Multilevel Norms for $H^{-1/2}$

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Abstract

We give some estimates related to finite element multilevel splittings and Sobolev norms of negative order. Basically, results for the positive order case are carried over by duality. In particular, semi-orthogonal splittings based on piecewise constants are studied for Sobolev spaces of order $-1/2$. Numerical experiments are provided for the screen problem.

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1 Introduction

Let Ω be a (bounded, open, connected) polyhedral domain in $\mathbb{R}^d$ equipped with a finite initial partition $\mathcal{T}_0$ into simplices. We assume that the extension property for Sobolev spaces holds (this assumption excludes domains with slits and similar degeneracies). For convenience, we also assume that simplices in $\mathcal{T}_0$ are of diameter $\asymp 1$. Let $\{\mathcal{T}_j\}_{j=0}^{\infty}$ be obtained by regular refinement from $\mathcal{T}_0$, i.e., the $\mathcal{T}_j$ should consist of shape-regular simplices of diameter $\asymp 2^{-j}$ and be quasi-uniform. Let V_j and $\tilde{V}_j$ denote the spaces of piecewise constant functions and linear continuous finite elements with respect to $\{\mathcal{T}_j\}$, respectively. If the latter is equipped with homogeneous boundary conditions we use the notation $\tilde{V}_{j,0}$.

Let Q_j denote the L_2-orthogonal projections onto V_j ($Q_{-1} \equiv 0$), and set $R_j = Q_j - Q_{j-1}$, $j \geq 0$. $\tilde{Q}_j$, $\tilde{Q}_{j,0}$, $\tilde{R}_j$, and $\tilde{R}_{j,0}$ are defined analogously. Norms based on these projectors (and their discrete counterparts defined on V_J) play a central role for the theory of optimal additive preconditioners for finite element discretizations of elliptic problems in Sobolev spaces $H^s(\Omega)$ (see [27]). Recall that we adopt the usual definition of

$$H^s(\Omega) = H^s(\mathbb{R}^d)|_\Omega \ , \quad -\infty < s < \infty \ ,$$

by restriction, i.e., $f \in H^s(\Omega)$ if there exists a $g \in H^s(\mathbb{R}^d)$ such that $g|_\Omega = f$ (in the sense of distributions), and

$$\|f\|_{H^s(\Omega)} = \inf_{g\,:\,f=g|_\Omega} \|g\|_{H^s(\mathbb{R}^d)} \ .$$

We also need the dual scale $\tilde{H}^s(\Omega) = (H^{-s})'(\Omega)$, $-\infty < s < \infty$. We assume that the reader is familiar with these spaces; for more detailed definitions and properties, see [32, Chapter 4], [18], and Section 2 below. A basic theoretical result of multilevel finite element approximation theory which is associated with the so-called BPX-preconditioner [7] can be stated as follows:

Theorem 1 *We have the following norm equivalences: If* $0 \le s < 3/2$ *then*

$$\|u\|_{H^s}^2 \asymp \sum_{j=0}^{\infty} 2^{2js} \|\tilde{R}_j u\|_{L_2}^2 \qquad \forall\, u \in H^s(\Omega)\ , \tag{1}$$

and if $1/2 < s < 3/2$ *then*

$$\|u\|_{H^s}^2 \asymp \sum_{j=0}^{\infty} 2^{2js} \|\tilde{R}_{j,0} u\|_{L_2}^2 \qquad \forall\, u \in H_0^s(\Omega)\ . \tag{2}$$

The proof for $s > 0$ is a consequence of Jackson-Bernstein inequalities of approximation theory and related results for Besov spaces [27, Sections 2-3]. Results of this type have a long history in connection with interpolation theory, see [8] for an early, abstract treatment of decomposition norms in scales of function spaces; compare also [3, 5]. The case $s = 0$ in (1) is obvious by orthogonality.

There are straightforward extensions of the above statements to the range $-3/2 < s < 3/2$ which follow by using interpolation and duality arguments. See, e.g., [9, 10] for an overview of these techniques and the generalization to the biorthogonal setting. In Section 2 below, we demonstrate this reduction, together with a short introduction into the scales of function spaces used in this paper.

