Smoothness of Nonlinear Subdivision Schemes

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Abstract. We introduce a new nonlinear subdivision scheme on $\mathbb{R}^1$ whose subdivision operator S_∞ is defined by a polynomial interpolation/imputation procedure based on best local L_∞ approximation by constants, and analyze its convergence and Hölder smoothness properties. The results grew out of a case study [9] for the Donoho-Yu median-interpolation subdivision scheme [4], and complement the theory presented in [2].

§1. Introduction

Nonlinear subdivision schemes on $\mathbb{R}^1$ occur in a number of applications, e.g., in connection with normal schemes for curve design [3], the capturing of singularities by ENO-schemes [7,2], denoising [4], shape-preserving data interpolation [10,8], etc. However, their theoretical investigation has only begun [3,2]. In the present paper which grew out of a case study [9] on the C^s smoothness properties of the median-interpolation subdivision scheme by Donoho and Yu [4], we introduce a new nonlinear subdivision operator S_∞, where the local interpolation/imputation procedure is based on best L_∞ approximation by constants rather than on medians, and investigate its Hölder smoothness. In Section 2, we explain terminology and notation, and state a corollary to results from [2,3,9] which applies when the nonlinear subdivision operator S is a higher order perturbation of a linear subdivision operator $\hat{S}$. In Section 3, we show in the dyadic case $r = 2$ that the nonlinear subdivision scheme S_∞ is a second order perturbation of an appropriately chosen linear scheme $\hat{S}_\infty$, and prove a close-to-optimal lower bound for its Hölder exponent.

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§2. Notation and General Results

We will define a stationary, shift-invariant, locally supported subdivision scheme with integer dilation factor $r \geq 2$ by fixing a function $\tau : \mathbb{R}^N \to \mathbb{R}^r$, and introducing the associated subdivision operator $S := S_\tau$ acting on sequences $m := \{m_i\} \in \ell_\infty(\mathbb{Z})$ according to

$$(Sm)_{ri+l-1} = \tau_l(m_{i-n+1}, \ldots, m_{i-n+N}), \qquad l = 1, \ldots, r, \quad i \in \mathbb{Z}. \quad (1)$$

The choices for τ and the integer parameters n, N depend on the particular application. If τ is a linear map then S is called linear subdivision operator. The subdivision scheme itself consists in repeatedly applying S to any initial $m \in \ell_\infty(\mathbb{Z})$. This leads to a sequence $\{m^j\}$, where $m^0 = m$, $m^{j+1} = Sm^j = S^{j+1}m$, $j \geq 0$. It is customary to associate with m^j either a piecewise constant function

$$g^j(x) = m_i^j, \qquad x \in [ir^{-j}, (i+1)r^{-j}), \quad i \in \mathbb{Z},$$

or a piecewise linear function f^j given by the interpolation conditions $f^j(ir^{-j}) = m_i^j$, $i \in \mathbb{Z}$, and call the subdivision scheme convergent if, for each $m \in \ell_\infty(\mathbb{Z})$, the sequence $\{f^j\}$ converges uniformly to some $f := S^\infty m \in C(\mathbb{R})$ as $j \to \infty$. Obviously, if $\{f^j\}$ converges to f so does $\{g^j\}$, vice versa.

In [2], the authors introduce a quasilinear (or data-dependent) subdivision scheme by specifying a family $\mathbf{\Psi} = \{\Psi(m) \; : \; m \in \ell_\infty(\mathbb{Z})\}$ of bounded linear mappings $\Psi(m) : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ with the following local support property: In the matrix representation $\Psi(m) = \{\psi_{i'i}(m)\}$ entries vanish if $|i' - ri| > L$ for some fixed L (which may vary from family to family but is independent of m). The associated $S := S_{\mathbf{\Psi}}$ is defined

$$Sm = \Psi(m)m, \qquad m \in \ell_\infty(\mathbb{Z}). \quad (2)$$

Our above definition (1) is slightly more restrictive due to the additional assumption of shift-invariance which, however, is satisfied in most examples.

