Smoothness of Nonlinear Subdivision Schemes

Peter Oswald

Abstract. We consider some examples of nonlinear subdivision schemes whose subdivision operator S is defined by a nonlinear local polynomial interpolation/imputation procedure, and analyze their Hölder smoothness properties. The results grew out of a case study [8] for the Donoho-Yu median-interpolation subdivision scheme [4], and complement the theory presented in [2].

§1. Introduction

We will deal with the C^s-smoothness analysis for nonlinear subdivision schemes on $\mathbb{R}^1$. Such schemes occur in a number of applications, e.g., in connection with normal schemes for curve design [3], the capturing of singularities by ENO-schemes [7,2], denoising [4], shape-preserving data interpolation [9], etc. However, their theoretical investigation has only begun [3,2]. The present paper grew out of a case study [8] on the smoothness properties of the Donoho-Yu scheme [4] for removing heavy-tail noise from one-dimensional data sets, and relates the specific results obtained in [8] to the general theory for quasilinear subdivision schemes developed in [2]. In Section 2, we introduce necessary notation, state a slightly improved version of the main result from [2], and introduce a subclass of subdivision schemes where S is given as a higher-order perturbation of a linear subdivision operator $\hat{S}$. In Section 3, we apply the abstract result to the Donoho-Yu scheme and a new nonlinear subdivision scheme, where the local interpolation/imputation procedure is based on fitting quadratic polynomials to the data by best constant approximation in L_∞.

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§2. Abstract Results

We will define a stationary, shift-invariant, locally supported subdivision scheme with integer dilation factor $r \geq 2$ by fixing a function $\tau : \mathbb{R}^N \to \mathbb{R}^r$, and introducing the associated subdivision operator $S := S_\tau$ acting on sequences $m := \{m_i\} \in \ell_\infty(\mathbb{Z})$ according to

$$(Sm)_{ri+l-1} = \tau_l(m_{i-n+1}, \ldots, m_{i-n+N}), \qquad l = 1, \ldots, r, \quad i \in \mathbb{Z}. \quad (1)$$

The choices for τ and the integer parameters n, N depend on the particular application. In our examples, the dilation factor is either $r = 2$ (dyadic subdivision) or $r = 3$ (triadic subdivision). If τ is a linear map then S is called linear subdivision operator. The subdivision scheme itself consists in repeatedly applying S to any initial $m \in \ell_\infty(\mathbb{Z})$. This leads to a sequence $\{m^j\}$, where $m^0 = m$, $m^{j+1} = Sm^j = S^{j+1}m$, $j \geq 0$. It is customary to associate with m^j either a piecewise constant function

$$g^j(x) = m_i^j, \qquad x \in [ir^{-j}, (i+1)r^{-j}), \quad i \in \mathbb{Z},$$

or a piecewise linear function f^j given by the interpolation conditions

$$f^j(ir^{-j}) = m_i^j, \qquad i \in \mathbb{Z},$$

and call the subdivision scheme convergent if, for each $m \in \ell_\infty(\mathbb{Z})$, the sequence $\{f^j\}$ converges uniformly to some $f := S^\infty m \in C(\mathbb{R})$ as $j \to \infty$. Obviously, if $\{f^j\}$ converges to f so does $\{g^j\}$, vice versa.

In [2], the authors introduce a quasilinear (or data-dependent) subdivision scheme by specifying a family $\boldsymbol{\Psi} = \{\Psi(m) : m \in \ell_\infty(\mathbb{Z})\}$ of bounded linear mappings $\Psi(m) : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z})$ with the following local support property: In the matrix representation $\Psi(m) = \{\psi_{i'i}(m)\}$ entries vanish if $|i' - ri| > L$ for some fixed L (which may vary from family to family but is independent of m). The associated $S := S_{\boldsymbol{\Psi}}$ is defined

$$Sm = \Psi(m)m, \qquad m \in \ell_\infty(\mathbb{Z}). \quad (2)$$

Obviously, the additional assumption of shift-invariance, although satisfied in most examples, makes our above definition (1) more particular.

The theory presented in [2] systematically extends the existing theory for linear subdivision operators S (see [1,5]) by requiring some properties to hold uniformly for the whole family $\boldsymbol{\Psi}$ instead of a single S. A family $\boldsymbol{\Psi}$ is called bounded if

$$\|\Psi(m)\|_\infty \leq C, \qquad m \in \ell_\infty(\mathbb{Z}), \quad (3)$$

and Lipschitz continuous if for all $m, \tilde{m} \in \ell_\infty(\mathbb{Z})$

$$\|\Psi(m) - \Psi(\tilde{m})\|_\infty \leq C(\max\{\|m\|_\infty, \|\tilde{m}\|_\infty\})\|m - \tilde{m}\|_\infty, \quad (4)$$

where $\|\cdot\|_\infty$ is simultaneously used for sequence and operator norms on $\ell_\infty(\mathbb{Z})$. The constant in (4) is allowed to continuously depend on its argument (when used without argument, C denotes a generic constant that depends on S but not on m). Due to the local support property, both (3) and (4) are equivalent to similar conditions on the entries $\psi_{i'i}(m)$. We say that $\mathbf{\Psi}$ has order of polynomial reproduction of at least $k \geq 1$ (in short, order k) if all $\Psi(m)$ have order k. For linear S, the order of polynomial reproduction can be defined in the following recursive way: S has order 1 if it reproduces constants, i.e.,

$$S\mathbf{1} = \mathbf{1}, \qquad \mathbf{1} = \{\ldots, 1, 1, 1, \ldots\}, \tag{5}$$

it then follows that

$$\Delta^1(Sm) = S_1(\Delta^1 m), \qquad m \in \ell_\infty(\mathbb{Z}), \tag{6}$$

for some other subdivision operator S_1. Here, Δ^1 is the forward difference operator defined by $(\Delta^1 m)_i = m_{i+1} - m_i$, as usual $\Delta^\mu = (\Delta^1)^\mu$, $\mu > 0$, and $\Delta^0 m = m$. Note that S_1 has again the local support property, and the entries of its matrix representation are linear combinations of the entries of S. Suppose now that S has order $k-1$, in which case $S_1, \ldots, S_{k-1}$ are already defined. Then S has order k if $r^{k-1}S_{k-1}$ reproduces constants, and as above we can define S_k such that

$$\Delta^k(Sm) = \Delta^{k-1}(S_1\Delta^1 m) = \ldots = \Delta^1(S_{k-1}\Delta^{k-1}m) = S_k\Delta^k m \tag{7}$$

for all $m \in \ell_\infty(\mathbb{Z})$. Note that if $\mathbf{\Psi}$ has order k and satisfies (3) or (4), then the derived families $\mathbf{\Psi}_\mu$, $\mu = 1, \ldots, k$, inherit those properties by construction.

Theorem 1. *Assume that the subdivision operator $S = S_{\mathbf{\Psi}}$ is given by (2), where $\mathbf{\Psi}$ has order k and is bounded (3). If*

$$\rho_k := \liminf_{j\to\infty} \sup_{m\in\ell_\infty(\mathbb{Z})} \|\Psi_k(S^{j-1}m)\ldots\Psi_k(Sm)\Psi_k(m)\|_\infty^{1/j} < 1, \tag{8}$$

then the associated subdivision scheme converges, the limit functions $f = S^\infty m$ belong to $C^s(\mathbb{R})$, and satisfy the estimate

$$\|f\|_{C^s} \leq C(s)\|m\|_\infty, \qquad m \in \ell_\infty(\mathbb{Z}), \tag{9}$$

for all $0 \leq s < s_k := \min\{\log_r(1/\rho_k), k\}$. If, in addition, $\mathbf{\Psi}$ is Lipschitz continuous (4) then the subdivision scheme is also C^s-stable, $0 \leq s < s_k$, i.e.,

$$\|f - \tilde{f}\|_{C^s} \leq C(s, \max\{\|m\|_\infty, \|\tilde{m}\|_\infty\})\|m - \tilde{m}\|_\infty \tag{10}$$

for all $m, \tilde{m} \in \ell_\infty(\mathbb{Z})$ $(\tilde{f} = S^\infty \tilde{m})$.

