_{j;(l,i)} = \tilde{\Phi}(\phi_{l,i}) \ .$$

If no additional precaution is taken it may happen even for integer s that $\tilde{A}_j$ is rather dense, and contains significantly more than $\mathrm{O}(n_j)$ nonzero elements where $n_j = \dim \tilde{V}_j$. We will postpone presenting a cure for this undesired feature of our above construction until Section 5.3, and concentrate first on preconditioning $\tilde{A}_j$. What is certain is that the spectral condition number of $\tilde{A}_j$ satisfies

$$\kappa(\tilde{A}_j) \preceq C 2^{2js} \ ,$$

which is typically attained in the asymptotic range, *i.e.* , for $j \to \infty$.

To construct the preconditioner, consider the additive Schwarz formulation (2.4) associated with the splitting (3.2.1) (compare also (2.10)):

$$\tilde{\mathcal{P}}_j \tilde{u}_j \equiv \sum_{l=0}^{j} \sum_{i \in \tilde{\omega}_l} \frac{\tilde{a}(\tilde{u}_j, \phi_{l,i})}{d_{l,i}} \phi_{l,i} = \tilde{\phi}_j \equiv \sum_{l=0}^{j} \sum_{i \in \tilde{\omega}_l} \frac{\tilde{\Phi}(\phi_{l,i})}{d_{l,i}} \phi_{l,i} \ . \tag{4.1.9}$$

For the scalings $d_{l,i}$, any choice satisfying

$$d_{l,i} \asymp 2^{2ls} \|\phi_{l,i}\|^2_{L_2(\mathbf{R}^d)} \tag{4.1.10}$$

will be appropriate. For practical implementations, a good choice is

$$d_{l,i} = \tilde{a}(\phi_{l,i}, \phi_{l,i}) \tag{4.1.11}$$

for which (4.1.10) usually holds (for spline basis functions and integer $s \leq r+1$ this can be verified directly).

As was shown in Section 2, the matrix representation of the equation (4.1.9) takes the form

$$\tilde{C}_j \tilde{A}_j \tilde{x}_j = \tilde{C}_j \tilde{f}_j \ , \tag{4.1.12}$$

where the multiplication by $\tilde{C}_j$ corresponds to a sparse matrix multiplication (the recursive structure of $\tilde{C}_j$ is slightly more complicated than shown in (2.20), due to the more complicated structure of the basis $\tilde{\mathcal{B}}_j$ defining the stiffness matrix $\tilde{A}_j$). This becomes obvious if one considers how the coefficient vector $\tilde{y}_j$ of $\tilde{\mathcal{P}}_j \tilde{u}_j$ is computed from the coefficient vector $\tilde{x}_j$ of $\tilde{u}_j$. To describe the details, let m_l resp. $m_l^* \le m_l$ denote the number of indices in $\tilde{\omega}_l$ resp. ω_l^*, $l = 0, 1, \ldots, j$. Note that $n_j = \sum_l m_l^*$, and set $d_l = m_l - m_l^*$. In a first step, $\tilde{A}_j \tilde{x}_j$ yields the values of $\tilde{a}(\tilde{u}_j, \phi_{l,i})$ for all $i \in \omega_l^*$ and $l = 0, \ldots, j$. After this, to compute all remaining coefficients in (4.1.9) and to get $\tilde{y}_j$ by suitable summation, one has to implement a V-cycle algorithm which corresponds to $\tilde{C}_j$. Since $\tilde{\omega}_j = \omega_j^*$ by construction, we have already all necessary $\tilde{a}(\tilde{u}_j, \phi_{l,i})$ in (4.1.9) for $l = j$. For any $i \in \tilde{\omega}_{j-1} \backslash \omega_{j-1}^*$, the support of $\phi_{j-1,i}$ is completely contained in $\tilde{\Omega}_j$, i.e., the unique basis representation of $\phi_{j-1,l}$ in the B-spline basis of V_j needs only basis functions $\phi_{j,i'}$ with $i' \in \tilde{\omega}_j$. Using the linearity of $\tilde{a}(\cdot,\cdot)$ in the second argument, we can compute the value $\tilde{a}(\tilde{u}_j, \phi_{j-1,i})$ from the available values $\tilde{a}(\tilde{u}_j, \phi_{j,i'})$. Thus, after this operation which can be described by a rectangular matrix $\tilde{S}_j$ of dimension $(n_j + d_{j-1}) \times n_j$, we get a vector containing all $\tilde{a}(\tilde{u}_j, \phi_{l,i})$ with $i \in \tilde{\omega}_l$ for $l \ge j-1$ and $i \in \omega_l^*$ for $l < j-1$. It is easy to see that a multiplication by $\tilde{S}_j$ can be performed in $\mathrm{O}(d_{j-1})$ arithmetical operations, and that there is no principal need fort storing this matrix since its entries come (by using the translation-dilation invariance of the bases) from a finite number of coefficient sets describing the expressions of the few types of different B-splines corresponding to V_0 in the basis of V_1.

This process can now be repeated leading after j steps to the vector $\tilde{S}_1 \cdots \tilde{S}_j \tilde{A}_j \tilde{x}_j$ of length $n_j + d_{l-1} + \ldots + d_0$ which contains the values $\tilde{a}(\tilde{u}_j, \phi_{l,i})$ for all $i \in \tilde{\omega}_l$ and $l \le j$. Applying a diagonal matrix $\tilde{D}$ serves for the scaling by $d_{l,i}$ (which can be precomputed and stored, if necessary). Finally, going the inverse direction from $l = 1$ to $l = j$, the vector $\tilde{y}_j$ can be computed by using again the expressions for $\phi_{l-1,i}$ with $i \in \tilde{\omega}_{l-1} \backslash \omega_{l-1}^*$ in terms of $\phi_{l,i'}$ with $i' \in \tilde{\omega}_l$ to eliminate all terms in (4.1.9) involving $\phi_{l,i} \notin \mathcal{B}_j$. A close look to this process yields

$$\tilde{y}_j = \tilde{S}_j^T \cdots \tilde{S}_1^T \tilde{D} \tilde{S}_1 \cdots \tilde{S}_j \tilde{A}_j \ ,$$

where the multiplication by the symmetric preconditioning matrix

$$\tilde{C}_j = \tilde{S}_j^T \cdots \tilde{S}_1^T \tilde{D} \tilde{S}_1 \cdots \tilde{S}_j \tag{4.1.13}$$

can be performed by $\mathrm{O}(n_j + d_0 + d_1 + \cdots + d_{j-1})$ operations. It is easy to see that under the conditions of Subsection 3.2, this operation count can be bounded by $\mathrm{O}(n_j)$. In general, one has to take care of the behavior of $\{d_l\}$ (and also about the complexity of a matrix-vector multiplication with $\tilde{A}_j$) by additional assumptions (see Subsection 5.3).

The general theory of subspace correction methods leads in conjunction with Lemma 4 to the following

Theorem 2. *Let the assumptions of Lemma 4 hold. Then, for any symmetric $H^s(\mathbb{R}^d)$-elliptic bilinear form $\tilde{a}(\cdot,\cdot)$ $(0 < s < r + 3/2)$, the additive Schwarz operator $\tilde{\mathcal{P}}_j$ in (4.1.9), (4.1.10), corresponding to the splitting (3.2.1) of $\tilde{V}_j$ is symmetric with respect to $\tilde{a}(\cdot,\cdot)$ and has uniformly bounded spectral condition number*

$$\kappa(\tilde{\mathcal{P}}_j) \le C \ ,$$

where the constant C depends on k, r, s, the ellipticity constants of $\tilde{a}(\cdot,\cdot)$, and the constants in **(G2)** *and (4.1.10).*

As a consequence, the number of iterations to reach a fixed error reduction in a Richardson iteration for (4.1.12) (algorithm **A***) or preconditioned conjugate gradient algorithm with preconditioning matrix $\tilde{C}_j$ for (4.1.8) is bounded independently of j and other specifics (except for the constant in* **(G2)***) of the construction of the subspaces $\tilde{V}_j$.*