The theory presented in [2] systematically extends the existing theory for linear subdivision operators S (see [1,5]) by requiring some properties to hold uniformly for the whole family $\mathbf{\Psi}$ instead of a single S. A family $\mathbf{\Psi}$ is called bounded if

$$\|\Psi(m)\|_\infty \leq C, \qquad m \in \ell_\infty(\mathbb{Z}). \quad (3)$$

where $\|\cdot\|_\infty$ is simultaneously used for sequence and operator norms on $\ell_\infty(\mathbb{Z})$. We say that $\mathbf{\Psi}$ has order of polynomial reproduction of at least $k \geq 1$ (in short, order k) if all $\Psi(m)$ have order k. For linear S, the order of polynomial reproduction can be defined in the following recursive way: S has order 1 if it reproduces constants, i.e., $S\mathbf{1} = \mathbf{1}$, $\mathbf{1} = \{\ldots, 1, 1, 1, \ldots\}$, it then follows that

$$\Delta^1(Sm) = S_1(\Delta^1 m), \qquad m \in \ell_\infty(\mathbb{Z}),$$

for some other linear subdivision operator S_1. Here, Δ^1 is the forward difference operator defined by $(\Delta^1 m)_i = m_{i+1} - m_i$, as usual $\Delta^\mu = (\Delta^1)^\mu$, $\mu > 0$, and $\Delta^0 m = m$. Note that S_1 has again the local support property, and the entries of its matrix representation are linear combinations of the entries of S. Suppose now that S has order $k-1$, in which case $S_1, \ldots, S_{k-1}$ are already defined. Then S has order k if $r^{k-1} S_{k-1}$ reproduces constants, and as above we can define S_k such that

$$\Delta^k(Sm) = \Delta^{k-1}(S_1 \Delta^1 m) = \ldots = \Delta^1(S_{k-1}\Delta^{k-1} m) = S_k \Delta^k m$$

for all $m \in \ell_\infty(\mathbb{Z})$. Note that if $\mathbf{\Psi}$ has order k and satisfies (3), then the derived families $\mathbf{\Psi}_\mu$, $\mu = 1, \ldots, k$, inherit this property by construction.

The theorems proved in [2] essentially assert that if $\mathbf{\Psi}$ is bounded, has order $k \geq 1$, and satisfies a bound on the joint ℓ_∞-spectral radius ρ_k of the derived family $\mathbf{\Psi}_k$,

$$\rho_k := \liminf_{j \to \infty} \sup_{m \in \ell_\infty(\mathbb{Z})} \|\Psi_k(S^{j-1}m) \ldots \Psi_k(Sm)\Psi_k(m)\|_\infty^{1/j},$$

then the subdivision scheme governed by $S_{\mathbf{\Psi}}$ converges, and its limit functions belong to $C^s(\mathbb{R})$ for all $0 \leq s < s_k := \min\{\log_r(1/\rho_k), k\}$.

Note that there is a slight ambiguity in the choice of the family $\mathbf{\Psi}$ since (2) does not uniquely determine $\mathbf{\Psi}$ if S is given. The choice for $\mathbf{\Psi}$ is made easy if S is a higher order perturbation of a linear subdivision operator $\hat{S}$, i.e., if it can be represented in the form

$$Sm = \hat{S}m + \Phi(m)\Delta^k m, \qquad m \in \ell_\infty(\mathbb{Z}), \tag{4}$$

$\mathbf{\Phi} = \{\Phi(m) : m \in \ell_\infty(\mathbb{Z})\}$ is a family of linear operators on $\ell_\infty(\mathbb{Z})$ with the local support property. This assumption came as natural in the analysis of the Donoho-Yu scheme [4,9], which appeared to be a second order perturbation of the midpoint-interpolation scheme. Obviously, if S is given by (4), it can also be written in the form (2) with $\Psi(m) = \hat{S} + \Phi(m)\Delta^k$. This $\mathbf{\Psi}$ has order k if $\hat{S}$ has order k, and is bounded if $\mathbf{\Phi}$ is bounded. Thus, we have the following corollary to [2] which is easy to apply in practical applications.