This theorem was essentially proved in [2], the above formulation incorporates two slight improvements: First, we have replaced the joint ℓ_∞-spectral radius definition

$$\tilde{\rho}_k := \liminf_{j\to\infty} \sup_{m^0,\ldots,m^{j-1}\in\ell_\infty(\mathbb{Z})} \|\Psi_k(m^{j-1})\ldots\Psi_k(m^1)\Psi_k(m^0)\|_\infty^{1/j}, \quad (11)$$

for $\mathbf{\Psi}_k$ used in [2] by ρ_k, which gives sometimes better bounds, and, secondly, we observed that the stability proof given in [2] does not require the existence of a linear (and locally supported) left-inverse T of S. The proof of Theorem 1 is given in Appendix A.

There is a slight ambiguity in the above approach: Since a nonlinear subdivision operator S does not uniquely determine $\mathbf{\Psi}$, the application of Theorem 1 may lead to non-optimal results if $\mathbf{\Psi}$ has not been chosen carefully. A particular situation where this potential problem goes away is when S can be represented in the form

$$Sm = \hat{S}m + \Phi(m)\Delta^k m, \qquad m \in \ell_\infty(\mathbb{Z}), \quad (12)$$

where $\hat{S}$ is a linear subdivision operator, and $\mathbf{\Phi} = \{\Phi(m) \,:\, m \in \ell_\infty(\mathbb{Z})\}$ is a family of linear operators on $\ell_\infty(\mathbb{Z})$ with the local support property. Roughly speaking, S is a higher order perturbation of a linear $\hat{S}$. This assumption came as natural in the analysis of the Donoho-Yu scheme [4,8], which appeared to be a perturbation of the midpoint-interpolation scheme. Obviously, if S is given by (12), it can also be written in the form (2) with $\Psi(m) = \hat{S} + \Phi(m)\Delta^k$. This $\mathbf{\Psi}$ has order k if $\hat{S}$ has order k, and is bounded if $\mathbf{\Phi}$ is bounded. Thus, we have the following corollary to the proof of Theorem 1, which is easy to apply in practical applications.

Theorem 2. *Assume that the subdivision operator S is given by (12), where the linear subdivision operator $\hat{S}$ has order k, $\mathbf{\Phi}$ has the local support property and is bounded. If for some $j_0 \geq 1$ the inequality*

$$\|\Delta^k(S^{j_0}m)\|_\infty \leq c_{k,j_0}\|\Delta^k m\|_\infty, \qquad m \in \ell_\infty(\mathbb{Z}), \quad (13)$$

holds with some $c_{k,j_0} < 1$, then the associated subdivision scheme converges, the limit functions $f = S^\infty m$ belong to $C^s(\mathbb{R})$, and satisfy the estimate

$$\|f\|_{C^s} \leq C(s)\|m\|_\infty, \qquad m \in \ell_\infty(\mathbb{Z}), \quad (14)$$

for all $0 \leq s < s_{k,j_0} := \min\{\log_r(1/c_{k,j_0})/j_0, k\}$.

In [8], for the particular example of the Donoho-Yu scheme, it was shown how to derive this result from a perturbation theorem proved in [3], under the assumption that the linear subdivision scheme governed by $\hat{S}$

has an Hölder exponent $\geq s_{k,j_0}$ (since $\hat{S}$ can be replaced by any other $\hat{S}'$ of order k in (12), this is not an essential constraint). From the identity

$$\Delta^k(S^j m) = \Psi_k(S^{j-1}m)\dots\Psi_k(m)\Delta^k m, \quad \Psi_k(m) = \hat{S}_k + \Delta^k\Phi(m),$$

we easily see that

$$\hat{s}_k := \sup_j s_{j,k} \geq s_k \geq \tilde{s}_k := \min\{\log_r(1/\tilde{\rho}_k), k\}. \tag{15}$$

Thus, we should get at least as good smoothness exponent estimates by proving (13) as by obtaining similar bounds for the joint spectral radii ρ_k resp. $\tilde{\rho}_k$. Note that to verify (13), only S (but not $\mathbf{\Phi}$ resp. $\mathbf{\Psi}$) needs to be specified.

§3. Examples

We first treat the Donoho-Yu scheme for quadratic median-interpolation, where $N = 3$, $n = -2$ in (1), and τ is constructed as follows. Given three arguments, say (m_{-1}, m_0, m_1), we determine a quadratic polynomial p whose median value on $I_i := [i, i+1]$ coincides with m_i, $i = -1, 0, 1$. The r components τ_l of τ are then defined as the medians of p with respect to the r subintervals $I'_{l-1} = [(l-1)/r, l/r]$, $l = 1, \dots, r$, of $I_0 = [0, 1]$. Similar definitions can be given using polynomials of arbitrary even degree, however, only the quadratic case can be treated by explicit analytical formulas. For basic results on median polynomial interpolation, see [6]. For $r = 3$, the associated subdivision scheme was investigated in [4] while in [8] the smoothness estimates were significantly improved upon, and extended to the dyadic case.

In the following, we show some details for $r = 2$. The corresponding subdivision operator is denoted by $S = S_{med}$. Let $p = p_{med}$ be parametrized in the form $p(t) = a(t-c)^2 + b$, where $a \neq 0$ iff $(\Delta^2 m)_{-1} = m_1 - 2m_0 + m_{-1} \neq 0$ (if $(\Delta^2 m)_{-1} = 0$ then the median-interpolant p is obviously linear, this exceptional case can be handled separately). Since for monotone and continuous f the median on an interval $I = [t_1, t_2]$ coincides with $f((t_1+t_2)/2)$, most of the time the median interpolation/imputation with p is essentially a linear operation (midpoint evaluation). An exception occurs only if c belongs to the middle portion $((3t_1 + t_2)/4, (3t_1 + t_2)/4)$ of the interval which explains the nonlinearity of S_{med}. It is therefore quite suggestive to compare S_{med} with the linear midpoint-interpolation subdivision operator $\hat{S} = S_{mid}$, where the quadratic polynomial $p = p_{mid}$ is defined by

$$p(-1/2) = m_{-1}, \quad p(1/2) = m_0, \quad p(3/2) = m_1,$$

and $\tau_1 = p(1/4), \tau_2 = p(3/4)$. Using elementary calculus, it has been found in [8], Proposition 1, that

$$(S_{med}m)_{2i+e} = (S_{mid}m)_{2i+e} - \phi_e(c_i)(\Delta^2 m)_{i-1}, \quad e = 0, 1, \qquad (16)$$

which establishes (12) with $k = 2$. Here, c_i is the c-value of the quadratic median-interpolant associated with the triple (m_{i-1}, m_i, m_{i+1}), note that in the exceptional case when $(\Delta^2 m)_{i-1} = 0$ and the associated quadratic median-interpolant becomes linear, we formally set $c_i = \infty$ and $\phi_e(c_i) = 0$). The explicit formulas for ϕ_e established in [8] show that the associated $\mathbf{\Phi}$ is bounded. Unfortunately, they also reveal that $\mathbf{\Phi}$ is not Lipschitz continuous with respect to $m \in l_\infty(\mathbb{Z})$ which leaves the stability question open.