A result analogous to (1) holds for the multilevel scale $\{V_j\}$ and $-1/2 < s < 1/2$:

$$\|u\|_{H^s}^2 \asymp \|u\|_{\tilde{H}^s}^2 \asymp \sum_{j=0}^{\infty} 2^{2js} \|R_j u\|_{L_2}^2 \qquad \forall\, u \in H^s(\Omega)\ ,\ -1/2 < s < 1/2\ . \tag{3}$$

Note that $\tilde{H}^s(\Omega) = H^s(\Omega)$ for this range of smoothness parameters s. While extensions to $s \ge 1/2$ seem to be hopeless since $V_J \not\subset H^{1/2}(\Omega)$, the case $s \le -1/2$ deserves additional thought. In Section 3, we treat in detail the limiting case $s = -1/2$ for which we show

Theorem 2 *We have the inequalities*

$$c\|u_J\|_{\tilde{H}^{-1/2}}^2 \le \sum_{j=0}^{J} 2^{-j} \|R_j u_J\|_{L_2}^2 \le C(J+1)^2 \|u_J\|_{\tilde{H}^{-1/2}}^2 \tag{4}$$

which holds with constants independent of $u_J \in V_J$ *and* $J \ge 0$. *The result remains valid if the* $\tilde{H}^{-1/2}$*-norm is replaced by the* $H^{-1/2}$*-norm.*

The analog for closed polyhedral surfaces is also true. With the counterexample at the end of Section 4, we also show that the J^2-behaviour of the upper estimate in (4) cannot be improved. Due to the well-known equivalence between two-sided norm estimates such as (4) and condition number estimates for multilevel preconditioners (see [27]), this also disproves Conjecture 2.2 in [25]. One can clearly generalize this result in many directions (e.g., to $s < -1/2$ or to the sequence $\{\tilde{V}_j\}$ and $s \le -3/2$).

The above results are of interest for applications to boundary integral equations (compare the multilevel preconditioners discussed in [6, 31, 15, 19, 25]) as well as for domain decomposition methods (in connection with interface preconditioners). We refer to [2, 17] for a general introduction to multigrid algorithms and other numerical methods for the solution of integral equations. A few numerical experiments for the screen problem ($d = 1, 2$) are provided in Section 4.

2 Duality

Duality is an important concept for operator equations (see, for example, [1, Chapter 2] for an introduction in the case of Hilbert spaces and, in particular, Sobolev spaces). We will discuss applications to orthogonal subspace splittings.

We have in mind the following abstract setting. Let $L_2(\Omega)$ (the basic Hilbert space in our considerations) be decomposed into the orthogonal sum of closed subspaces W_j :

$$L_2(\Omega) = \bigoplus_{j \ge 0} W_j$$

such that

$$V_J = W_0 \oplus W_1 \oplus \ldots \oplus W_J \ , \quad J \ge 0 \ .$$

One may think of $\{W_j\}$ as given, and $\{V_j\}$ produced by the above formula (here we consider the V_j as subspaces of $L_2(\Omega)$), or, alternatively, start from a strictly increasing, dense sequence of closed subspaces $\{V_j\}$ of $L_2(\Omega)$ from which the complement spaces $W_j = V_j \ominus_{L_2} V_{j-1}$, $j \ge 1$, $W_0 = V_0$, are obtained. In the above notation, W_j coincides with the range of the orthogonal projection R_j.