Theorem 1. *Assume that the subdivision operator S is given by (4), where the linear subdivision operator $\hat{S}$ has order k, $\mathbf{\Phi}$ has the local support property and is bounded. If for some $j \geq 1$ the inequality*

$$\|\Delta^k(S^j m)\|_\infty \leq \kappa_{k,j} \|\Delta^k m\|_\infty, \qquad m \in \ell_\infty(\mathbb{Z}), \tag{5}$$

holds with some constant $\kappa_{k,j} < 1$, then the associated subdivision scheme converges, the limit functions $f = S^\infty m$ belong to $C^s(\mathbb{R})$, and satisfy the estimate

$$\|f\|_{C^s} \le C(s)\|m\|_\infty, \qquad m \in \ell_\infty(\mathbb{Z}),$$

for all $0 \le s < s_{k,j_0} := \min\{\log_r(1/\kappa_{k,j_0})/j_0, k\}$.

In [9], in connection with the particular example of the Donoho-Yu scheme, an alternative derivation of this result was given based on a perturbation theorem proved in [3]. For a more detailed discussion, we refer the interested reader to [2,3,9], and the extended version of this paper posted at `http://cm.bell-labs.com/who/poswald`.

§3. Hölder regularity estimates for S_∞

We define the nonlinear subdivision operator S_∞ by (1), where $N = 3$, $n = 2$, and τ is constructed as follows. Given three arguments, say (m_{-1}, m_0, m_1), we determine a polynomial $p = p_\infty$ of degree ≤ 2 whose best L_∞-approximating constants on the intervals $I_i := [i, i+1]$ coincide with m_i, $i = -1, 0, 1$. In other words, the interpolation rule is formally given by $c_\infty(p_\infty; I_i) = m_i$, $i = -1, 0, 1$, where the function $c_\infty(f; I)$ is defined for any continuous function f and closed interval I by

$$\|f - c_\infty(f; I)\|_{L_\infty(I)} = \min_{c \in \mathbb{R}} \|f - c\|_{L_\infty(I)},$$

or, in more explicit form, by

$$c(f; I)_\infty = (\max_{x \in I} f(x) + \min_{x \in I} f(x))/2. \tag{6}$$

Below, we will use the parametrization $p_\infty(t) = a(t-c)^2 + b$ with parameter $c \in \mathbb{R}$ if $(\Delta^2 m)_{-1} \ne 0$, otherwise we set $c = \infty$. Similarly, in the imputation step we set

$$(Sm)_{l-1} = \tau_l(m_{-1}, m_0, m_1) := c(p_\infty; I'_{l-1}), \quad l = 1, \ldots, r,$$

where $I'_{l-1} = [(l-1)/r, l/r]$ denote the r equal subintervals of $I_0 = [0, 1]$. The only difference with the median-interpolation scheme analyzed in [4,9] is that the latter is defined by medians which represent best L_1 approximating constants.

Although S_∞ could also be defined by using higher-degree polynomials, compare [4,9], we present its Hölder regularity analysis only for the above quadratic case. From now on, fix $r = 2$ for simplicity. Since $c(f, [\alpha, \beta])_\infty = (f(\alpha) + f(\beta))/2$ for monotone f and quadratic polynomials consist of two monotonic pieces, we suggest to define the associated linear subdivision operator $\hat{S}_\infty$ by end-point averaging interpolation. I.e., we find a quadratic $p := \hat{p}$ according to

$$\hat{p}(-1) + \hat{p}(0) = 2m_{-1}, \quad \hat{p}(0) + \hat{p}(1) = 2m_0, \quad \hat{p}(1) + \hat{p}(2) = 2m_1, \tag{7}$$

and impute new values by using the same rule:

$$(\hat{S}_\infty m)_0 = (\hat{p}(0) + \hat{p}(1/2))/2, \quad (\hat{S}_\infty m)_1 = (\hat{p}(1/2) + \hat{p}(1))/2. \tag{8}$$

Note that (7), (8) imply that

$$\hat{p}(t) = m_{-1}p_{-1}(t) + m_0 p_0(t) + m_1 p_1(t),$$

where $p_{-1}(t) = (2(t-1)^2 - 1)/4$, $p_0(t) = (5-(2t-1)^2)/4$, $p_1(t) = (2t^2-1)/4$ are the Lagrange polynomials associated with (7), and

$$\begin{aligned} (\hat{S}_\infty m)_{2i} &= \frac{m_{i-1}}{16} + \frac{9m_i}{8} - \frac{3m_{i+1}}{16}, \\ (\hat{S}_\infty m)_{2i+1} &= -\frac{3m_{i-1}}{16} + \frac{9m_i}{8} + \frac{m_{i+1}}{16}, \end{aligned} \qquad i \in \mathbb{Z},\ m \in \ell_\infty(\mathbb{Z}). \tag{9}$$