In Theorem 1 of [8] it was rigorously established that $c_{2,1} = 1/2$, and $c_{2,2} < 31/128 < c_{2,1}^2$ for $j_0 = 1, 2$ in (13). Thus, Theorem 2 implies that S_{med} has C^s-limit functions where $0 \le s \le \log_4(128/31) = 1.0229....$ An open conjecture supported by numerical evidence (see [4,8]) is to prove that the Hölder exponent $s_\infty(S_{med})$, i.e., the supremum of the set of all $s > 0$ for which the limit functions $S_{med}^\infty m$ belong to $C^s(\mathbb{R})$ independently of m, coincides with the Hölder exponent $s_\infty(S_{mid}) = \log_2(16/7) = 1.1982...$ of S_{mid}.

Note that the formulation of Theorem 2 suggests a straightforward way to collect numerical evidence on the potential C^s-smoothness. E.g., using the shift-invariance property and the fact that $S_{med}(\lambda m + \beta) = \lambda S_{med} m + \beta$ for arbitrary constants λ, β, the computation of c_{2,j_0}, $j_0 \ge 1$, can essentially be reduced to computing the global maximum of the functional

$$F_{j_0}(x_1, x_2, x_3) = \Big(\max_{i'=-2,\ldots,r^{j_0}-1} |(\Delta^2 S^{j_0} m)_{i'}|\Big) / \Big(\max_{i=-2,-1,0} |(\Delta^2 m)_i|\Big) \qquad (17)$$

where $m = \{\ldots, m_{-2}, m_{-1}, m_0, m_1, m_2, \ldots\} \equiv \{\ldots, x_1, x_2, 0, 1, x_3, \ldots\}$. This leads to the following numerically obtained estimates for s_{2,j_0}:

$$\begin{aligned} r = 2 \ &: \quad s_{2,2} = 1.0963..., \quad s_{2,3} = 1.1141..., \quad s_{2,4} = 1.1293...\,, \\ r = 3 \ &: \quad s_{2,2} = 0.9255..., \quad s_{2,3} = 0.9504..., \quad s_{2,4} = 0.9628...\,. \end{aligned}$$

The values shown for the triadic case have to be compared to the bound $s_{2,1} = \log_3(135/53) = 0.8510...$ rigorously established in [8], and to their conjectured limit of $s_\infty(S_{mid}) = 1$.

As a second example we consider a variation of S_{med} where the median rule (or best local L_1 approximation by constants) is replaced by best local L_∞ approximation by constants, i.e., the interpolation rule now amounts to finding a quadratic polynomial $p = p_\infty$ such that

$$\|p - m_i\|_{C(I_i)} = \min_\gamma \|p - \gamma\|_{C(I_i)} \iff 2m_i = \min_{t \in I_i} p(t) + \max_{t \in I_i} p(t) \quad (18)$$

for $i = -1, 0, 1$, similarly for the imputation step. As above, if $(\Delta^2 m)_{-1} \neq 0$ we use the parametrization $p_\infty(t) = a(t-c)^2 + b$, otherwise we set $c = \infty$. The resulting scheme is denoted by S_{L_∞}. Since for monotone f the best approximating constant in $L_\infty(I)$ is given by $(f(t_1) + f(t_2))/2$, we suggest to define the linear associated linear subdivision operator $\hat{S} := S_{ep}$ by end-point averaging interpolation, i.e., to find $p = p_{ep}$ the interpolation rule would be

$$p(-1) + p(0) = 2m_{-1}, \quad p(0) + p(1) = 2m_0, \quad p(1) + p(2) = 2m_1, \quad (19)$$

and the imputation (for $r = 2$) would give

$$(S_{ep}m)_0 = (p(0) + p(1/2))/2, \quad (S_{ep}m)_1 = (p(1/2) + p(1))/2. \quad (20)$$

Note that (19), (20) imply that

$$p_{ep}(t) = m_{-1}p_{-1}(t) + m_0 p_0(t) + m_1 p_1(t), \quad (21)$$

where $p_{-1}(t) = (2(t-1)^2 - 1)/4$, $p_0(t) = (5 - (2t-1)^2)/4$, $p_1(t) = (2t^2 - 1)/4$ are the Lagrange polynomials associated with (19), and

$$\begin{aligned} (S_{ep}m)_{2i} &= \frac{m_{i-1}}{16} + \frac{9m_i}{8} - \frac{3m_{i+1}}{16}, \\ (S_{ep}m)_{2i+1} &= -\frac{3m_{i-1}}{16} + \frac{9m_i}{8} + \frac{m_{i+1}}{16}, \end{aligned} \qquad i \in \mathbb{Z},\ m \in \ell_\infty(\mathbb{Z}). \quad (22)$$

Lemma 3. *For $r = 2$, we have*

$$(S_{L_\infty}m)_{2i+e} = (S_{ep}m)_{2i+e} + \hat{\phi}_e(c_i)(\Delta_2 m)_{i-1}, \quad e = 0, 1, \quad (23)$$

with bounded, Lipschitz continuous functions $\hat{\phi}_e(\cdot)$ (c_i has a similar meaning as before).

Proof: The proof is similar to that of Proposition 1 in [8]. We give the main steps. We can set $i = 0$, and assume that $(\Delta^2 m)_{-1} \neq 0$, i.e., $c = c_0$ is finite (the case $(\Delta^2 m)_{-1} = 0$ is trivial, and $i \neq 0$ follows by shift-invariance). By symmetry, we have

$$\hat{\phi}_0(c) = \hat{\phi}_1(1-c), \quad \hat{\phi}_1(c) = \hat{\phi}_0(1-c), \quad (24)$$

and we can restrict our attention to $c \leq 1/2$. If $c \notin [-1, 2]$ then p_∞ satisfies the same interpolation conditions (19) as p_{ep}, which gives $p_\infty = p_{ep}$, and since the imputation steps are also identical we arrive at (23) with $\hat{\phi}_e(c) \equiv 0$.

We next consider $c \in [-1, -1/2]$. Then according to (18), p_∞ satisfies on I_{-1} the condition $p(c) + p(0) = 2m_{-1}$ while on the other intervals it satisfies the same interpolation conditions as p_{ep} (see (19)). Consequently, $\tilde{p} := p_\infty - p_{ep} = \beta_0 p_{-1}$. The constant β_0 can be found as follows: We have

$$\tilde{p}(c) + \tilde{p}(0) = \beta_0(p_{-1}(c) + p_{-1}(0)) = -(c-1)^2\beta_0/2.$$

On the other hand, using the interpolation condition for p_∞ and p_{ep} on I_{-1}, we see that

$$\tilde{p}(c) + \tilde{p}(0) = p_{ep}(-1) - p_{ep}(c) = -(\Delta^2 m)_{-1}(1+c)((c-1)/2 + q),$$

where we have used the identity

$$p_{ep}(t) - p_{ep}(t') = (\Delta^2 m)_{-1}(t-t')((t+t')/2 + q), \quad t, t' \in \mathbb{R}, \tag{25}$$

where $q := (\Delta^1 m)_{-1}/(\Delta^2 m)_{-1}$. Since

$$\begin{aligned} 2(m_0 - m_{-1}) &= (p_\infty(0) + p_\infty(1)) - (p_\infty(c) + p_\infty(0)) = a(c-1)^2, \\ 2(m_1 - m_0) &= (p_\infty(2) + p_\infty(1)) - (p_\infty(1) + p_\infty(0)) = -4a(c-1), \end{aligned}$$

and $1/q + 1 = (m_1 - m_0)/(m_0 - m_{-1})$ by definition of q, we get $q = (1-c)/(3+c)$ as a function of c. Substituting into the above expressions, we find

$$\beta_0 = \frac{2(\Delta^2 m)_{-1}(1+c)}{(c-1)^2}\left(\frac{c-1}{2} - \frac{c-1}{c+3}\right) = -(\Delta^2 m)_{-1}\frac{(c+1)^2}{(3+c)(1-c)}. \tag{26}$$