In the sequel, we assume that

$$X'(\Omega) \supset L_2(\Omega) \supset X(\Omega) \ ,$$

where $X(\Omega)$ is another Hilbert space, with norm $\| \cdot \|_X$, which is dense in $L_2(\Omega)$. Here, $X'(\Omega)$ denotes the dual of $X(\Omega)$, equipped with the norm

$$\|v\|_{X'} = \sup_{0 \ne u \in X(\Omega)} \frac{\langle v, u \rangle_{X' \times X}}{\|u\|_X} \qquad \forall \ v \in X'(\Omega) \ .$$

The duality pairing is defined as the natural extension to $X'(\Omega) \times X(\Omega)$ of the $L_2(\Omega)$ scalar product:

$$\langle v, u\rangle_{X'\times X} = (v,u)_{L_2} \quad \forall\, u \in X(\Omega)\,,\ \forall\, v \in L_2(\Omega)\,.$$

From this, we have $(L_2)'(\Omega) = L_2(\Omega)$ which explains the name *basic* Hilbert space. Note, that $X''(\Omega) = X(\Omega)$ which means in particular that

$$\|u\|_X = \sup_{0\neq v\in X'(\Omega)} \frac{\langle v, u\rangle_{X'\times X}}{\|v\|_{X'}} \qquad \forall\, u \in X(\Omega)\,.$$

Sobolev spaces on $\mathbb{R}^d$ and on domains are extensively treated in [22, 32, 18]. In the introduction, we defined $H^s(\Omega)$ for bounded domains $\Omega \subset \mathbb{R}^d$ by restriction from their well-studied counterparts $H^s(\mathbb{R}^d)$, see [32, Chapter 2 and 4] for this common approach. Furthermore, denote

$$H_0^s(\Omega) = \overline{C_0^\infty(\Omega)}|_{H^s(\Omega)}\ ,$$

which is a proper closed subspace of $H^s(\Omega)$ for $s > 1/2$ while $H_0^s(\Omega) = H^s(\Omega)$ for $s \le 1/2$. Finally, we give an alternative, more explicit definition of the $\tilde{H}^s$-spaces:

$$\tilde{H}^s(\Omega) = \{f = g|_\Omega\ :\ g \in H^s(\mathbb{R}^d)\,,\ \operatorname{supp} g \subset \bar{\Omega}\}$$

(the norm of f is given by the $H^s(\mathbb{R}^d)$-norm of g). It turns out that

$$\tilde{H}^s(\Omega) \subset H_0^s(\Omega)\,, \quad -\infty < s < \infty\,,$$

the embedding being continuous and dense. The spaces coincide as sets and possess equivalent norms if $s > -1/2$ and $s - 1/2 \neq$ integer, in all other cases the embedding is strict. Since

$$(H^s)'(\Omega) = \tilde{H}^{-s}(\Omega)\,, \quad (\tilde{H}^s)'(\Omega) = H^{-s}(\Omega)\,, \qquad -\infty < s < \infty\,, \tag{5}$$

the above explicit definition of $\tilde{H}^s$-spaces is indeed compatible with the definition mentioned in the introduction. Proofs of all these facts can be found in [32, Section 2 and 4] for C^∞-domains. For the polyhedral domains under considerations, the above-mentioned facts remain valid for the range $|s| < 3/2$, at least, see [18] for $d = 2$, [14], [22].

In the special case $s = \pm 1/2$ considered in Section 3 in connection with the proof of Theorem 2 we have

$$\tilde{H}^{1/2}(\Omega) \subset H^{1/2}(\Omega)(= H_0^{1/2}(\Omega))$$

and

$$\tilde{H}^{-1/2}(\Omega) = (H^{1/2})'(\Omega) \subset H^{-1/2}(\Omega) = (\tilde{H}^{1/2})'(\Omega)\,, \tag{6}$$

where the embeddings are continuous and dense, and cannot be replaced by equality. Thus, on domains different from $\mathbb{R}^d$, the two possibilities to define $H^{-1/2}$ (via restriction or as dual to $H^{1/2}(\Omega) = H_0^{1/2}(\Omega)$) lead to close but different results. It depends on the application which of the $H^{\pm 1/2}(\Omega)$ spaces is appropriate. Note that the $\tilde{H}^{-1/2}$ case arises in connection with screen problems (see [28] or [4, Chapter 7]) which lead to the study of the single layer potential operator of order -1. For a flat screen embedded into $\mathbb{R}^3$, it takes the form

$$(Su)(x) = \int_\Omega \frac{u(y)}{|x-y|}\, dy\ , \quad x \in \Omega \subset \mathbb{R}^2\ .$$

The differences between $\tilde{H}^s$ and H^s spaces also show up if multilevel norms with respect to $\{\tilde{V}_j\}$ and $\{\tilde{V}_{j,0}\}$ are considered, see below.