Lemma 2. *For $r = 2$, we have*

$$(S_\infty m)_{2i+e} = (\hat{S}_\infty m)_{2i+e} + \hat{\phi}_e(c_i)(\Delta_2 m)_{i-1}, \quad e = 0, 1, \tag{10}$$

where $\hat{\phi}_e(\cdot)$ are bounded, Lipschitz continuous functions, and c_i denotes the parameter c associated with p_∞ and the data m_{i-1}, m_i, m_{i+1}.

Proof: The proof is similar to that of Proposition 1 in [9]. We give the main steps. We can set $i = 0$, and assume that $(\Delta^2 m)_{-1} \neq 0$, i.e., $c = c_0$ is finite (the case $(\Delta^2 m)_{-1} = 0$ is trivial, and $i \neq 0$ follows by shift-invariance). By symmetry, we have

$$\hat{\phi}_0(c) = \hat{\phi}_1(1-c), \quad \hat{\phi}_1(c) = \hat{\phi}_0(1-c), \tag{11}$$

and we can restrict our attention to $c \le 1/2$. The proof of (11) is left to the reader.

If $c \notin [-1, 2]$ then p_∞ and $\hat{p}$ satisfy identical interpolation conditions which gives $p_\infty = \hat{p}$. Since the imputation steps are also identical we arrive at (10) with $\hat{\phi}_e(c) \equiv 0$.

We next consider $c \in [-1, -1/2]$. According to (6), p_∞ satisfies on I_{-1} the condition $p_\infty(c) + p_\infty(0) = 2m_{-1}$ while on the other intervals it satisfies the same interpolation conditions as $\hat{p}$. Consequently, $\tilde{p} := p_\infty - \hat{p} = \beta_0 p_{-1}$, where the constant β_0 can be found as follows. On the one hand,

$$\tilde{p}(c) + \tilde{p}(0) = \beta_0(p_{-1}(c) + p_{-1}(0)) = -(c-1)^2\beta_0/2.$$

On the other hand, using the interpolation condition for p_∞ and $\hat{p}$ on I_{-1}, we see that

$$\tilde{p}(c) + \tilde{p}(0) = \hat{p}(-1) - \hat{p}(c) = -(\Delta^2 m)_{-1}(1+c)((c-1)/2 + q),$$

where we have used the identity

$$\hat{p}(t) - \hat{p}(t') = (\Delta^2 m)_{-1}(t-t')((t+t')/2 + q), \quad t, t' \in \mathbb{R}, \tag{12}$$

with $q := (\Delta^1 m)_{-1}/(\Delta^2 m)_{-1}$ (we postpone the proof of (12) until the end of the paragraph). Since

$$\begin{aligned} 2(m_0 - m_{-1}) &= (p_\infty(0) + p_\infty(1)) - (p_\infty(c) + p_\infty(0)) = a(c-1)^2, \\ 2(m_1 - m_0) &= (p_\infty(2) + p_\infty(1)) - (p_\infty(1) + p_\infty(0)) = -4a(c-1), \end{aligned}$$

and $1/q + 1 = (m_1 - m_0)/(m_0 - m_{-1})$ by definition of q, we get $q = (1-c)/(3+c)$ as a function of c. Substituting into the above expressions, we find

$$\beta_0 = \frac{2(\Delta^2 m)_{-1}(1+c)}{(c-1)^2}\left(\frac{c-1}{2} - \frac{c-1}{c+3}\right) = -(\Delta^2 m)_{-1}\frac{(c+1)^2}{(3+c)(1-c)}. \tag{13}$$

Since for $c \in [-1, -1/2]$ the imputation rules for S_{L_∞} and S_{ep} are the same, we conclude that

$$\begin{aligned} (S_\infty m)_0 - (S_{ep} m)_0 &= \beta_0(p_{-1}(0) + p_{-1}(1/2)) = \beta_0/16, \\ (S_\infty m)_1 - (S_{ep} m)_1 &= \beta_0(p_{-1}(1/2) + p_{-1}(1)) = -3\beta_0/16. \end{aligned} \tag{14}$$