Since for $c \in [-1, -1/2]$ the imputation rules for S_{L_∞} and S_{ep} are the same, we conclude that

$$\begin{aligned} (S_{L_\infty} m)_0 - (S_{ep} m)_0 &= \beta_0(p_{-1}(0) + p_{-1}(1/2)) = \beta_0/16, \\ (S_{L_\infty} m)_1 - (S_{ep} m)_1 &= \beta_0(p_{-1}(1/2) + p_{-1}(1)) = -3\beta_0/16. \end{aligned} \tag{27}$$

Together with (26) this establishes (23) for $c \in [-1, -1/2]$. For $c \in [-1/2, 0]$, the only change is that now $p = p_\infty$ satisfies on I_{-1} the condition $p(-1) + p(c) = 2m_{-1}$. Similar considerations lead to $\tilde{p} = \beta_1 p_{-1}$, where

$$\beta_1 = -(\Delta^2 m)_{-1} c^2/(4 - c^2), \quad c \in [-1/2, 0], \tag{28}$$

and again to (27), with β_0 replaced by β_1 for these c.

Finally, we consider $c \in [0, 1/2]$. This time, p_∞ satisfies the same interpolation conditions as p_{ep} on I_{-1} and I_1, while on I_0 we have $(p(1) + p(c)) = 2m_0$. Thus, we conclude that $\tilde{p} = \beta_2 p_0$, and obtain

$$\tilde{p}(c) + \tilde{p}(1) = \beta_2(p_0(c) + p_0(1)) = (2 + c - c^2)\beta_2. \tag{29}$$

On the other hand, as above

$$\tilde{p}(c) + \tilde{p}(1) = p_{ep}(0) - p_{ep}(c) = -(\Delta^2 m)_{-1} c(c/2 + q), \tag{30}$$

and from

$$2(m_0 - m_{-1}) = (p_\infty(1) + p_\infty(c)) - (p_\infty(-1) + p_\infty(0)) = -a(c^2 + 4c),$$
$$2(m_1 - m_0) = (p_\infty(2) + p_\infty(1)) - (p_\infty(1) + p_\infty(c)) = a(c-2)^2,$$

we compute $q = -(c^2 + 4c)/(2c^2 + 4)$ and

$$\beta_2 = (\Delta^2 m)_{-1} c^2/(2c^2 + 4), \quad c \in [0, 1/2]. \tag{31}$$

The imputation step has to be performed carefully. Since $c \notin I_1'$ we have

$$(S_{L_\infty} m)_1 - (S_{ep} m)_1 = \beta_2 (p_0(1/2) + p_0(1))/2 = 9\beta_2/8. \tag{32}$$

For $c \in [0, 1/4]$ we get

$$\begin{aligned}(S_{L_\infty} m)_0 &= (p_\infty(c) + p_\infty(1/2))/2 \\ &= (S_{ep} m)_0 + (p_{ep}(c) - p_{ep}(0))/2 + \beta_2(p_0(c) + p_0(1/2))/2 \\ &= (S_{ep} m)_0 + \beta_2(p_0(1/2) - p_0(1))/2,\end{aligned}$$

the last equality comes from (30),(29). By substituting (31) we get

$$(S_{L_\infty} m)_0 - (S_{ep} m)_0 = c^2 (\Delta^2 m)_{-1}/(16(c^2 + 2)), \quad c \in [0, 1/4]. \tag{33}$$

A similar formula holds if $c \in [1/4, 1/2]$:

$$(S_{L_\infty} m)_0 - (S_{ep} m)_0 = (c^2 + 8c - 2)(\Delta^2 m)_{-1}/(16(c^2 + 2)). \tag{34}$$

This concludes the proof of Lemma 3. □

From Lemma 3 we see that S_{L_∞} can be written in the form required by Theorem 2, i.e., we have (12) with $k = 2$. Each row of $\Phi(m)$ has at most one non-zero entry given by the corresponding $\hat{\phi}_e(c_i)$. According to (23), (22), the associated $\Psi_2(m) = \hat{S}_2 + \Delta^2 \Phi(m)$ is given by

$$\begin{aligned}(\Delta^2 S_{L_\infty} m)_{2i} &= (\frac{7}{16} + \hat{\psi}_0(c_i))(\Delta^2 m)_{i-1} - (\frac{3}{16} - \hat{\phi}_0(c_{i+1}))(\Delta^2 m)_i, \\ (\Delta^2 S_{L_\infty} m)_{2i+1} &= -(\frac{3}{16} - \hat{\phi}_1(c_i))(\Delta^2 m)_{i-1} + (\frac{7}{16} + \hat{\psi}_1(c_{i+1}))(\Delta^2 m)_i.\end{aligned} \tag{35}$$

where $\hat{\psi}_0(c) = \hat{\phi}_0(c) - 2\hat{\phi}_1(c)$, $\hat{\psi}_1(c) = \hat{\phi}_1(c) - 2\hat{\phi}_0(c)$. Explicit expressions and estimates for the ranges of $\hat{\phi}_e(c)$ and $\hat{\psi}_e(c)$ can be found from the above (see (27),(26),(28) for $c \in [-1, 0]$, (32),(33),(34),(31) for $c \in [0, 1/2]$, and (24) for $c \in [1/2, 2]$, for all other c the functions vanish). In particular, we obtain

$$-3/240 \le \hat{\phi}_e(c) \le 1/16, \qquad -1/16 \le \hat{\psi}_e(c) \le 7/240. \tag{36}$$

By definition of the $\ell_\infty(\mathbb{Z})$ operator norm and the symmetry property (24) we find

$$c_{2,1} \le \sup_{m \in \ell_\infty(\mathbb{Z})} \|\Psi_2(m)\|_\infty = \max_{c,c'} \{ |\frac{7}{16} + \hat{\psi}_0(c)| + |\frac{3}{16} - \hat{\phi}_0(c')| = \frac{2}{3}.$$

More details are given in Appendix B. As a result, we arrive at

Theorem 4. *In the dyadic case, the nonlinear subdivision scheme defined by S_{L_∞} converges and has C^s-limit functions for all s satisfying $0 < s < \log_2(3/2) = 0.5850....$*

To obtain improved bounds, one could again try to verify (13) for some $j_0 > 1$. Using a functional similar to (17), we numerically calculated values of $s_{2,2} = 0.6202...$, $s_{2,3} = 0.6335...$, $s_{2,4} = 0.6446....$ Together with other numerical evidence, this lets us conjecture that the exact Hölder exponent of S_{L_∞} coincides with $s_\infty(S_{ep}) = \log_2(8/5) = 0.6781....$

§Appendix A: Proof of Theorem 1

Our proof of Theorem 1 closely follows the proofs for Theorems 1-4 in [2]. In comparison with [2], we have incorporated slight changes in some steps, especially in the proof of (10), to avoid some additional assumptions made in [2]. The interested reader is recommended to compare with [2].

For simplicity, we will not explicitly display coefficient dependencies but rather use C' if the constant may also depend on $\max\{\|m\|_\infty, \|\tilde{m}\|_\infty\}$, otherwise if C is used it generally depends on S and s only. We freely use the notation introduced in Section 2. In particular, $\mathbf{\Psi}(=\mathbf{\Psi}_0)$ has order k, and the families $\mathbf{\Psi}_\mu$, $\mu = 1, \ldots, k$, are recursively defined by (6), (7). We will always silently assume that $\mathbf{\Psi}$ and thus all $\mathbf{\Psi}_\mu$, $\mu = 0, \ldots, k$, are bounded.