To prove Theorem 2, we need to establish the stability of the splitting

$$\{\tilde{V}_j; \tilde{a}(\cdot,\cdot)\} = \sum_{l=0}^{j} \sum_{i\in\tilde{\omega}_l} \{V_{l,i}; 2^{2js}(\cdot,\cdot)_{L_2(\mathbb{R}^d)}\}$$

expressed by the norm equivalence

$$\tilde{a}(\tilde{u}_j,\tilde{u}_j) \asymp \inf_{u_{l,i}\in V_{l,i}\,:\,\tilde{u}_j=\sum_l\sum_i u_{l,i}} \sum_{l=0}^{j}\sum_{i\in\tilde{\omega}_l} 2^{2js}\|u_{l,j}\|^2_{L_2(\mathbb{R}^d)} \tag{4.1.14}$$

which has to be verified for all $\tilde{u}_j \in \tilde{V}_j$ (compare Section 2). Since any

$$\hat{u}_l = \sum_{i\in\tilde{\omega}_l} c_{l,i}\phi_{l,i} \equiv \sum_{i\in\tilde{\omega}_l} u_{l,i}$$

from $\hat{V}_l$ satisfies

$$\|\hat{u}_l\|^2_{L_2(\mathbb{R}^d)} \asymp \sum_{i\in\tilde{\omega}_l} 2^{-ld}c^2_{l,i} \asymp \sum_{i\in\tilde{\omega}_l} \|u_{l,i}\|^2_{L_2(\mathbb{R}^d)} \ ,$$

we get, using the L_2-stability of the B-spline basis,

$$\inf_{u_{l,i}\in V_{l,i}\,:\,\tilde{u}_j=\sum_l\sum_i u_{l,i}} \sum_{l=0}^{j}\sum_{i\in\tilde{\omega}_l} 2^{2js}\|u_{l,j}\|^2_{L_2(\mathbb{R}^d)}$$

$$\asymp \inf_{\hat{u}_l\in\hat{V}_l\,:\,\tilde{u}_j=\sum_l \hat{u}_l} \sum_{l=0}^{j} 2^{2js}\|\hat{u}_l\|^2_{L_2(\mathbb{R}^d)} = (|||\tilde{u}_j|||\hat{_s})^2 \ .$$

Since by the H^s-ellipticity $\tilde{a}(\tilde{u}_j,\tilde{u}_j) \asymp \|\tilde{u}_j\|^2_{H^s(\mathbb{R}^d)}$, Lemma 4 gives the result.

Theorem 2 can be interpreted as a result about *subframes* generated from the set of all B-splines $\{\phi_{l,i}\}$ (see Section 2 for the definition of frames and their connection with stable subspace splittings).

Proposition 1. *Under the assumptions of Theorem 1, the system*

$$\tilde{\mathcal{F}}_j^s = \{\frac{1}{2^{ls}\|\phi_{l,i}\|_{L_2(\mathbb{R}^d)}}\phi_{l,i} \ : \ i \in \tilde{\omega}_l \ , \ l = 0,\ldots,j\}$$

is a frame in $\tilde{V}_j$ equipped with the H^s-elliptic scalar product $\tilde{a}(\cdot,\cdot)$. The frame constants are bounded independently of the specific $\tilde{V}_j$. They depend on the same quantities as the condition number of $\tilde{\mathcal{P}}_j$. The statement holds also for infinite-dimensional spaces $\tilde{V}$ produced by the above construction if $j \to \infty$.

Note that the frames $\tilde{\mathcal{F}}_j^s$ are obtained from the subset $\tilde{\mathcal{F}}_j$ of the set of all B-splines $\tilde{\mathcal{F}} = \{\phi_{l,i}\}$ by suitable scaling, and that the scaled version $\tilde{\mathcal{F}}^s$ of the infinite set $\tilde{\mathcal{F}}$ forms a frame in $H^s(\mathbb{R}^d)$, $0 < s < r + 3/2$. This is a consequence of the general theory and Lemma 2. Thus, the conditions formulated for the construction of $\tilde{V}_j$ are sufficient conditions for a subset of $\tilde{\mathcal{F}}^s$ to form a frame in its linear span considered as a subspace of $H^s(\mathbb{R}^d)$, without disturbing the frame constants too much. Thus, it is easy to construct simple frames with nice properties in subspaces of $H^s(\mathbb{R}^d)$.

A last comment: If the finite sequences $\{\Omega_l^* \ : \ l \le j\}$ the construction of this subsection starts with, are defined (for different j) as sections of a fixed infinite sequence $\{\Omega_l^*\}$, then the resulting sequence of subspaces $\tilde{V}_j$ from (3.2.1) as well as the index sets $\tilde{\omega}_j$ are increasing in j. Another obvious case where the subspaces $\tilde{V}_j$ as well as the frames $\tilde{\mathcal{F}}_j^s$ are monotone in j is the construction of Subsection 3.2. There are more general situations arising, *e.g.* , in practical adaptive refinement applications (see [62]), where the resulting frames are obtained by adding and, possibly, deleting new functions. These have not yet been studied in a rigorous way.

4.2 H^s-preconditioner for $V_{j,\Omega}$

We present now a preconditioner for the linear system

$$A_j x_j = f_j \tag{4.2.1}$$

which is the discretization of the variational problem (3.1.16) with respect to the basis

$$\mathcal{B}_{j,\Omega} = \{\phi_{j,i,\Omega} \ : \ i \in \omega_j\}$$

of the discretization space $V_{j,\Omega}$ described in Subsection 3.1. Recall that we have two types of basis functions: the "boundary-adapted" basis functions $\phi_{j,i,\Omega} = M_{\partial\omega_\Box \cup \omega_\Box} \phi_{\Box,i}|_\Omega$ if $i \in \omega_\Box$ for some $\Box \in C_j$, and "unmodified" basis functions $\phi_{j,i,\Omega} = \phi_{j,i}|_\Omega$ if $i \in \omega_j'$. The solution vector x_j of (4.2.1) represents the coefficient vector of the solution u_j of (3.1.16):

$$u_j = \sum_{i \in \omega_j} x_{j,i} \phi_{j,i,\Omega} \, .$$

We do not detail the assembly process of the matrix A_j and the vector f_j which involves the typical integrals (over Ω resp. $\partial\Omega$) for products of basis functions $\phi_{j,i}$ resp. input functions from the variational problem plus some local transformations corresponding to the newly introduced functions $\phi_{\Box,i}$, $i \in \omega_\Box$. Since the partitions $\mathcal{R}_j$ are not adapted to the boundary $\partial\Omega$, it seems to be necessary to modify existing codes for uniform rectangular grids by some "boundary integration" rules. But this might be the only serious change in this part compared to the situation of a rectangular domain.

Now we put together the results of Subsection 3.1, 3.2, and 4.1. Roughly speaking, our preconditioner for (4.2.1) is the result of switching from the $V_{j,\Omega}$-discretization to the associated $\tilde{V}_j$-discretization as described in Subsection 3.2. The advantage is that for the latter an asymptotically optimal multilevel preconditioner $\tilde{C}_j$ is already available (see Subsection 4.1), and that the "switch" is just a two-level method and easy to understand. Let us comment that the idea of switching from a given discretization to a close-by discretization of similar complexity for which fast solvers are available is by now standard in the field, and can be successfully used for theoretical and implementational purposes (see [14, 61, 13]). Let us preserve the notation R_j for the matrix representation (with respect to the bases $\tilde{\mathcal{B}}_j$ in $\tilde{V}_j$ resp. $\mathcal{B}_{j,\Omega}$ in $V_{j,\Omega}$) of the restriction operator defined by (3.2.5) and (3.1.9). This matrix is rather sparse and can be implemented by half of a V-cycle (to compute the B-spline-coefficients in V_j corresponding to $\tilde{u}_j|_{\Omega_j}$ from the given $\tilde{x}_j$) and some local transformations involving the values of the biorthogonal functions $\eta_{\Box,i}$ for $i \in \omega_\Box$, $\Box \in \mathcal{C}_j$. The first part can be avoided if the control parameter D is chosen sufficently large: compare the construction of $\{\tilde{\Omega}_l\}$ in Subsection 3.2.