Together with (13) this establishes (10) for $c \in [-1, -1/2]$. For $c \in [-1/2, 0]$, the only change is that now p_∞ satisfies on I_{-1} the condition $p(-1) + p(c) = 2m_{-1}$. Similar considerations lead to $\tilde{p} = \beta_1 p_{-1}$, where

$$\beta_1 = -(\Delta^2 m)_{-1} \cdot c^2/(4 - c^2), \quad c \in [-1/2, 0], \tag{15}$$

and again to (14), with β_0 replaced by β_1 for these c.

This completes the argument for $c \in [-1, 0]$ if we establish (12). Assume that $(\Delta^2 m)_{-1} \neq 0$, and write

$$\hat{p}(t) = \hat{a}(t - \hat{c})^2 + \hat{b} \quad \hat{a} \neq 0.$$

From the interpolation conditions (7) we derive

$$\begin{aligned} 2(m_0 - m_{-1}) &= \hat{p}(1) - \hat{p}(-1) = -4\hat{a}\hat{c}, \\ 2(m_1 - m_0) &= \hat{p}(2) - \hat{p}(0) = -4\hat{a}(\hat{c} - 1), \end{aligned}$$

which gives

$$\hat{a} = (\Delta^2 m)_{-1}/2, \qquad \tilde{c} = -(\Delta^1 m)_{-1}/(\Delta^2 m)_{-1} = -q.$$

Since $\hat{p}(t) - \hat{p}(t') = \hat{a}(t-t')(t+t'-2\hat{c})$, (12) follows by substitution.

Finally, we consider $c \in [0, 1/2]$. This time, p_∞ satisfies the same interpolation conditions as $\hat{p}$ on I_{-1} and I_1, while on I_0 we have $(p(1) + p(c)) = 2m_0$. Thus, we conclude that $\tilde{p} = \beta_2 p_0$, and obtain

$$\tilde{p}(c) + \tilde{p}(1) = \beta_2(p_0(c) + p_0(1)) = (2 + c - c^2)\beta_2. \tag{16}$$

On the other hand, as above

$$\tilde{p}(c) + \tilde{p}(1) = \hat{p}(0) - \hat{p}(c) = -(\Delta^2 m)_{-1} c(c/2 + q), \tag{17}$$

and from

$$\begin{aligned} 2(m_0 - m_{-1}) &= (p_\infty(1) + p_\infty(c)) - (p_\infty(-1) + p_\infty(0)) = -a(c^2 + 4c), \\ 2(m_1 - m_0) &= (p_\infty(2) + p_\infty(1)) - (p_\infty(1) + p_\infty(c)) = a(c-2)^2, \end{aligned}$$

we compute $q = -(c^2 + 4c)/(2c^2 + 4)$ and

$$\beta_2 = (\Delta^2 m)_{-1} c^2/(2c^2 + 4), \quad c \in [0, 1/2]. \tag{18}$$

The imputation step has to be performed carefully. Since $c \notin I_1'$ we have

$$(S_\infty m)_1 - (\hat{S}_\infty m)_1 = \beta_2(p_0(1/2) + p_0(1))/2 = 9\beta_2/8. \tag{19}$$

For $c \in [0, 1/4]$ we get

$$\begin{aligned} (S_\infty m)_0 &= (p_\infty(c) + p_\infty(1/2))/2 \\ &= (\hat{S}_\infty m)_0 + (\hat{p}(c) - \hat{p}(0))/2 + \beta_2(p_0(c) + p_0(1/2))/2 \\ &= (\hat{S}_\infty m)_0 + \beta_2(p_0(1/2) - p_0(1))/2, \end{aligned}$$

the last equality comes from (17),(16). By substituting (18) we get

$$(S_\infty m)_0 - (\hat{S}_\infty m)_0 = c^2(\Delta^2 m)_{-1}/(16(c^2 + 2)), \quad c \in [0, 1/4]. \tag{20}$$

A similar formula holds if $c \in [1/4, 1/2]$:

$$(S_\infty m)_0 - (\hat{S}_\infty m)_0 = (c^2 + 8c - 2)(\Delta^2 m)_{-1}/(16(c^2 + 2)). \tag{21}$$