Let us recall the following elementary fact. Suppose that T is a bounded linear operator acting on $\ell_\infty(\mathbb{Z})$ which satisfies the local support property (i.e., $t_{i'i} = 0$ if $|i' - ir| > L$) and whose null-space contains $\mathbf{1}$, i.e., $T\mathbf{1} = \mathbf{0} := (\ldots, 0, 0, 0, \ldots)$. Then, if the entries of the matrix representation of $\hat{T}$ are defined by

$$\hat{t}_{i'i} = -\sum_{l=-\infty}^{i} t_{i'l} = -\sum_{(i'-L)/r \le l \le i} t_{i'l},$$

we see that $Tm = \hat{T}\Delta^1 m$, and that $\hat{T}$ satisfies the local support property with the same (or a smaller) L. Indeed, $\hat{t}_{i'i} = 0$ for all i, i' with $i' \ge ir + L$ because of $T\mathbf{1} = \mathbf{0}$, and obviously for all i, i' with $i' < ir - L$. Moreover,

$$(Tm)_{i'} = \sum_i -(\hat{t}_{i'i} - \hat{t}_{i',i-1})m_i = \sum_i \hat{t}_{i'i}(m_{i+1} - m_i) = (\hat{T}\Delta^1 m)_{i'}.$$

Note that a similar derivation is used to define S_μ resp. $\Psi_\mu(m)$ via (6), (7).

Applying the above to $T := \Psi_\mu(m) - \Psi_\mu(\tilde{m})$ for arbitrarily given $m, \tilde{m} \in \ell_\infty(\mathbb{Z})$ and $\mu = 0, \ldots, k-1$, we deduce

$$\|(\Psi_\mu(m) - \Psi_\mu(\tilde{m}))x\|_\infty \le C'\|m - \tilde{m}\|_\infty \|\Delta^1 x\|_\infty, \quad x \in \ell_\infty(\mathbb{Z}), \qquad (37)$$

under the assumption that also (4) holds. Similarly, if we set $T := \Psi_\mu(\hat{m}) - r^{-\mu}\bar{S}$, $\mu = 0, \ldots, k-1$, where the linear subdivision operator $\bar{S}$ is defined by $(\bar{S}x)_{ir+l} = x_i$, $l = 0, \ldots, r-1$, $i \in \mathbb{Z}$, we find

$$\begin{aligned} \|(\Psi_\mu(\hat{m})x\|_\infty &\le r^{-\mu}\|\bar{S}x\|_\infty + \|(\Psi_\mu(\hat{m}) - r^{-\mu}\bar{S})x\|_\infty \\ &\le r^{-\mu}\|x\|_\infty + C\|\Delta^1 x\|_\infty, \quad x, \hat{m} \in \ell_\infty(\mathbb{Z}). \end{aligned} \tag{38}$$

(37),(38) will be frequently used below.

We first derive the following geometric decay estimates for $\alpha_\mu^j := \|\Delta^\mu m^j\|_\infty$, $\mu = 0, \ldots, k$:

$$\alpha_\mu^j \le C\rho^j \|\Delta^\mu m\|_\infty, \quad \rho_k < \rho \le 1, \quad r^{-\mu} \le \rho, \quad j \ge 0, \tag{39}$$

where C depends on the actual choice of ρ. Fix an arbitrary $\rho_k < \rho \le 1$. By the definition (8) of ρ_k, there is a j_0 (which generally depends on ρ), such that

$$\|\Psi_k(S^{j_0-1}m)\ldots\Psi_k(m)\|_\infty^{1/j_0} \le \rho' := \frac{\rho+\rho_k}{2} \ (<\rho), \quad m \in \ell_\infty(\mathbb{Z}). \tag{40}$$

Writing $j = nj_0 + l$, where $l = 0, \ldots, j_0 - 1$, we find from (7), (40), and (3) that

$$\begin{aligned} \alpha_k^j &= \|\Psi_k(S^{j_0-1}m)\ldots\Psi_k(m)\Delta^k m\|_\infty \\ &\le (\rho')^{nj_0}\|\Psi_k(S^{l-1}m)\ldots\Psi_k(m)\Delta^k m\|_\infty \\ &\le C(\rho')^j\|\Delta^k m\|_\infty \le C\rho^j\|\Delta^k m\|_\infty, \qquad j \ge 0. \end{aligned}$$

This is the geometric decay estimate (39) for $\mu = k$. To get the result for $\mu = k-1, \ldots, 0$, we use induction in μ. Suppose (39) holds for $\mu = \nu + 1$. If $\rho_k < r^{-\nu}$ then we take any ρ such that $\max\{r^{-(\nu+1)}, \rho_k\} < \rho < r^{-\nu}$, otherwise, let $\rho_k < \rho \le 1$ be arbitrary. Then, since $\alpha_\nu^{j+1} = \|\Psi_\nu(m^j)\Delta^\nu m^j\|_\infty$, from setting $\mu = \nu$, $\hat{m} = m^j$, $x = \Delta^\nu m^j$ in (38) we get

$$\begin{aligned} \alpha_\nu^{j+1} &\le r^{-\nu}\alpha_\nu^j + C\alpha_{\nu+1}^j \le r^{-\nu}\alpha_\nu^j + C\rho^j\|\Delta^{\nu+1}m\|_\infty \\ &\le r^{-2\nu}\alpha_\nu^{j-1} + C(\rho^j + r^{-\nu}\rho^{j-1})\|\Delta^{\nu+1}m\|_\infty \\ &\ldots \\ &\le C(\sum_{l=0}^{j} r^{-l\nu}\rho^{j-l})\|\Delta^\nu m\|_\infty \\ &\le C(\max\{r^{-\nu}, \rho\})^{j+1}\|\Delta^\nu m\|_\infty, \end{aligned}$$

where in the last step we have used the special choice of ρ. This establishes (39) for $\mu = \nu$, and concludes the induction step. If $\mu = 0$, (39) states that the sequence $\{m^j\}$ is uniformly bounded in $\ell_\infty(\mathbb{Z})$.

Note that the geometric decay estimates remain true if (8) is replaced by the assumption (13) (clearly, in this case ρ_k should be replaced by $c_{k,j_0}^{1/j_0}$). Indeed, this follows from initially estimating

$$\begin{aligned}\alpha_k^j &= \|\Delta^k(S^j m)\|_\infty \le c_{k,j_0}\|\Delta^k(S^{j-j_0}m)\|_\infty \\ &\le c_{k,j_0}^n\|\Psi_k(S^{l-1}m)\dots\Psi_k(m)\Delta^k m\|_\infty, \quad j = nj_0 + l,\end{aligned}$$

and then repeating the above derivation. This explains how Theorem 2 can be obtained as a corollary to the proof of Theorem 1.