For any sequence $a = \{a_j\}_{j\geq 0}$ of positive weights $a_j > 0$, let

$$\|u\|_{A^a} = \left(\sum_{j=0}^{\infty} a_j \|R_j u\|_{L_2}^2 \right)^{1/2}\ .$$

This multilevel norm is well-defined for all $u \in \cup_{j\geq 0} V_j$, at least (recall that this set is a dense subspace of $L_2(\Omega)$). The associated classes are

$$A^a(\Omega) = \overline{\{\cup_{j\geq 0} V_j\}}\Big|_{\|\cdot\|_{A^a}}\ .$$

This definition serves as a particular example of how to inherit Hilbert space structures from the Cartesian product of W_j spaces. One can view these spaces as generalizations of weighted l_2-spaces.

Note some simple facts. We have

$$L_2(\Omega) \supset A^a(\Omega) \quad \Longleftrightarrow \quad \inf_{j\geq 0} a_j > 0\ ,$$

the embedding being continuous and dense. As shown in [16] (see also [26]), the condition

$$\sum_{j\geq J} a_j^{-1} = \mathrm{O}(a_J^{-1})\ , \quad J \to \infty\ ,$$

implies the equivalence of norms

$$\|u\|_{A^a} \asymp |||u|||_{A^a} = \inf_{v_j \in V_j\,:\, u = \sum_{j\geq 0}^{\infty} v_j} \left(\sum_{j=0}^{\infty} a_j \|v_j\|_{L_2}^2 \right)^{1/2} \tag{7}$$

on $A^a(\Omega)$. If $a_j = 2^{js}$ then the above condition on a is satisfied for $s > 0$. This norm equivalence is helpful since it leads to an optimality result for the additive Schwarz preconditioners associated with the multilevel subspace splitting $V_J = \sum_{j=0}^{J} V_j$ which is related to the BPX-method (compare [27, Section 4]).

However, (7) cannot hold for $s \leq 0$ which implies the non-optimality of the BPX-method for elliptic problems in Sobolev spaces of negative order.

Duality is very simple in the scale of A^a-spaces. Denote the sequence $\{1/a_j\}$ by $1/a$.

Lemma 1 *For all weight sequences a, we have $(A^a)'(\Omega) = A^{1/a}(\Omega)$, with identical norms.*

Proof. It is enough to verify the coincidence of the norms for an arbitrary $u_J \in V_J$, $J \geq 0$. By definition of the A^a-spaces and of the dual norm, the mutual orthogonality of the W_j, the density of $\cup V_j$ in all spaces, and the assumption that dual pairings are inherited from the L_2 scalar product, we obtain

$$\|u_J\|_{(A^a)'} = \sup_{0 \neq v_J \in V_J} \frac{(u_J, v_J)_{L_2}}{\|v_J\|_{A^a}} = \sup_{0 \neq v_J \in V_J} \frac{\sum_{j=0}^J (R_j u_J, R_j v_J)_{L_2}}{(\sum_{j=0}^J a_j \|R_j v_J\|_{L_2}^2)^{1/2}} \ .$$

Now,

$$\sum_{j=0}^J (R_j u_J, R_j v_J)_{L_2} \leq (\sum_{j=0}^J a_j^{-1} \|R_j u_J\|_{L_2}^2)^{1/2} (\sum_{j=0}^J a_j \|R_j v_J\|_{L_2}^2)^{1/2}$$

implies $\|u_J\|_{(A^a)'} \leq \|u_J\|_{A^{1/a}}$, while the special choice $R_j v_J = a_j^{-1} R_j u_J$ leads to a $v_J \in V_J$ which shows the opposite inequality. This gives the lemma.