This concludes the proof of Lemma 2 (explicit expressions for $\hat{\phi}_e$ and their properties are given below). □

From Lemma 2 we see that S_∞ can be written in the form (4) required by Theorem 1, with $k=2$. Each row of $\Phi(m)$ has at most one non-zero entry given by the corresponding $\hat\phi_e(c_i)$. According to (10), (9), the associated $\Psi_2(m) = (\hat S_\infty)_2 + \Delta^2\Phi(m)$ is given by

$$(\Delta^2 S_\infty m)_{2i} = (\frac{7}{16} + \hat\psi_0(c_i))(\Delta^2 m)_{i-1} - (\frac{3}{16} - \hat\phi_0(c_{i+1}))(\Delta^2 m)_i,$$
$$(\Delta^2 S_\infty m)_{2i+1} = -(\frac{3}{16} - \hat\phi_1(c_i))(\Delta^2 m)_{i-1} + (\frac{7}{16} + \hat\psi_1(c_{i+1}))(\Delta^2 m)_i.$$

where $\hat\psi_0(c) = \hat\phi_0(c) - 2\hat\phi_1(c)$, $\hat\psi_1(c) = \hat\phi_1(c) - 2\hat\phi_0(c)$. Explicit expressions and estimates for the ranges of $\hat\phi_e(c)$ and $\hat\psi_e(c)$ can be found from the above (see (14),(13),(15) for $c\in[-1,0]$, (19),(20),(21),(18) for $c\in[0,1/2]$, and (11) for $c\in[1/2,2]$, for all other c the functions vanish). E.g., for $c\le 1/2$ we find

$$16\hat\phi_0(c) = \begin{cases} 0, & c\le -1,\\ (c+1)^2/(4-(c+1)^2), & -1\le c\le -1/2,\\ c^2/(4-c^2), & -1/2\le c\le 0,\\ c^2/(2+c^2), & 0\le c\le 1/4,\\ (c^2+8c-2)/(2+c^2), & 1/4\le c\le 1/2, \end{cases}$$

and

$$16\hat\phi_1(c) = \begin{cases} 0, & c\le -1,\\ -3(c+1)^2/(4-(c+1)^2), & -1\le c\le -1/2,\\ -3c^2/(4-c^2), & -1/2\le c\le 0,\\ 9c^2/(2+c^2), & 0\le c\le 1/2. \end{cases}$$

Together with (11), this shows that $\hat\phi_e(c)$ is Lipschitz continuous for $c\in\mathbb{R}$, monotone on each of the intervals $I'_l = [l/2,(l+1)/2]$, $l=-2,\ldots,3$, and vanishes outside $[-1,2]$. Thus, Φ is bounded. Similarly,

$$16\hat\psi_0(c) = \begin{cases} 0, & c\le -1,\\ 7(c+1)^2/(4-(c+1)^2), & -1\le c\le -1/2,\\ 7c^2/(4-c^2), & -1/2\le c\le 0,\\ -17c^2/(2+c^2), & 0\le c\le 1/4,\\ (-17c^2+8c-2)/(2+c^2), & 1/4\le c\le 1/2, \end{cases}$$

and

$$16\hat\psi_1(c) = \begin{cases} 0, & c\le -1,\\ -5(c+1)^2/(4-(c+1)^2), & -1\le c\le -1/2,\\ -5c^2/(4-c^2), & -1/2\le c\le 0,\\ 7c^2/(2+c^2), & 0\le c\le 1/4,\\ (7c^2-16c+4)/(2+c^2), & 1/4\le c\le 1/2. \end{cases}$$