The geometric decay estimates (39) imply the C^s-property in a standard way. Suppose first that $r^{-1} \le \rho_k < 1$. By definition of the sequence $\{g^j\}$ and of the auxiliary subdivision operator $\bar{S}$ (see the beginning of this section), from (38) we have

$$\|g^{j+1} - g^j\|_{L_\infty} = \|\Psi(m^j)m^j - \bar{S}m^j\|_\infty \le C\|\Delta^1 m^j\|_\infty,$$

which together with (39) for $\mu = 1$ implies that

$$\|g^{j+1} - g^j\|_{L_\infty} \le C\rho^j\|\Delta^1 m\|_\infty, \quad j \ge 0,$$

for any $\rho_k < \rho < 1$. Since obviously $\|g^j - f^j\|_{L_\infty} \le \|\Delta^1 m^j\|_\infty$, the same geometric decay estimate holds for $\{f^j\}$, i.e., both sequences $\{g^j\}$ and $\{f^j\}$ uniformly converge to some $f = S^\infty m \in C(\mathbb{R})$ at geometric rate:

$$\max\{\|f - g^j\|_{L_\infty}, \|f - f^j\|_C\} \le C\rho^j\|\Delta^1 m\|_\infty, \quad j \ge 0, \tag{41}$$

for the same range of ρ. Furthermore, for arbitrary x, y with $|x - y| \le 1$ we can find a j such that $r^{-(j+1)} < |x - y| \le r^{-j}$ and get to

$$\begin{aligned}|f(x) - f(y)| &\le 2\|f - g^j\|_{L_\infty} + |g^j(x) - g^j(y)| \\ &\le 2\|f - g^j\|_{L_\infty} + C\|\Delta^1 m^j\|_\infty \\ &\le C\rho^j\|\Delta^1 m\|_\infty \le C|x - y|^s\|\Delta^1 m\|_\infty, \quad s = \log_r(1/\rho).\end{aligned}$$

By varying ρ within the bounds allowed, this shows (9) for $0 \le s < s_k = \log_r(1/\rho_k) \le 1$ since for this range

$$\|f\|_{C^s} = \|f\|_C + \max_{0<|x-y|\le 1} |x - y|^{-s}|f(x) - f(y)|.$$

Note that the necessary bound for $\|f\|_C$ follows from setting $j = 0$ in (41), and using $\|g^0\|_{L_\infty} = \|m\|_\infty$.

If $r^{-2} \le \rho_k < r^{-1}$ then necessarily $k \ge 2$ (for $r = 2$, compare Proposition 2 and (26) in [2]). Since (39) holds for $\mu = 1$ and any $r^{-1} < \rho < 1$ we already know that the subdivision scheme converges, and that the C^s-property holds for all $0 < s < 1$. In particular, $\{f^j\}$ converges uniformly to f. Let g_1^j be the step function which is obtained if we replace m^j by $r^j\Delta^1 m^j$ in the definition of g^j, $j \geq 0$, similarly for f_1^j. Using (39) for $\mu = 2$, we obtain in exactly the same way as above the uniform convergence $g_1^j, f_1^j \to f_1 \in C(\mathbb{R})$ and the C^t-property of f_1 for all $0 < t < s_k - 1$ (since by assumption $s_k \in (1, 2]$ this range is non-empty). We leave this to the reader. Since by construction the derivative of f^j coincides with g_1^j, we can easily establish that f' coincides with f_1. This proves (9) for all $1 < s < s_k \leq 2$ since for this range

$$\|f\|_{C^s} = \|f\|_C + \|f'\|_C + \max_{0<|x-y|\leq 1} |x-y|^{-s-1}|f'(x) - f'(y)|$$

is a suitable norm definition for $C^s(\mathbb{R})$ (note that $s = 1$ follows by interpolation resp. by direct inspection depending on which definition of Hölder spaces is adopted for integer s). Finally, to get the result for arbitrary $r^{-k} \leq \rho_k < 1$, this argument can be used inductively.

To prove the stability result we again closely follow [2]. From now on, both (3) and (4) will be assumed. Lemma 2 from [2] can be replaced by the estimate

$$\beta_k^{j+1} \leq C'\rho^j \sum_{l=0}^{j} \beta_0^l, \quad j \geq 0, \tag{42}$$

where $\rho_k < \rho < 1$, and the abbreviation $\beta_\mu^j = \|\Delta^\mu(m^j - \tilde{m}^j)\|_\infty$ is used for $j \geq 0$ and $\mu = 0, \ldots, k$. Indeed, with the same notation as in the proof of the case $\mu = k$ of (39) we have

$$\begin{aligned}
\beta_k^{j+1} &= \|\Psi_k(m^j)\ldots\Psi_k(m^{j+1-j_0})\Delta^k m^{j+1-j_0} \\
&\qquad - \Psi_k(\tilde{m}^j)\ldots\Psi_k(\tilde{m}^{j+1-j_0})\Delta^k \tilde{m}^{j+1-j_0}\|_\infty \\
&\leq \|\Psi_k(\tilde{m}^j)\ldots\Psi_k(\tilde{m}^{j+1-j_0})\Delta^k(m^{j+1-j_0} - \tilde{m}^{j+1-j_0})\|_\infty \\
&\quad + \|(\Psi_k(m^j)\ldots\Psi_k(m^{j+1-j_0}) - \Psi_k(\tilde{m}^j)\ldots\Psi_k(\tilde{m}^{j+1-j_0})\Delta^k m^{j+1-j_0}\|_\infty \\
&\equiv A^j + B^j.
\end{aligned}$$

We have

$$\begin{aligned}
A^j &= \|\Psi_k(S^{j_0}\tilde{m}^{j+1-j_0})\ldots\Psi_k(\tilde{m}^{j+1-j_0})\Delta^k(m^{j+1-j_0} - \tilde{m}^{j+1-j_0})\|_\infty \\
&\leq (\rho')^{j_0}\beta_k^{j+1-j_0} \leq \rho^{j_0}\beta_k^{j+1-j_0},
\end{aligned}$$

while for estimating B^j we use the notation

$$\begin{aligned}
B^{j'} &:= \|(\Psi_k(m^{j'})\ldots\Psi_k(m^{j+1-j_0}) \\
&\qquad - \Psi_k(\tilde{m}^{j'})\ldots\Psi_k(\tilde{m}^{j+1-j_0}))\Delta^k m^{j+1-j_0}\|_\infty, \\
K^{j'} &:= \|(\Psi_k(m^{j'}) - \Psi_k(\tilde{m}^{j'}))\Psi_k(m^{j'-1})\ldots\Psi_k(m^{j+1-j_0})\Delta^k m^{j+1-j_0}\|_\infty, \\
L^{j'} &:= \|\Psi_k(\tilde{m}^{j'})(\Psi_k(m^{j'-1})\ldots\Psi_k(m^{j+1-j_0}) \\
&\qquad - \Psi_k(\tilde{m}^{j'-1})\ldots\Psi_k(\tilde{m}^{j+1-j_0}))\Delta^k m^{j+1-j_0}\|_\infty,
\end{aligned}$$

where $j-j_0+1 \le j' \le j$. Obviously, $B^{j'} \le K^{j'} + L^{j'}$, and by the boundedness (with constant $\bar{C}$) and Lipschitz continuity (with constant $\bar{C}'$) of the family $\mathbf{\Psi}_k$ we get $L^{j'} \le \bar{C}B^{j'-1}$, and

$$\begin{aligned}K^{j'} &\le \|\Psi_k(m^{j'}) - \Psi_k(\tilde{m}^{j'})\|_\infty \|\Psi_k(m^{j'-1})\ldots\Psi_k(m^{j+1-j_0})\Delta^k m^{j+1-j_0}\|_\infty \\ &\le \bar{C}'\beta_0^{j'}\bar{C}^{j'-(j+1-j_0)}\alpha_k^{j+1-j_0}.\end{aligned}$$

Substituting iteratively and taking into account $B^{j-j_0}=0$ as well as (39) for $\mu=k$, we get

$$\begin{aligned}B^j &\le K^j + \bar{C}B^{j-1} \le \ldots \le \sum_{l=j+1-j_0}^{j} \bar{C}^{j-l}K^l \\ &\le \bar{C}'\bar{C}^{j_0-1}\alpha_k^{j+1-j_0}\sum_{l=j+1-j_0}^{j}\beta_0^l \le C'\rho^j\sum_{l=j+1-j_0}^{j}\beta_0^l,\end{aligned}$$

where C' is a new constant which depends on S, ρ, and $\max\{\|m\|_\infty, \|\tilde{m}\|_\infty\}$. Thus, together with the estimate for A^j, we get

$$\beta_k^{j+1} \le \rho^{j_0}\beta_k^{j+1-j_0} + C'\rho^j\sum_{l=j+1-j_0}^{j}\beta_0^l.$$

Iteration leads to (42).