Theorem 3. *Let the bilinear form $a(\cdot,\cdot)$ be symmetric and $H^s(\Omega)$-elliptic $(0 < s \le r+1)$, and let the remaining assumptions of Lemma 3 be satisfied. Then*

$$C_j = R_j \tilde{C}_j R_j^T$$

is a symmetric preconditioning matrix for A_j which satisfies

$$\kappa(C_jA_j) \equiv \frac{\lambda_{\max}(C_jA_j)}{\lambda_{\min}(C_jA_j)} \le C \ , \tag{4.2.2}$$

where the constant C depends on k, r, s, the ellipticity constants of $a(\cdot,\cdot)$, and the constant in **(G1)**. *Thus, applying the pcg-algorithm to (4.2.1) yields uniformly bounded iteration numbers for a fixed error reduction if the constant in* **(G1)** *is independent of $j \to \infty$. The operation count for a matrix-vector multiplication with C_j is $O(\dim V_{j,\Omega})$.*

Proof: This is a simple consequence of the fictitious space lemma of Nepomnyaschikh, see, *e.g.* , [38, 66], or [60, Theorem 17], and the norm estimates of the previous subsections. We give the full argument. Let us fix any $H^s(\mathbb{R}^d)$-elliptic bilinear form $\tilde{a}(\cdot,\cdot)$. For example, take the scalar product of $H^s(\mathbb{R}^d)$. Since the construction of Subsection 3.2 automatically yields **(G2)**, we have the result of Theorem 2:

$$((\tilde{A}_j\tilde{C}_j\tilde{A}_j\tilde{x}_j,\tilde{x}_j)) = \tilde{a}(\tilde{\mathcal{P}}_j\tilde{u}_j,\tilde{u}_j) \asymp \tilde{a}_j(\tilde{u}_j,\tilde{x}_j) = ((\tilde{A}_j\tilde{x}_j,\tilde{x}_j))$$

which yields (by substituting $\tilde{A}_j^{-1}\tilde{x}_j$ instead of $\tilde{x}_j$)

$$((\tilde{C}_j\tilde{x}_j,\tilde{x}_j)) \asymp ((\tilde{A}_j^{-1}\tilde{x}_j,\tilde{x}_j)) \ , \qquad \forall\, \tilde{x}_j \ .$$

Here and in the sequel, $((\cdot,\cdot))$ denotes the Euclidean product of $\mathbb{R}^n$-vectors where the dimension n should be clear from the context. It is easy to observe that $X = R_j\tilde{A}_j^{-1}R_j^TA_j$ is invertible (to this end, show the positivity of

$$((A_jXx_j,x_j)) = ((\tilde{A}_j^{-1}R_j^TA_jx_j, R_j^TA_jx_j)) \ge \|R_j^TA_jx_j\|^2$$

by using the surjectivity of R_j). By the above spectral equivalence of $\tilde{C}_j$ and $\tilde{A}_j^{-1}$, we have

$$\begin{aligned}((A_jR_j\tilde{C}_jR_j^TA_jx_j,x_j)) &= ((\tilde{C}_j\tilde{y}_j,\tilde{y}_j)) \\ &\asymp ((\tilde{A}_j^{-1}\tilde{y}_j,\tilde{y}_j)) = ((A_jXx_j,x_j)) \ ,\end{aligned} \tag{4.2.3}$$

where $\tilde{y}_j = R_j^TA_jx_j$.

Since R_j is onto, for each x_j there is at least one $\tilde{x}_j$ such that $x_j = R_j\tilde{x}_j$. Thus,

$$((A_jX^{-1}x_j,x_j)) = ((\tilde{A}_j\tilde{y}_j,\tilde{x}_j)) \le \sqrt{((\tilde{A}_j\tilde{y}_j,\tilde{y}_j))}\sqrt{((\tilde{A}_j\tilde{x}_j,\tilde{x}_j))} \ ,$$

where $\tilde{y}_j = \tilde{A}_j^{-1}R_j^TA_jX^{-1}R_j\tilde{x}_j$. Since

$$(\tilde{A}_j\tilde{y}_j,\tilde{y}_j)) = ((R_j\tilde{A}_j^{-1}R_jA_jX^{-1}x_j, A_jX^{-1}x_j)) = ((A_jX^{-1}x_j,x_j)) \ ,$$

we obtain $((A_j X^{-1} x_j, x_j)) \leq ((\tilde{A}_j \tilde{x}_j, \tilde{x}_j))$ for any of these $\tilde{x}_j$. The particular choice $\tilde{x}_j = \tilde{y}_j$ which indeed satisfies $R_j \tilde{y}_j = R_j \tilde{A}_j^{-1} R_j^T A_j X^{-1} x_j = x_j$ shows that equality is attained. This implies

$$((A_j X^{-1} x_j, x_j)) = \inf_{\tilde{x}_j \,:\, x_j = R_j \tilde{x}_j} ((\tilde{A}_j \tilde{x}_j, \tilde{x}_j)) \; . \tag{4.2.4}$$

Now Lemma 3 and the assumed H^s-ellipticity of the bilinear forms come into play. Consider all $\tilde{u}_j \in \tilde{V}_j$ such that $u_j = R_j \tilde{u}_j$. By (3.2.12) we have

$$a(u_j, u_j) = a(R_j \tilde{u}_j, R_j \tilde{u}_j) \leq C \tilde{a}(\tilde{u}_j, \tilde{u}_j) \; ,$$

while the particular choice $\tilde{u}_j = E_j u_j$ satisfies $u_j = R_j \tilde{u}_j$ according to (3.2.11), and (3.2.13) gives

$$\tilde{a}(\tilde{u}_j, \tilde{u}_j) = \tilde{a}(E_j u_j, E_j u_j) \leq C a(u_j, u_j) \; .$$

Altogether, we arrive at

$$a(u_j, u_j) \asymp \inf_{\tilde{u}_j \,:\, u_j = R_j \tilde{u}_j} \tilde{a}(\tilde{u}_j, \tilde{u}_j) \; ,$$

which by (4.2.3) and (4.2.4) yields

$$\kappa(C_j A_j) \asymp \kappa(X) = \kappa(X^{-1}) \leq C \; .$$

Theorem 3 is established.

§5 Extensions

5.1 General multiresolution analyses

When analyzing the considerations of the above sections, we see that not too many "spline-specific" properties have been used. A generalization to other types of scaling functions and multiresolution analyses on $\mathbb{R}^d$ is straightforward under the following conditions.

(A) The scaling functions $\phi^1, \ldots, \phi^L \in L_2(\mathbb{R})^d$ possess local support and are refinable:

$$\phi^l(x) = \sum_{l'=1}^{L} \sum_{\beta \in \mathbf{Z}^d} a_\beta^{l,l'} \phi^{l'}(2x - \beta) \; , \quad l = 1, \ldots, L \; .$$

(B) We have $\phi^l \in H^t(\mathbb{R}^d)$, $l = 1, \ldots, L$, for some $t > 1$.

(C) The integer translates of the scaling functions form a Riesz basis $\mathcal{N}_0$ in the L_2-closure V_0 of their span, *i.e.* ,

$$u = \sum_{l=1}^{L} \sum_{\beta \in \mathbf{Z}^d} c^l_\beta \phi^l(\cdot - \beta) \in V_0 \Longleftrightarrow \|u\|^2_{L_2(\mathbf{R}^d)} \asymp \sum_{l=1}^{L} \sum_{\beta \in \mathbf{Z}^d} (c^l_\beta)^2$$

for all $(l_2(\mathbf{Z}^d))^L$-sequences.

(D) Polynomials of total degree $\leq k$ can be represented by $\mathcal{N}_0$, *i.e.* , there exist coefficient sequences $\{c^{l,\alpha}_\beta\}$ such that

$$x^\alpha = \sum_{l=1}^{L} \sum_{\beta \in \mathbf{Z}^d} c^{l,\alpha}_\beta \phi^l(\cdot - \beta) \ , \quad |\alpha| \leq k \ .$$

(E) The functions in $\mathcal{N}_0$ are locally linearly independent, *i.e.* , if

$$\sum_{l=1}^{L} \sum_{\beta \in \mathbf{Z}^d} c^l_\beta \phi^l(\cdot - \beta) \equiv 0$$

on the unit cube $\Box_0$ then $c^l_\beta = 0$ for all index pairs (l, β) such that $\operatorname{supp} \phi^l(\cdot - \beta) \cap \Box_0 \neq \emptyset$.