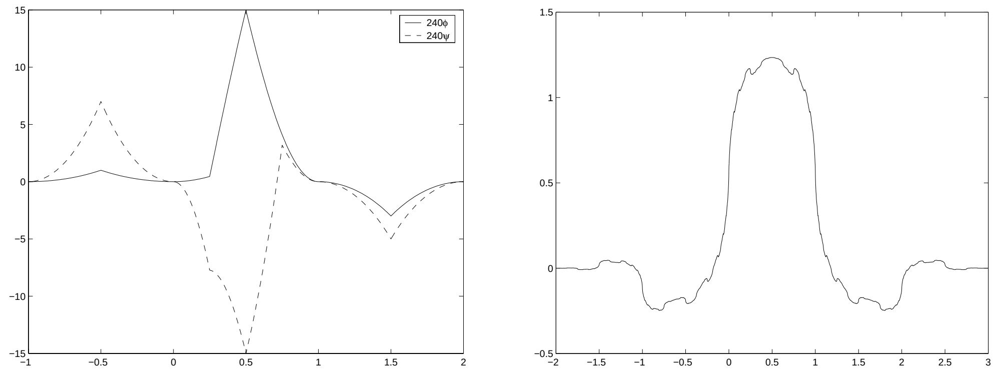

Fig. 1. Graphs of $240\hat{\phi}_0$, $240\hat{\psi}_0$ (on the left), and $S_\infty^\infty \boldsymbol{\delta}$ (on the right).

The graphs of $\hat{\phi}_0$ and $\hat{\psi}_0$ are displayed in Figure 1. We also show the graph of the limit function of the subdivision process with initial sequence $\boldsymbol{\delta} = (\ldots, 0, 0, 1, 0, 0, \ldots)$.

From the above formulae, we easily obtain

$$-3/240 \le \hat{\phi}_e(c) \le 1/16, \qquad -1/16 \le \hat{\psi}_e(c) \le 7/240. \tag{22}$$

According to the above recursions for $\Delta^2 S_\infty m$ and (22), we have

$$\begin{aligned} |(\Psi_2(m)x)_{2i}| &\le |7/16 + \hat{\psi}_0(c_i)||x_{i-1}| + |3/16 - \hat{\phi}_0(c_{i+1})||x_i| \\ &\le (5/8 + \hat{\psi}_0(c_i) - \hat{\phi}_0(c_{i+1}))\|x\|_\infty \\ &\le (5/8 + (7+3)/240)\|x\|_\infty = 2/3\|x\|_\infty . \end{aligned} \tag{23}$$

Due to (11), the bound for $|(\Psi_2(m)x)_{2i+1}|$ will be the same. Thus, we have proved

$$\kappa_{2,1} \le \sup_{m \in \ell_\infty(\mathbb{Z})} \|\Psi_2(m)\|_\infty \le 2/3. \tag{24}$$

The example

$$(\ldots, m_{-1}, m_0, m_1, m_2, \ldots) = (\ldots, -7, -4, 4, 7, \ldots)/5$$

shows that the inequalities in (24) actually turn into equalities. The remaining entries of m are chosen such that $m \in \ell_\infty(\mathbb{Z})$ and $\|\Delta^2 m\|_\infty = 1$, this can obviously be done (note that $(\Delta^2 m)_{-1} = -(\Delta^2 m)_0 = 1$ by the choice of the scaling factor 5). For such an m, we get $c_0 = -1/2$ since $p_\infty(t) = 2(-11 + (2c+1)^2)/15$ satisfies the interpolation conditions for the data $(m_{-1}, m_0, m_1) = (-7, -4, 4)/5$. By symmetry, we have $c_1 = 3/2$. Thus, for this m and by setting $i = 0$ in (23), we get

$$(\Psi_2(m)\Delta^2 m)_0 = \frac{7}{16} + \hat{\psi}_0(-\frac{1}{2}) + \frac{3}{16} - \hat{\phi}_0(\frac{3}{2}) = \frac{5}{8} + \frac{10}{240} = \frac{2}{3}.$$

This establishes equality in (24), and by Theorem 1 we arrive at

Theorem 3. *The dyadic nonlinear subdivision scheme defined by* S_∞ *converges and has limit functions in the Hölder space* $C^s(\mathbb{R})$ *for all* s *satisfying* $0 < s < \log_2(3/2) = 0.5850....$

To obtain improved bounds, one could try to verify (5) for some $j > 1$. We numerically calculated values of $s_{2,2} = 0.6202...$, $s_{2,3} = 0.6335...$, $s_{2,4} = 0.6446....$ Together with other numerical evidence, this lets us conjecture that the exact Hölder exponent of S_∞ coincides with the corresponding value $s_\infty(\hat{S}_\infty) = \log_2(8/5) = 0.6781...$ for $\hat{S}_\infty$.

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