Let us proceed with an obvious consequence of this lemma which is very useful in many applictions to multilevel schemes. Assume that $\cup V_j$ is also dense in the Hilbert space $X(\Omega)$, satisfying the basic requirement formulated above. Let us assume that for some weight sequence a and positive constants A_J, B_J, we have

$$A_J \|u_J\|_X \leq \|u_J\|_{A^a} \leq B_J \|u_J\|_X \qquad \forall \ u_J \in V_J \ . \tag{8}$$

As long as the subspaces V_J are finite-dimensional, this is not a restrictive requirement. However, when designing fast iterative solvers based on subspace corrections, we aim at proving inequality (8) with tight constants A_J, B_J. Especially, $B_J/A_J = \mathrm{O}(1)$ would be desirable. The next statement is therefore of central importance for optimal preconditioning of discretizations of $X(\Omega)$- or $X'(\Omega)$-elliptic problems.

Theorem 3 *The following statements are equivalent:*

a) $X(\Omega) = A^a(\Omega)$, *with equivalent norms:*

$$A\|u\|_X \leq \|u\|_{A^a} \leq B\|u\|_X \qquad \forall \ u \in X(\Omega); .$$

b) $X'(\Omega) = A^{1/a}(\Omega)$*, with equivalent norms:*

$$B^{-1}\|u\|_{X'} \leq \|u\|_{A^{1/a}} \leq A^{-1}\|u\|_{X'} \qquad \forall\, u \in X'(\Omega)\ .$$

c) The constants A_J, B_J *in (8) are uniformly bounded away from* 0 *and* ∞*, i.e.,* $0 < A \leq A_J \leq B_J \leq B < \infty$*,* $J \geq 0$*, for some positive constants* A, B*.*

d) We have

$$B^{-1}\|u_J\|_{X'} \leq \|u_J\|_{A^{1/a}} \leq A^{-1}\|u_J\|_{X'} \qquad \forall\, u_J \in V_J\ ,\ J \geq 0\ . \tag{9}$$

Proof. The direction a) $\Longrightarrow$ c) is obvious, the opposite direction c) $\Longrightarrow$ a) follows by a density argument (note that the two norms under consideration are by assumption in c) equivalent on the dense set $\cup_{j\geq 0} V_j$). An analogous argument gives b) $\Longleftrightarrow$ d). The equivalence between a) and b) follows from the definition of dual spaces and by Lemma 1.

Theorem 3, together with the above properties of Sobolev spaces leads to the following extension of Theorem 1.

Theorem 4 *We have the following norm equivalences:*

$$\|u\|_{H^s}^2 \asymp \sum_{j=0}^{\infty} 2^{2js}\|\tilde{R}_j u\|_{L_2}^2 \qquad \begin{cases} \forall\, u \in H^s(\Omega) & \text{if } 0 \leq s < 3/2 \\ \\ \forall\, u \in \tilde{H}^s(\Omega) & \text{if } -3/2 < s < 0 \end{cases}\ , \tag{10}$$

and

$$\|u\|_{H^s}^2 \asymp \sum_{j=0}^{\infty} 2^{2js}\|\tilde{R}_{j,0} u\|_{L_2}^2 \qquad \begin{cases} \forall\, u \in \tilde{H}^s(\Omega) & \text{if } 0 \leq s < 3/2 \\ \\ \forall\, u \in H^s(\Omega) & \text{if } -3/2 < s < 0 \end{cases}\ . \tag{11}$$

Proof. The statements for non-negative s follow from Theorem 1 (in the second case, observe that $\tilde{H}^s(\Omega) = H_0^s(\Omega)$ for $1/2 < s < 3/2$, that the case $s = 0$ holds trivially, the intermediate range follows by real interpolation). To get the cases $s < 0$, apply Theorem 3 in conjunction with (5).