Under these sufficient (and partly overlapping) conditions, the above results on approximating and solving $H^s(\Omega)$-elliptic problems remain valid under the same geometric assumptions on Ω, at least, for integer $1 \leq s \leq s_0$, where s_0 is the largest integer less than both t and $k+1$. The practical implementation is restricted to examples where the support of the scaling functions is reasonably small, and the control parameter D is not too large. Otherwise, boundary modifications and the geometric set-up will become increasingly difficult.

The crucial assumption which makes our geometric, cube-partition-oriented approach possible is **(E)** which is stronger than **(C)**. For results on **(E)**, see [47, 9]. Assumptions **(A)**, **(C)**, **(D)** are standard, and can be studied on the basis of the refinement equation form **(A)** (see [16, 30]). The calculation of the optimal smoothness parameter t in **(B)** is a delicate issue, compare [30, 31, 50]. Most of the papers on multiresolution analyses study the case $L = 1$, for the so-called multiwavelet case $L > 1$, see [28].

Assumptions **(A)**-**(D)** yield characterizations of Sobolev spaces $H^s(\mathrm{I\!R}^d)$, $0 < s \leq s_0$, analogous to Lemma 2 (see [25] for a survey on multilevel approximations and related function spaces). **(E)** is essentially used in the definition of the spaces on domains and for the corresponding quasi-interpolants. It is also advisable to assume **(E)** in order to avoid a very complicated implementation. We leave it to the reader to fill the details (some minor changes are necessary such as replacing the L_∞-estimates like in (3.1.7) by L_2-bounds).

5.2 Essential boundary conditions

For applications to boundary value problems for elliptic partial differential equations of second or fourth order, the case of *essential boundary conditions* is of importance. This leads to subspaces of H^s-spaces characterized by trace conditions on part of $\partial\Omega$. For simplicity, we only discuss the case of a $H_0^s(\Omega)$-problem (pure homogeneous Dirichlet boundary conditions) and integer $s = 1, \ldots, r+1$. Also, we avoid discussing regularity problems arising from the geometry of the domain by assuming that Ω is a C^∞ domain.

If we wish to stay within the framework of conforming discretizations, we need to modify the construction of $V_{j,\Omega}$ to make it a subspace of $H_0^s(\Omega)$. A natural approach is to consider only scaling functions with support in Ω as generating functions for $V_{j,\Omega}$. Then, in order to preserve the approximation order, we need to include in the definition (3.2.1) standard basis functions of level $> j$ near the boundary. The wavelet counterpart of this approach has been considered in [44, Section 3] (however, the author of [44] did not elaborate on solving the resulting linear systems in practice and on other efficiency and robustness issues).

Unfortunately, the construction leads to satisfactory results only for $d = 2$ while for $d \geq 3$ the newly constructed subspaces have either significantly larger dimension or still reduced approximation power. This coincides with analogous results of [44]. The alternative would be a more subtile boundary modification or a hybrid construction which uses more flexible finite element partitions and functions to better approximate the problem in a boundary strip. As a whole, the problem of general boundary conditions requires additional thought.

Let us again consider the sets $\Omega_j \subset \Omega \subset \Omega_j^e$ as defined in Subsection 3.1. Fix integers j, $J > j$, $D \geq 1$, and define Ω_l^*, $l \leq J$ as follows. For $l = J$, set

$$\Omega_J^* = \left(\cup_{\Box \subset \Omega_J \setminus \Omega_{J-1}} \Box\right) \cup \left(\cup_{\Box \subset \Omega_{J-1} \,:\, \mathrm{dist}(\Box, \partial\Omega_{J-1}) < D 2^{-J}} \Box\right) .$$

Now, if Ω_{l+1}^* is already constructed $(l = J-1, \ldots, j+1)$, set

$$\Omega_l^* = \left(\cup_{\Box \subset \Omega_l \setminus (\Omega_{l-1} \cup \Omega_{l+1}^*)} \Box\right) \cup \left(\cup_{\Box \subset \Omega_{l-1} \,:\, \mathrm{dist}(\Box, \partial\Omega_{l+1}^*) < D 2^{-l}} \Box\right) .$$

In these definitions, it is agreed that the requirement $\Box \subset \Omega_l$ automatically assumes that $\Box$ is a l-cube. Finally,

$$\Omega_j^* = \Omega_j \setminus \Omega_{j+1}^* , \qquad \Omega_l^* = \emptyset , \; l < j .$$

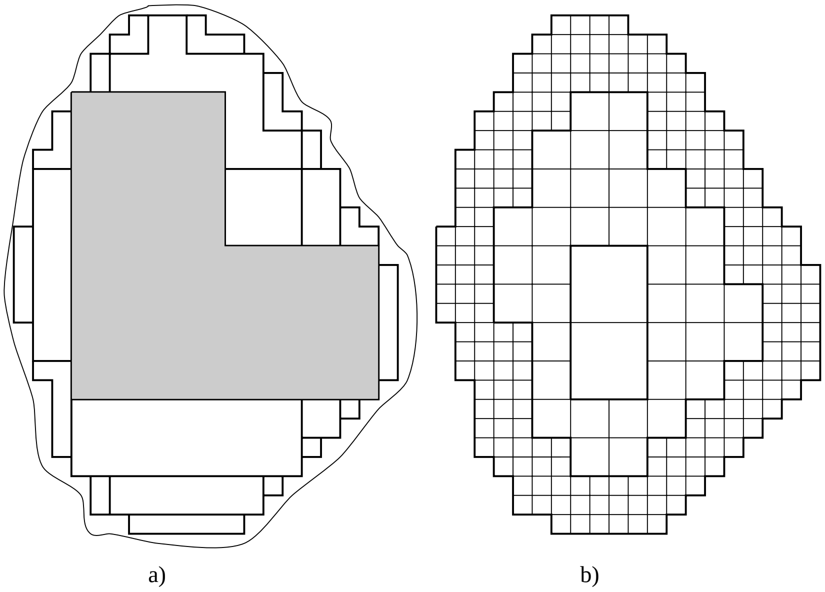


Figure 5. Boundary refinement: a) $\{\Omega_l\}$ and b) $\{\Omega_l^*\}$

The idea of this construction is to introduce a layer of regular refinement up to level J near the boundary of Ω_j. The choice of J is dictated by complexity considerations and will be discussed later. Figure 5 shows the construction for $j = 2$, $J = 5$, and $D = 1$. Note that in this hypothetical example $\Omega_l = \emptyset$ for $l = 0, 1$ and $\Omega_l^* = \emptyset$ for $l \leq 2$, resp.. The conforming subspace $V_{j,0} \equiv \tilde{V}_J \subset H_0^s(\Omega)$ is now defined via (3.2.1). In contrast to Subsection 3.1, a boundary modification by introducing new types of basis functions is avoided, at the price of enlarging the subspace by standard basis functions from levels $l > j$. What we claim is that for $d = 2$, under appropriate conditions on Ω, this subspace $V_{j,0}$ provides good approximation rates under the usual regularity assumptions on $u \in H_0^s(\Omega)$, satisfies the theory of subsection 4.1, and still has dimension comparable with the dimension of a (simplicial and boundary-adapted) finite element subspace of the same approximation power.

We will assume **(G2)** from Subsection 4.1. Since, by construction, $\tilde{\Omega}_0 = \Omega_J$, this is actually a condition on the thickness of the discrete boundaries $\Omega_l^e \backslash \Omega_l$, $l \leq J$. It is asymptotically satisfied for Lipschitz domains but the constant C in **(G2)** might be very large (due to the macroshape of the domain) for the computationally interesting range of small j and J. With this assumption, we can use Theorem 2 to get optimal condition number estimates for the additive Schwarz operator associated with the splitting from (3.2.1) for the $V_{j,0}$ discretization of a generic symmetric $H_0^s(\Omega)$-elliptic problem. All one has to do is to formally extend the symmetric bilinear form from $H_0^s(\Omega)$ to $H^s(\mathbb{R}^d)$.