For $\mu=k-1,\ldots,0$, we use (38) and (37) to estimate

$$\begin{aligned}\beta_\mu^{j+1} &\le \|\Psi_k(\tilde{m}^j)(\Delta^\mu(m^j-\tilde{m}^j))\|_\infty + \|(\Psi_k(m^j)-\Psi_k(\tilde{m}^j))\Delta^\mu m^j\|_\infty \\ &\le r^{-\mu}\beta_\mu^j + \bar{C}\|\Delta^{\mu+1}(m^j-\tilde{m}^j)\|_\infty + \bar{C}'\|m^j-\tilde{m}^j\|_\infty\|\Delta^{\mu+1}m^j\|_\infty \\ &\le r^{-\mu}\beta_\mu^j + C\beta_{\mu+1}^j + C'\rho^j\beta_0^j,\end{aligned} \tag{43}$$

which according to (39) holds for any ρ such that $\rho > \rho_k$ and $\rho \ge r^{-(\mu+1)}$.

In analogy to [2], for $j \ge 0$ we consider the recursion

$$\bar{\beta}_\mu^{j+1} = r^{-\mu}\bar{\beta}_\mu^j + A\bar{\beta}_{\mu+1}^j + A'(\rho'_\mu)^j\bar{\beta}_0^j, \tag{44}$$

if $0 \le \mu < k$, and

$$\bar{\beta}_k^{j+1} = A(\rho')^j\sum_{l=0}^{j}\bar{\beta}_0^l, \tag{45}$$

where $\bar{\beta}_\mu^0 = \beta_\mu^0$ for $0 \le \mu \le k$, and appropriately fixed constants A, A', ρ', and $\rho'_\mu \ge r^\mu$, $\mu=0,\ldots,k-1$, for which we assume

$$\rho_k < \rho'_k := \rho' < \rho'_{k-1} < \ldots < \rho_1 < \rho_0 = 1.$$

Obviously, if the constants A, A' are properly chosen for a given set of admissible ρ'_μ, by comparing (42), (43) with (45), (44), we have $\beta^j_\mu \le \bar\beta^j_\mu$ for all j and μ. From (44) with $\mu = 0$ it follows that $\bar\beta^j_0$ is an increasing sequence, thus, we get from (45)

$$\bar\beta^{j+1}_k \le C'(j+1)(\rho')^j \bar\beta^j_0, \quad j \ge 0. \tag{46}$$

Substituting into (44) for $\mu = k-1$, and taking into account that $\rho' = \rho'_k$, we obtain

$$\begin{aligned}
\bar\beta^{j+1}_{k-1} &\le r^{-(k-1)}\bar\beta^j_{k-1} + C'(j+1)(\rho'_k)^j \bar\beta^j_0 \\
&\le r^{-2(k-1)}\bar\beta^j_{k-1} + C'((j+1)(\rho'_k)^j + jr^{-(k-1)}(\rho'_k)^{j-1})\bar\beta^j_0 \\
&\quad \dots \\
&\le r^{-(j+1)(k-1)}\bar\beta^0_{k-1} + C'(\sum_{l=0}^{j}(j+1-l)r^{-l(k-1)}(\rho'_k)^{j-l})\bar\beta^j_0.
\end{aligned}$$

Since $\bar\beta^0_\mu = \beta^0_\mu \le 2^\mu \beta^0_0 \le 2^\mu \bar\beta^j_0$ and $\rho'_k < \rho'_{k-1}$ as well as $r^{-(k-1)} \le \rho'_{k-1}$, this implies the following bound for $\mu = k-1$ and $j \ge 0$:

$$\bar\beta^j_{k-1} \le C' \max\{(j+1)^2(\rho'_k)^j, r^{-j(k-1)}\}\bar\beta^j_0 \le C'(\rho'_{k-1})^j \bar\beta^j_0, \quad j \ge 0.$$

Repeating the same estimation techniques by induction for $\mu = k-2, \dots, 1$, we get

$$\bar\beta^j_\mu \le C'(\rho'_\mu)^j \bar\beta^j_0, \quad j \ge 0, \tag{47}$$

for all $\mu = 1, \dots, k-1$. In particular, $\bar\beta^j_1 \le C'(\rho'_1)^j \bar\beta^j_0$, where $\rho'_1 < 1$. Substituting the latter inequality into (44) for $\mu = 0$, we get

$$\bar\beta^{j+1}_0 \le (1 + C'(\rho'_1)^j)\bar\beta^j_0 \le [\prod_{l=0}^{j}(1 + C'(\rho'_1)^l)]\bar\beta^0_0 \le C'\bar\beta^0_0 = C'\beta^0_0$$

for all $j \ge 0$. Therefore, the sequence $\{\bar\beta^j_0\}$ is bounded by $C\|m - \tilde m\|_\infty$, by backsubstitution into (47) and (46) (and varying the chosen ρ'_μ, if necessary), we arrive at

$$\|\Delta^\mu(m^j - \tilde m^j)\|_\infty \le C'\tilde\rho^j_\mu \|m - \tilde m\|_\infty, \quad j \ge 0, \tag{48}$$

which holds for arbitrary for arbitrary $\tilde\rho_\mu$ satisfying $\tilde\rho_\mu \ge r^{-\mu}$ and $\tilde\rho_\mu > \rho_k$, and each $\mu = 0, 1, \dots, k$. Since these are exactly the same inequalities that were used for the C^s-smoothness proof (with m replaced by $m - \tilde m$), the stability assertion (10) follows now in the same way. □

§Appendix B: Proof of Theorem 4

We fill the missing technical details in the proof of Theorem 4. Lemma 3 was established in sufficient detail, with the exception of (24), (25), and explicit formulas for $\hat{\phi}_e(c)$ which we need later on. The symmetry argument is as follows: If $p_\infty(t) = a(t-c)^2 + b$ is associated with the data (m_{-1}, m_0, m_1) then $\bar{p}_\infty(t) = p_\infty(1-t) = a(t-(1-c))^2 + b$ is the polynomial associated with the data $(\bar{m}_{-1}, \bar{m}_0, \bar{m}_1) = (m_1, m_0, m_{-1})$. Thus, $\bar{c} = 1-c$, $(S\bar{m})_0 = (Sm)_1$, $(S\bar{m})_1 = (Sm)_0$, and since $(\Delta^2\bar{m})_{-1} = (\Delta^2 m)_{-1}$ we arrive at (24). By definition of $\hat{\psi}_e(c)$, we also have $\hat{\psi}_1(c) = \hat{\psi}_0(1-c)$ and $\hat{\psi}_0(c) = \hat{\psi}_1(1-c)$.

Concerning (25), let us assume that $(\Delta^2 m)_{-1} \neq 0$, and write

$$p_{ep}(t) = \tilde{a}(t-\tilde{c}) + \tilde{b}, \quad \tilde{a} \neq 0.$$

From the interpolation conditions (19) we derive

$$\begin{aligned} 2(m_0 - m_{-1}) &= p_{ep}(1) - p_{ep}(-1) = -4\tilde{a}\tilde{c}, \\ 2(m_1 - m_0) &= p_{ep}(2) - p_{ep}(0) = -4\tilde{a}(\tilde{c}-1), \end{aligned}$$

which gives

$$\tilde{a} = (\Delta^2 m)_{-1}/2, \qquad \tilde{c} = -(\Delta^1 m)_{-1}/(\Delta^2 m)_{-1} = -q.$$

Since $p_{ep}(t) - p_{ep}(t') = \tilde{a}(t-t')(t+t'-2\tilde{c})$, (25) follows by substitution.