3 The piecewise constant case and $s = -1/2$

Throughout this section, let us concentrate on the case $s = -1/2$ and multilevel norms with respect to $\{V_j\}$. We prove Theorem 2, for both norms $\|\cdot\|_{H^{-1/2}}$ and $\|\cdot\|_{\tilde{H}^{-1/2}}$ as introduced above. Since, by (6) and the definitions of Sobolev spaces in Section 2,

$$\cup_{j\geq 0} V_j \subset L_2(\Omega) \subset \tilde{H}^{-1/2}(\Omega) \subset H^{-1/2}(\Omega)$$

and

$$\|u\|_{H^{-1/2}} \le C\|u\|_{\tilde{H}^{-1/2}} \qquad \forall\, u \in \tilde{H}^{-1/2}(\Omega)\ ,$$

we actually have to prove

$$c\|u_J\|^2_{\tilde{H}^{-1/2}} \le \|u_J\|^2_{A^{-1/2}} \le C(J+1)^2\|u_J\|^2_{H^{-1/2}} \quad \forall\, u_J \in V_J\ ,\ J \ge 0\ . \qquad (12)$$

Here, by $A^s(\Omega)$ we denote the space $A^a(\Omega)$ defined with respect to $\{V_j\}$ by the weight sequence $a_j = 2^{2js}$.

We start with the observation that, according to

$$\|u_J\|_{X'} = \sup_{0\ne v\in X(\Omega)} \frac{(u_J,v)_{L_2}}{\|v\|_X} = \sup_{0\ne v_J\in V_J} \frac{(u_J,v_J)_{L_2}}{\inf_{v\in X(\Omega)\,:\,Q_Jv=v_J}\|v\|_X}$$

which holds for any densely embedded Hilbert space $X(\Omega)\subset L_2(\Omega)$, the two-sided inequality (12) is established if we prove

$$\|v_J\|^2_{A^{1/2}} \equiv \sum_{j=0}^{J} 2^j\|R_jv_J\|^2_{L_2} \le C \inf_{v\in H^{1/2}(\Omega)\,:\,Q_Jv=v_J} \|v\|^2_{H^{1/2}} \qquad \forall\, v_J \in V_J\ , \quad (13)$$

and

$$\inf_{v\in \tilde{H}^{1/2}(\Omega)\,:\,Q_Jv=v_J} \|v\|^2_{\tilde{H}^{1/2}} \le C(J+1)^2\|v_J\|^2_{A^{1/2}} \qquad \forall\, v_J \in V_J\ . \qquad (14)$$

The easy part is (13). For any $v \in H^{1/2}(\Omega)$ with $Q_Jv = v_J$ we have

$$\begin{aligned}\|v_J\|^2_{A^{1/2}} &= \sum_{j=0}^{J} 2^j\|R_jv\|^2_{L_2} \le C(\|v\|^2_{L_2} + \sum_{j=0}^{J} 2^jE_j(v)^2_{L_2})\\ &\le C(\|v\|^2_{L_2} + \sum_{j=0}^{J} 2^j\omega_1(2^{-j},v)^2_{L_2}) \le C\|v\|^2_{B^{1/2}_{2,2}} \le C\|v\|^2_{H^{1/2}}\ .\end{aligned}$$

Here,

$$E_j(v)_{L_2} = \|v - Q_jv\|_{L_2}$$

are the L_2-best approximations of v with respect to $\{V_j\}$, and we have used the usual Jackson-type estimate

$$E_j(v)_{L_2} \le C\omega_1(2^{-j},v)_{L_2}\ , \quad v \in L_2(\Omega)\ ,\ j \ge 0\ ,$$

valid for piecewise constant approximation (compare [27, Section 2] for properties of the modulus of continuity

$$\omega_1(t,v)_{L_2} = \sup_{|h|\le t} \|f(\cdot+h) - f(\cdot)\|_{L_2(\Omega_h)}\ , \quad \Omega_h = \{x\in\Omega\,:\,[x,x+h]\subset\Omega\}\ ,$$

$t > 0$, as well as for information on the direct and inverse inequalities of approximation theory with piecewise polynomial functions and on Besov spaces). Taking the infimum with respect to all those v, we arrive at (13).