Obviously, the dimension of $V_{j,0}$ will depend on the number of additional boundary layers $j < l \leq J$, and the size and shape of Ω itself. We assume again that $\Omega_0 \neq \emptyset$. This automatically yields $\dim V_{j,0} \succeq 2^{jd}$. As a by-product of **(G2)**, it can easily be seen that the number of l-cubes in Ω_l^* is proportional to $2^{l(d-1)}$. The constants may depend on D, the constant in **(G2)**, and the $(d-1)$-dimensional measure of $\partial\Omega$. This yields

$$2^{jd} \preceq \dim V_{j,0} \preceq (2^{jd} + \sum_{l=j+1}^{J} 2^{l(d-1)}) \ . \tag{5.2.1}$$

The upper bound, which asymptotically represents the correct dimension if D is large enough, yields the desired $\dim V_{j,0} \asymp 2^{jd}$ only if $J \leq jd/(d-1)$. For larger J, the bound is $\preceq 2^{J(d-1)}$.

Now we come to the more difficult part, the estimation of the approximation power of $V_{j,0}$. Under the above assumptions (recall that $s = 1, \ldots, r+1$ is an integer) we have for $m = 0, 1, \ldots, s$ the estimate

$$||u||_{H^m(\Omega_\delta)} \leq C\delta^{m-s+1/2}||u||_{H^{s+1}(\Omega)} \ , \qquad u \in H_0^s(\Omega) \cap H^{s+1}(\Omega) \ , \tag{5.2.2}$$

in a boundary strip $\Omega_\delta = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) \leq \delta)$. This estimate does not improve if u is more regular. Since functions from $V_{j,0}$ may vanish or are of very limited flexibility in a boundary strip of size $\asymp 2^{-J}$, we cannot expect to have error estimates better than $\mathrm{O}(2^{-J/2})$ in the energy norm (*i.e.* , in the $H^s(\Omega)$-norm), even for very smooth solutions of a boundary value problem in $H_0^s(\Omega)$.

Lemma 5. *Let the solution u of a $H_0^s(\Omega)$-elliptic variational problem belong to $H_0^s(\Omega) \cap H^{s+1}(\Omega)$. Suppose that u_j solves the variational problem discretized with respect to the subspace $V_{j,0} \subset H_0^s(\Omega)$ as constructed above. Then we have, under the above assumptions and for sufficiently large D, the error estimate*

$$||u - u_j||_{H^s(\Omega)} \leq C(2^{-J/2} + 2^{-j})||u||_{H^{s+1}(\Omega)} \ , \tag{5.2.3}$$

analogous estimates holding in other norms.

Proof: Again we construct a specific quasi-interpolant operator. To do this we enlarge Ω_J^* by adding the set $\mathbb{R}^d \backslash \Omega_J$. After this enlargement, the union of pairwise disjoint sets Ω_l^*, $l = j, \ldots, J$, is $\mathbb{R}^d$. This leads, via (3.2.1), to a space $\tilde{V}_{j,0} \subset V_J$ which contains both V_j and $V_{j,0}$. The algebraic bases for $V_{j,0}$ resp. $\tilde{V}_{j,0}$ are introduced along the lines of Subsection 4.2, and will be denoted by $\mathcal{B}_{j,0}$ resp. $\tilde{\mathcal{B}}_{j,0}$. Furthermore, by taking a sufficiently large D in the above construction, we may ensure the following condition to hold:

(G3) If $\phi_{l,i}$ and $\phi_{l',i'}$ belong to $\tilde{\mathcal{B}}_{j,0}$ and if their supports have a nontrivial intersection (with positive d-dimensional measure) then necessarily $|l-l'| \leq 1$ $(l, l' = j, j+1, \ldots, J)$.

Note that one cannot expect local linear independence of the basis $\tilde{\mathcal{B}}_{j,0}$ on all l-cubes $\square \in \Omega_l^*$. However, any polynomial of degree $\leq k$ has a unique representation with respect to $\tilde{\mathcal{B}}_{j,0}$ where, according to **(G3)**, the coefficients can be determined from local information. This will be used below. The condition **(G3)** also guarantees a good sparsity of the stiffness matrix with respect to $\mathcal{B}_{j,0}$, and is important to handle the implementation of the preconditioner in a simple way.

With each $\phi_{l,i} \in \tilde{\mathcal{B}}_{j,0}$, we associate an l-cube $\square_{l,i} \subset \Omega_l^*$ inside or at closest distance to $\operatorname{supp} \phi_{l,i}$, and such that $\square_{l,i} \subset \Omega_J$ whenever $\operatorname{supp} \phi_{l,i}$ intersects with Ω. By the construction of $\{\Omega_l^*\}$ and $\tilde{\mathcal{B}}_{j,0}$, the latter condition may lead to a choice for $\square_{l,i}$ outside $\operatorname{supp} \phi_{l,i}$ only if $l = J$ (and near the boundary of Ω). By **(G2)** we therefore have $\operatorname{dist}(\operatorname{supp} \phi_{l,i}, \square_{l,i}) \leq C 2^{-l}$ for all indices of interest.

We define

$$\tilde{Q}_{j,0} \; : \; u \in L_1(\mathbb{R}^d) \longmapsto \tilde{Q}_{j,0} u = \sum_{l=j}^{J} \sum_{i \in \omega_l^*} \lambda_{l,i}(u) \phi_{l,i} \in \tilde{V}_{j,0} \; , \qquad (5.2.4)$$

where the functionals $\lambda_{l,i}$ are defined via (3.2.9). The definition of $\eta_{l,i}$ in (3.2.9) is such that $\lambda_{l,i}(p)$ reproduces the coefficient in front of $\phi_{l,i}$ in the basis representation with respect to $\tilde{\mathcal{B}}_{j,0}$ for any polynomial p of degree $\leq k$, and satisfies the L_∞-bound

$$||\eta_{l,i}||_{L_\infty(\square_{l,i})} \prec 2^{ld} \; ,$$

uniformly in i, l.

The following considerations show that this is possible. Fix any of the cubes $\square_{l,i}$, and the affine transformation that transfers it into the unit cube $\square_0$. The function $\phi_{l,i}$ corresponds now to a certain function ϕ_{0,i_0} the support of which either contains $\square_0$ or is at distance $\leq C$ from the origin. Each monomial x^α with $|\alpha| \leq k$ possesses a unique representation

$$x^\alpha = \sum_{l'=j-l}^{J-l} \sum_{i}^{\prime} c_{l',i'}^{\alpha} \phi_{l',i'} \; ,$$

where the summation has to be carried out with respect to the basis obtained from $\tilde{\mathcal{B}}_{j,0}$ by the fixed affine transformation. By assumption **(G3)** and the local linear independence property (compare the construction of the basis (4.1.1)), the coefficients c^{α}_{0,i_0}, $|\alpha| \le k$, of interest depend only on the local geometry of the transferred cube partition associated with Ω_l and Ω_{l-1} and, if $l = J$, from the above distance bound. Since there are only finitely many different possibilities for these local cube partitions, we have an overall bound $|c^{\alpha}_{0,i_0}| \le C$. By the same arguments as at the beginning of Subsection 3.1, we find bounded functions η^{α} such that

$$p = \sum_{|\alpha| \le k} c^{\alpha} x^{\alpha} \quad \Longrightarrow \quad \int_{\Box_0} p \eta^{\alpha}\, dx = c^{\alpha} \quad (|\alpha| \le k) \; .$$

If we set $\eta'_{0,i_0} = \sum_{|\alpha| \le k} c^{\alpha}_{0,i_0} \eta^{\alpha}$, then

$$\int_{\Box_0} \eta'_{0,i_0} p \, dx = \sum_{|\alpha| \le k} c^{\alpha} c^{\alpha}_{0,i_0}$$

coincides with the coefficient of ϕ_{0,i_0} in the basis representation of p with respect to the transferred basis. Moreover, $||\eta'_{0,i_0}||_{L_\infty(\Box_0)} \prec 1$. Transferring back, we see that for some $\beta \in \mathbf{Z}^d$ the function $\eta_{l,i}(x) = 2^{ld} \eta'_{0,i_0}(2^l x - \beta)$ has the desired properties.