The explicit formula for $\hat{\phi}_e(c)$ and $c \le 1/2$ are as follows:

$$16\hat{\phi}_0(c) = \begin{cases} 0, & c \le -1, \\ (c+1)^2/(4-(c+1)^2), & -1 \le c \le -1/2, \\ c^2/(4-c^2), & -1/2 \le c \le 0, \\ c^2/(2+c^2), & 0 \le c \le 1/4, \\ (c^2+8c-2)/(2+c^2), & 1/4 \le c \le 1/2, \end{cases}$$

and

$$16\hat{\phi}_1(c) = \begin{cases} 0, & c \le -1, \\ -3(c+1)^2/(4-(c+1)^2), & -1 \le c \le -1/2, \\ -3c^2/(4-c^2), & -1/2 \le c \le 0, \\ 9c^2/(2+c^2), & 0 \le c \le 1/2. \end{cases}$$

Together with (24), this shows that $\hat{\phi}_e(c)$ is Lipschitz continuous for $c \in \mathbb{R}$, monotone on each of the intervals $I'_l = [l/2, (l+1)/2]$, $l = -2, \ldots, 3$, and vanishes outside $[-1, 2]$. A straightforward computation of the values $\phi_e(l/2)$ gives the following range for $\hat{\phi}_e(c)$, $c \in \mathbb{R}$, as

$$-3/240 = \hat{\phi}_1(-1/2) \le \hat{\phi}_e(c) \le \hat{\phi}_0(1/2) = 1/16, \quad e = 0, 1. \tag{49}$$

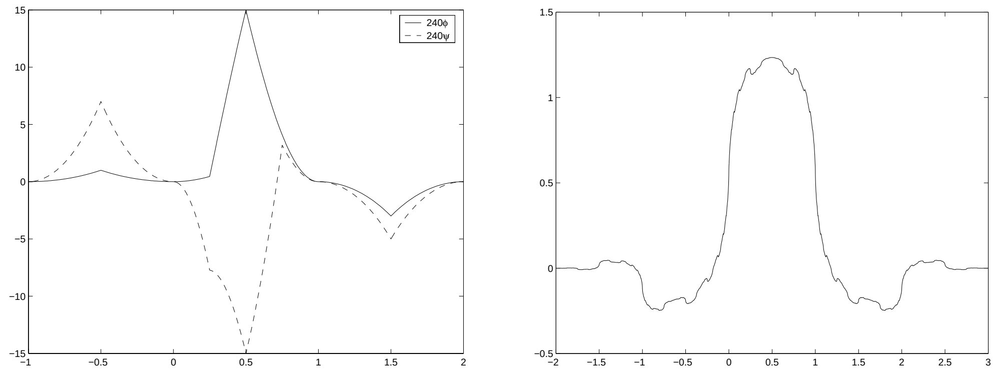

Fig. 1. Graphs of $\hat{\phi}_0$, $\hat{\psi}_0$ (on the left), and $S^{\infty}_{L_\infty}\boldsymbol{\delta}$ (on the right).

Similarly,

$$16\hat{\psi}_0(c) = \begin{cases} 0, & c \le -1, \\ 7(c+1)^2/(4-(c+1)^2), & -1 \le c \le -1/2, \\ 7c^2/(4-c^2), & -1/2 \le c \le 0, \\ -17c^2/(2+c^2), & 0 \le c \le 1/4, \\ (-17c^2+8c-2)/(2+c^2), & 1/4 \le c \le 1/2, \end{cases}$$

and

$$16\hat{\psi}_1(c) = \begin{cases} 0, & c \le -1, \\ -5(c+1)^2/(4-(c+1)^2), & -1 \le c \le -1/2, \\ -5c^2/(4-c^2), & -1/2 \le c \le 0, \\ 7c^2/(2+c^2), & 0 \le c \le 1/4, \\ (7c^2-16c+4)/(2+c^2), & 1/4 \le c \le 1/2, \end{cases}$$

and

$$-1/16 = \hat{\psi}_0(1/2) \le \hat{\psi}_e(c) \le \hat{\psi}_0(-1/2) = 7/240, \quad e = 0, 1. \tag{50}$$

The graphs of $\hat{\phi}_0$ and $\hat{\psi}_0$ are displayed in Figure 1. We also show the graph of the limit function associated with the delta-sequence $\boldsymbol{\delta} = (\ldots, 0, , 0, 1, 0, 0, \ldots)$.

We come to the estimation of $\|\Psi_2(m)\|_\infty$. According to (35) and the above bounds for $\hat{\phi}_0(c)$ and $\hat{\psi}_0(c)$, we have

$$\begin{aligned} |(\Psi_2(m)x)_{2i}| &\le |\frac{7}{16} + \hat{\psi}_0(c_i)||x_{i-1}| + |\frac{3}{16} - \hat{\phi}_0(c_{i+1})||x_i| \\ &\le (5/8 + \hat{\psi}_0(c_i) - \hat{\phi}_0(c_{i+1}))\|x\|_\infty \\ &\le (5/8 + (7+3)/240)\|x\|_\infty = 2/3\|x\|_\infty. \end{aligned} \tag{51}$$

Due to (24), the bound for $|(\Psi_2(m)x)_{2i+1}|$ will be the same. Thus, we have proved

$$c_{2,1} \le \sup_{m \in \ell_\infty(\mathbb{Z})} \|\Psi_2(m)\|_\infty \le \frac{2}{3}, \tag{52}$$

which according to Theorem 2 establishes the C^s-property of S_{L_∞} stated in Theorem 4. The example

$$(\ldots, m_{-1}, m_0, m_1, m_2, \ldots) = (\ldots, -7, -4, 4, 7, \ldots)/5$$

shows that the inequalities in (52) actually turn into equalities. The remaining entries of m are chosen such that $m \in \ell_\infty(\mathbb{Z})$ and $\|\Delta^2 m\|_\infty = 1$, this can obviously be done (note that $(\Delta^2 m)_{-1} = -(\Delta^2 m)_0 = 1$ by the choice of the scaling factor 5). For such an m, we get $c_0 = -1/2$ since $p_\infty(t) = 2(-11 + (2c+1)^2)/15$ satisfies (18) for the data $(m_{-1}, m_0, m_1) = (-7, -4, 4)/5$. By symmetry, we have $c_1 = 3/2$. Thus, for this m and by setting $i = 0$ in (51), we get

$$(\Psi_2(m)\Delta^2 m)_0 = \frac{7}{16} + \hat{\psi}_0(-1/2) + \frac{3}{16} - \hat{\phi}_0(3/2) = 5/8 + 10/240 = 2/3.$$

This establishes equality in (52). The same result can be obtained by directly verifying that for the above m

$$(\ldots, (Sm)_0, (Sm)_1, (Sm)_2, (Sm)_3, \ldots) = (\ldots, -17, -9, 9, 17, \ldots)/15.$$

Finally, let us mention without proof that the above example can be used to establish

$$\tilde{\rho}_2 = \frac{2}{3} > \rho_2,$$

pretty much in the same way as this was done for the Donoho-Yu scheme in [8]. I.e., the spectral radius definition used in [2] does not always allow to capture the maximal possible C^s-smoothness of a nonlinear subdivision scheme.

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Peter Oswald
Bell Laboratories - Lucent Technologies
Murray HilL, NJ 07974
`poswald@research.bell-labs.com`
`http://cm.bell-labs.com/who/poswald`