Now, the usual machinery of estimating the approximation error of quasi-interpolant operators can be applied to demonstrate Lemma 5. We omit the details, compare Subsection 3.1 for the main steps of the reasoning in a similar situation.

A closer look at the above arguments shows that only for $d = 2$ and minimal extra regularity $u \in H^{s+1}(\Omega)$ do we get the desired, optimal result when using $V_{j,0}$ as discretization space for an $H^s_0(\Omega)$-elliptic problem:

Theorem 4. *Let $d = 2$, and s be a positive integer. In the above construction of $V_{j,0}$, set $J = 2j$, assume* **(G2)**, *$\Omega_0 \ne \emptyset$, and choose D sufficiently large to guarantee* **(G3)** *and the desired "thickness" of the boundary strip Ω^*_J. Then*

a) *The dimension of $V_{j,0}$ satisfies $\dim V_{j,0} \asymp 2^{2j}$.*

b) *Let $u \in H^s_0(\Omega)$ resp. $u_{j,0} \in V_{j,0}$ denote the solutions of a symmetric H^s_Ω-elliptic problem resp. of its discretization with respect to $V_{j,0}$. If $u \in H^{s+1}(\Omega)$ then*

$$||u - u_j||_{H^s(\Omega)} \le C 2^{-j} ||u||_{H^{s+1}(\Omega)} \; . \qquad (5.2.5)$$

c) *The frame-based splitting (3.2.1) associated with $V_{j,0}$ leads to a preconditioner $C_{j,0}$ for the linear system*

$$A_{j,0} x_{j,0} = f_{j,0}$$

representing the $V_{j,0}$-discretization of the variational problem which satisfies

$$\kappa(C_{j,0}A_{j,0}) \le C \ . \tag{5.2.6}$$

The multiplications by $A_{j,0}$ and $C_{j,0}$ can be performed in $O(2^{2j})$ arithmetical operations.

For $d > 2$ the choice $J = 2j$ would still lead to (5.2.5) and (5.2.6), at the expense of a dimension of $V_{j,0}$ and work estimates which are larger than the desired $\asymp 2^{jd}$ bound by an exponentially growing factor. Alternatively, the choice $J = [dj/(d-1)]$ would asymptotically lead to the correct dimension. But then the convergence rate in (5.2.5) is expected to deteriorate from 2^{-j} to $2^{-jd/(2(d-1))}$.

5.3 Adaptivity

A formal advantage of multilevel Riesz basis and frame decompositions is that they provide *isomorphisms to coefficient spaces* for a number of Besov-Sobolev spaces on domains. This can be used for optimal approximation resp. compression purposes (see [53, 32, 33, 20] for some information in this direction). Roughly speaking, adaptivity strategies based on coefficient information are successful (and can be justified theoretically) if a suitable decomposition of the function under consideration is available at low cost. This is the case for many applications in signal and image processing but much harder to implement for the solution of operator equations (see, however, [21, 20] for a possible strategy).

As a compromise, we outline here an approach which is partly based on heuristic arguments. It is implemented in several adaptive finite element codes [3, 6] (see also [62] for a somewhat different variant), and has led to satisfactory results in standard applications. The ideas will be presented within the framework of Section 3 and 4, *i.e.* , for approximations to a symmetric $H^s(\Omega)$-elliptic problem by tensor-product splines, and integer $s = 1, \ldots, r+1$. Due to the lack of space, we do not give full proofs.

In order to have the flexibility necessary for an adaptive solver based on local refinement, we present the following construction. Let us start with an auxiliary family $\{\hat\Omega_l,\, l = j_0, \ldots, J\}$ of disjoint sets where each $\hat\Omega_l$ is a union of l-cubes from Ω_l. Here, j_0 is the number of the "coarsest" level, where the adaptive algorithm starts from (this number is fixed throughout the following exposition), while J indicates the number of the current "finest" level of refinement. Let $\{\hat\Omega_l^e,\, l = j_0, \ldots, J\}$ be another sequence, where $\hat\Omega_l^e$ is obtained from $\hat\Omega_l$ by *adding* some l-cubes from the boundary strip $\Omega_l^e \backslash \Omega_l$. Thus,

$$\hat\Omega_l \subset \hat\Omega_l^e \subset \Omega_l^e \ , \quad \hat\Omega_l^e \backslash \hat\Omega_l \subset \Omega_l^e \backslash \Omega_l \ , \qquad l = j_0, \ldots, J \ .$$

As before, we produce the sets

$$\tilde\Omega_j = \cup_{l=j}^J \hat\Omega_l\ , \quad \tilde\Omega_j^e = \cup_{l=j}^J \hat\Omega_l^e\ , \qquad j = j_0, \dots, J\ ,$$

which stand for the regions of j-th level refinement, and require that Ω is covered by $\tilde\Omega^e_{j_0}$. Note that by these definitions, $\tilde\Omega_{j_0}\backslash\tilde\Omega_j$ is a union of $(j-1)$-cubes from Ω_{j-1}.

To each $\hat\Omega_j$ we associate the index set

$$\hat\omega_j = \{i\ :\ \mathrm{supp}\,\phi_{j,i}\cap\hat\Omega_j \neq \emptyset\ ,\ \mathrm{supp}\,\phi_{j,i}\cap\tilde\Omega_{j_0}\backslash\tilde\Omega_j = \emptyset\} \tag{5.3.1}$$

and define sets of basis functions

$$\hat{\mathcal{B}}_j = \{\phi_{j,i}\ :\ i \in \hat\omega_j\}\ , \quad \hat{\mathcal{B}} = \cup_{j=j_0}^J \hat{\mathcal{B}}_j\ . \tag{5.3.2}$$

Analogously we introduce $\hat\omega_j^e$, $\hat{\mathcal{B}}_j^e$, and $\hat{\mathcal{B}}^e$. For an illustration we refer the reader to Figure 6 where an example with $J = 2, j_0 = 0$ is shown (the meaning of the gray squares will be explained below).

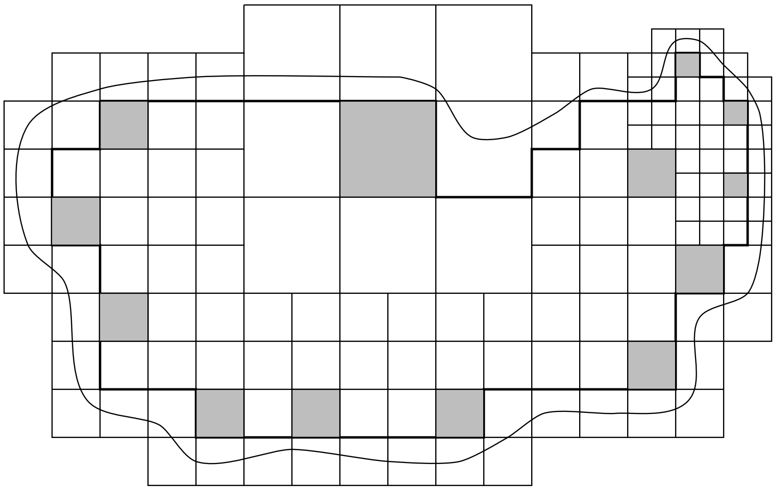


Figure 6. Cube partition for adaptive refinement

It can be shown by induction (see the beginning of Subsection 4.1 for analogous considerations) that $\hat{\mathcal{B}}$ resp. $\hat{\mathcal{B}}^e$ are linearly independent sets of functions, and form algebraic bases in

$$\hat V = \mathrm{span}\,\hat{\mathcal{B}}|_{\tilde\Omega_{j_0}} \quad \text{resp.} \quad \hat V^e = \mathrm{span}\,\hat{\mathcal{B}}^e|_{\tilde\Omega^e_{j_0}}\ .$$

However, to guarantee the existence of suitable biorthogonal functions $\hat{\eta}_{j,i}$, with supports near $\operatorname{supp}\phi_{j,i}$ ($i \in \hat{\omega}_j$, $j = j_0, \ldots, J$), and reasonable L_∞-bounds, we need to assume another geometric condition which plays a role similar to **(G3)** in Subsection 5.2:

(G3)' If the supports of two basis functions $\phi_{j,i}$, $\phi_{j',i'}$ from $\hat{\mathcal{B}}^e$ intersect at interior points then $|j - j'| \leq 1$.

To make a boundary modification in analogy with Subsection 3.1 possible, we introduce a stronger substitute for **(G1)**:

(G1)' For each l-cube $\Box \subset \hat{\Omega}_l^e$ and any $j_0 \leq l \leq J$ there is an l-cube $\Box' \subset \hat{\Omega}_l$ such that
a) $\omega_{\Box'} \subset \hat{\omega}_l$, and no function from $\hat{\mathcal{B}}_{l-1}$ contains $\Box'$ in its support,
b) $\operatorname{dist}(\Box, \Box') \leq C2^{-l}$.

The above conditions are satisfied in the example of Figure 6 if the bilinear case is considered. Since our sequences $\{\hat{\Omega}_l\}$ resp. $\{\hat{\Omega}_l^e\}$ describe the actual refinement process, the requirement **(G1)'** b) follows from **(G1)** if the refinement near the boundary is properly implemented. Assumption **(G1)'** a) is related to the implementation of **(G3)'** near the boundary. We do not know whether it can be neglected. Together, **(G1)'** and **(G3)'** imply that the corridors $\tilde{\Omega}_j \backslash \tilde{\Omega}_{j+1} = \hat{\Omega}_j$ are sufficiently "thick" which is a certain restriction for the refinement process (rapid, local changes of the refinement level seem to be impossible).

With **(G1)'** and **(G3)'** at hand, we can now finish the construction. According to **(G1)'**, we can select families $\hat{\mathcal{C}}_j$ of l-cubes $\Box \subset \hat{\Omega}_l$ near the domain boundary such that

- assumption **(G1)'** a) (with $\Box'$ replaced by $\Box$) is satisfied,
- the index sets $\omega_\Box$ are pairwise disjoint, and
- for all $i \in \partial\hat{\omega}_l \equiv \hat{\omega}_l^e \backslash \hat{\omega}_l$, there is a cube $\Box \in \hat{\mathcal{C}}_l$ at distance $\leq C2^{-l}$ from $\operatorname{supp}\phi_{l,i}$.

The gray squares in Figure 6 indicate a possible choice for $\hat{\mathcal{C}}_l$, $l = 0, 1, 2$ in the bilinear case. As in Subsection 3.1 we now modify the basis functions $\phi_{l,i}$ associated with $\Box \in \hat{\mathcal{C}}_l$ by replacing them by certain locally extended

$$\phi_{\Box,i} = \sum_{i' \in \hat{\omega}_\Box} c_{i,i'} \phi_{l,i'} \ .$$

The union of the sets $\omega_\Box \subset \hat{\omega}_\Box \subset \omega_\Box \cup \partial\hat{\omega}_l$, $\Box \in \hat{\mathcal{C}}_l$, which are defined by using the third of the above properties of $\hat{\mathcal{C}}_l$, cover $\partial\hat{\omega}_l$. To guarantee local polynomial reproduction, one has to change some of the $c_{i,i'}$ in comparison to the construction of Subsection 3.1. This is the case if the basis function

$\phi_{\Box,i}$ corresponds to some (shifted and dilated) monomial x^α, and if $\phi_{l,i'}$ is in the support of some (unchanged) basis function ϕ_{l-1,i^*} from $\hat{\mathcal{B}}_{l-1}$. Note that, by our above assumptions, no interference with basis functions of levels $\leq l-2$ and with modified basis functions of levels $\leq l-1$ is possible. Figure 7 shows the values of the locally extended functions corresponding to the monomials x^α, $|\alpha| \leq 1$ for the bilinear case. For simplicity, we assume that $l=0$ and that the shadowed cube $\Box$ coincides with $\Box_0$. The nodal points in $\hat{\omega}_\Box$ are indicated by circles, the modified values at P are due to the influence of the basis function from $\hat{\mathcal{B}}_{l-1}$ corresponding to Q.

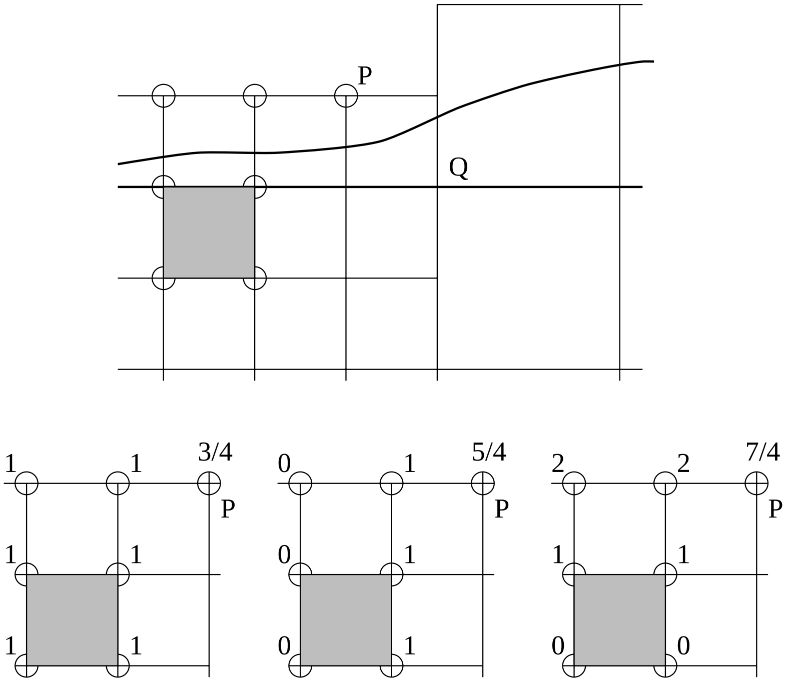


Figure 7. Boundary modification for an adaptive refinement example

Now, we introduce the subspace $\hat{V}_\Omega$ (which is the counterpart to $V_{j,\Omega}$ from Subsection 3.1) spanned by the basis $\hat{\mathcal{B}}_\Omega = \{\hat{\phi}_{l,i,\Omega}\}$, where the basis functions $\hat{\phi}_{l,i,\Omega}$ are defined as the restrictions to Ω of all previous basis functions (the modified $\phi_{\Box,i}$, $\Box \in \hat{\mathcal{C}}_l$ and the remaining, unmodified $\phi_{l,i} \in \hat{\mathcal{B}}_l$), from all levels $l = j_0, \ldots, J$. Again, functions from $\hat{V}_\Omega$ are uniquely determined by their values on $\tilde{\Omega}_{j_0}$, and the restrictions of the spaces $\hat{V}_\Omega$ and $\hat{V}$ to $\tilde{\Omega}_{j_0}$ coincide. The space $\hat{V}$ can be used to derive preconditioners for a $\hat{V}_\Omega$-discretization along the lines of Section 3 and 4.

We state the approximation result generalizing Lemma 1 for $s = 0$. The proof which is omitted shows implicitly that refinement in regions of low regularity of a function u can reduce the overall error, here with respect to the L_2-norm.

Lemma 6. *Assume that all assumptions (especially* **(G1)'** *and* **(G3)'***) are satisfied for the construction outlined above. Then, there exists a quasi-interpolant operator*

$$\hat{Q}^e \; : \; L_2(\tilde{\Omega}_{j_0}) \longmapsto \hat{V}^e$$

such that for any $u \in H^m(\Omega)$, $m = 1, \ldots, k+1$, and for its bounded extension $\hat{u} = Eu \in H^m(\mathbb{R}^d)$, we have the estimate

$$\sum_{l=j_0}^{J} \sum_{\Box \subset \hat{\Omega}_l^e} 2^{2lm} \|\hat{u} - \hat{Q}^e u\|^2_{L_2(\Box)} \le C \|u\|^2_{H^m(\Omega)} \; . \tag{5.3.3}$$

Moreover, the restricted operator $\hat{Q}_{\hat{V}_\Omega} \; : \; u \in L_2(\Omega) \longmapsto \hat{Q}^e u|_\Omega$ defines a projection onto $\hat{V}_\Omega$, and satisfies an analogous estimate.

This result is complemented by other standard inequalities which easily follow from the construction (especially the properties of the the locally defined biorthogonal functions $\hat{\eta}_{l,i,\Omega}$ are of importance), such as the inverse inequality

$$\|\hat{u}_\Omega\|^2_{H^m(\Omega)} \prec \sum_{l=j_0}^{J} \sum_{\Box \subset \hat{\Omega}_l} 2^{2lm} \|\hat{u}_\Omega\|^2_{L_2(\Box)} \tag{5.3.4}$$

and the L_2-stability of the corresponding basis $\hat{\mathcal{B}}_\Omega$ expressed by

$$\sum_{l=j_0}^{J} \sum_{\Box \subset \hat{\Omega}_l} 2^{2lm} \|\hat{u}_\Omega\|^2_{L_2(\Box)} \asymp \sum_{l=j_0}^{J} \sum_{i \in \hat{\omega}_l} 2^{l(2m-d)} c_{l,i}^2 \; , \tag{5.3.5}$$

which are valid for all

$$\hat{u}_\Omega = \sum_{l=j_0}^{J} \sum_{i \in \hat{\omega}_l} c_{l,i} \hat{\phi}_{l,i,\Omega} \in \hat{V}_\Omega \; ,$$

and all $m = 0, \ldots, r+1$.

These inequalities also provide us with a heuristic *a posteriori error estimator* built from local error indicators which are based on conforming higher-order local approximations. This concept has widely been used within the finite-element community (see the recent paper [4] for some references). In our context, let $\hat{W}_\Omega$ be the subspace constructed in the same way as $\hat{V}_\Omega$ but for tensor-product splines of degree $k+1$ (and the same smoothness order r). We do not change the "geometrical part" of the above construction. It can be assumed that

$$\hat{V}_\Omega \subset \hat{W}_\Omega \ , \tag{5.3.6}$$

and that the basis $\hat{\mathcal{B}}_W$ of $\hat{W}_\Omega$ can be obtained in hierarchical manner from $\hat{\mathcal{B}}_V \equiv \hat{\mathcal{B}}_\Omega$ by adding new functions $\hat{\phi}_{l,i,W}$, $i \in \hat{\omega}_{l,W}$, $l = j_0, \ldots, J$, *i.e.* ,

$$\hat{\mathcal{B}}_W = \bigcup_{l=j_0}^{J} \{\hat{\phi}_{l,i,\Omega} \ : \ i \in \hat{\omega}_l\} \cup \{\hat{\phi}_{l,i,W} \ : \ i \in \hat{\omega}_{l,W}\} \ . \tag{5.3.7}$$

Indeed, this can be achieved by switching at the very beginning of the whole construction from the B-spline basis to another, hierarchical basis in $S^r_{k+1}(\mathcal{R}_j)$ which contains as a subset the B-spline basis functions $\phi_{j,i}$ of $S^r_k(\mathcal{R}_j)$. For the bilinear case, the additional functions of such a "hierarchical" basis would simply consist of tensor products of a quadratic bubble function in one direction with quadratic bubble resp. piecewise linear hat functions in the other direction.

The error estimator is based on the following, *heuristic assumption*: Let $\hat{u}_V$ resp. $\hat{u}_W$ be the solutions of the $\hat{V}_\Omega$- resp. $\hat{W}_\Omega$-discretization of a symmetric $H^s(\Omega)$-elliptic variational problem given by the bilinear form $a(\cdot,\cdot)$ and a linear functional Φ defined on $H^s(\Omega)$. We assume that $s = 1, \ldots, r+1$ is integer. Then, at least for smooth exact solutions u of the variational problem, we may expect that $\hat{u}_W$ provides a better approximation to u than $\hat{u}_V$ does. Compare Lemma 1 resp. 6, and note that $\hat{W}_\Omega$ locally contains polynomials of degree up to $k+1$ which $\hat{V}_\Omega$ does not. Thus, the heuristic assumption that there is a $q < 1$ such that

$$\|u - \hat{u}_W\|_a \leq q \|u - \hat{u}_V\|_a \ , \tag{5.3.8}$$

seems to be plausible (hopefully, q is close to 0 and not to 1, the choice $q = 1$ being trivial since $\hat{V}_\Omega \subset \hat{W}_\Omega$). Using the triangle inequality for the energy norm $\|\cdot\|_a$, and the $H^s(\Omega)$-ellipticity of $a(\cdot,\cdot)$, (5.3.8) yields

$$\|u - \hat{u}_V\|^2_{H^s(\Omega)} \asymp a(u - \hat{u}_V, u - \hat{u}_V) \asymp a(\hat{u}_W - \hat{u}_V, \hat{u}_W - \hat{u}_V) \ , \tag{5.3.9}$$

with constants depending on q, and the ellipticity constants of $a(\cdot,\cdot)$.

Theorem 5. *Under the above assumptions for the construction of $\hat{V}_\Omega, \hat{W}_\Omega$ and the variational problem, we have*

$$a(\hat{u}_W - \hat{u}_V, \hat{u}_W - \hat{u}_V) \asymp \sum_{l=j_0}^{J} \sum_{i \in \hat{\omega}_{l,W}} \frac{(\Phi(\hat{\phi}_{l,i,W}) - a(\hat{u}_V, \hat{\phi}_{l,i,W}))^2}{d_{l,i,W}} \ ,$$

where the scaling factors satisfy $d_{l,i,W} \asymp 2^{j(2s-d)}$*. Together with the heuristic assumption (5.3.8), this two-sided estimate leads to an a posteriori error estimator for the error of the approximate solution* $\hat{u}_V$ *in the energy norm. The error estimator is based on the residual contributions*

$$r_{l,i,W} = \Phi(\hat{\phi}_{l,i,W}) - a(\hat{u}_V, \hat{\phi}_{l,i,W})$$

and does not require the computation of $\hat{u}_W$*.*

The proof uses Lemma 6, the inequalities (5.3.4), (5.3.5), and the hierarchical definition of the bases (5.3.7) in conjunction with the approach to hierarchical error estimation of [4] or [61, Section 3.5].

To use Theorem 5 in practice, one first computes (with sufficiently high accuracy) all residuals $r_{l,i,W}$ from the available current approximation to $\hat{u}_V$. For a typical elliptic PDE problem, this is a local procedure and can easily be parallelized. Formally, this step yields local error indicators, *e.g.* , we may define

$$e_{\square}^2 = \sum_{(l,i)\,:\,\square\cap\mathrm{supp}\,\phi_{l,i,W}\neq\emptyset} \frac{r_{l,i,W}^2}{d_{l,i,W}}$$

for all j-cubes $\square \subset \hat{\Omega}_j^e$ and $j = j_0, \ldots, J$. This choice of an error indicator is not unique. One has to experiment also with the scaling factors $d_{l,i,W}$. Obviously,

$$c\sum_{\square} e_{\square}^2 \le \sum_{l=j_0}^{J} \sum_{i\in\hat{\omega}_{j,W}} \frac{r_{l,i,W}^2}{d_{l,i,W}} \le \sum_{\square} e_{\square}^2 ,$$

where the summation is with respect to all above described $\square$. We do not have theoretical access to the constant $c > 0$. Its value may significantly influence the quality of error estimation. Note that the above splitting of the global error estimate into local components does not necessarily reflect the true local error of $\hat{u}_V$, however, the use of such local error indicators to monitor the refinement process is a well-established numerical practice.

In a second step, those cubes $\square$ are refined for which $e_{\square}$ exhibits a large error contribution. *Error equidistribution* is a common approach, however, many other issues might influence the decision to refine a given cube. After this, one usually cares about keeping the refinement structure sufficiently regular. In our framework, this could lead to additional refinement in order to stay close to the geometric assumptions **(G1)'**, **(G2)** (which is needed to guarantee an efficient multilevel scheme in the next solution step), and **(G3)'**. This enables us to set up the new cube structures $\{\hat{\Omega}_j\}$ resp. $\{\hat{\Omega}_j^e\}$, and so on.

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