Figure 1: Definition of $\tilde{v}_{J+1}$: $d = 1$

To give a clear presentation of the technicalities which lead to the more involved estimate (14), we first discuss the case $d = 1$. Let $\Omega = (0,1)$. For an arbitrarily fixed $v_J \in V_J$, we define a piecewise linear function $\tilde{v}_{J+1} \in \tilde{V}_{J+1,0}$ via the nodal values $\tilde{v}_{J+1}(P)$ at the gridpoints of $\mathcal{T}_{J+1}$ as follows: For P which are interior gridpoints of $\mathcal{T}_J$, we take the average of the values $c_\Delta = v_J|_\Delta$ of the piecewise constant function v_J on the two intervals $\Delta \in \mathcal{T}_J$ attached to P. In addition, we set $\tilde{v}_{J+1}(0) = \tilde{v}_{J+1}(1) = 0$. For the remaining P (one point per interval $\Delta \in \mathcal{T}_J$), the nodal value is fixed such that

$$\int_\Delta \tilde{v}_{J+1}\, dx = \int_\Delta v_J\, dx \qquad \forall\, \Delta \in \mathcal{T}_J \ .$$

This condition yields $Q_J \tilde{v}_{J+1} = v_J$ as desired. For uniform partitions, this construction is illustrated in Figure 1.

It is quite obvious from the above construction that

$$\begin{aligned} \|\tilde{v}_{J+1} - v_J\|_{L_2}^2 &\le C 2^{-J}(v_J(0)^2 + v_J(1)^2 + \textstyle\sum_{\bar{\Delta}\cap\bar{\Delta}'\neq\emptyset} |c_\Delta - c_{\Delta'}|^2) \\ &\le C(2^{-J}(v_J(0)^2 + v_J(1)^2) + \omega_1(2^{-J}, v_J)_{L_2}^2) \ . \end{aligned} \tag{15}$$

We have

$$\begin{aligned} \inf_{v\in\tilde{H}^{1/2}(\Omega)\,:\,Q_J v = v_J} \|v\|_{\tilde{H}^{1/2}}^2 &\le \|\tilde{v}_{J+1}\|_{\tilde{H}^{1/2}}^2 \le C\sum_{j=0}^{J+1} 2^j \|\tilde{R}_{j,0}\tilde{v}_{J+1}\|_{L_2}^2 \\ &\le C(2^J\|v_J - \tilde{v}_{J+1}\|_{L_2}^2 + \sum_{j=0}^{J+1} 2^j \|\tilde{R}_{j,0} v_J\|_{L_2}^2) \\ &\le C(2^J\|v_J - \tilde{v}_{J+1}\|_{L_2}^2 + \|v_J\|_{L_2}^2 + \sum_{j=0}^{J} 2^j \tilde{E}_{j,0}(v_J)_{L_2}^2) \end{aligned}$$

(see Theorem 4, the quantities $\tilde{E}_{j,0}(v)_{L_2}$ denote L_2-best approximations of $v \in L_2(\Omega)$ with respect to $\tilde{V}_{j,0}$, $j \geq 0$). Unfortunately, we cannot switch immediately to Besov norms by replacing the above best approximations $\tilde{E}_{j,0}(\cdot)_{L_2}$ since a Jackson type inequality with the above modulus of continuity cannot hold, due to the essential boundary conditions required.

Thus, we have to be more careful. We first try to replace the best approximations $\tilde{E}_{j,0}(v_J)_{L_2}$ by best approximations with respect to $\{\tilde{V}_j\}$ for which Jackson type estimates are valid. To this end, let $R_j v_J \equiv w_j \in W_j$, and define

$$v_j \equiv \sum_{k=0}^{j} w_k = v_j^{(0)} + v_j^* + v_j^{(1)} , \quad j = 0, \ldots, J .$$

where $v^{(0)}$ and $v_j^{(1)}$ coincide with v_j on the left-most and right-most interval in $\mathcal{T}_j$, respectively, and vanish outside the corresponding interval. Thus, we have

$$\|v_j^0\|_{L_2}^2 \leq C 2^{-j} |v_j(0)|^2 , \quad \|v_j^1\|_{L_2}^2 \leq C 2^{-j} |v_j(1)|^2 ,$$

and, since v_j^* vanishes on the two boundary intervals, also

$$\begin{aligned} \tilde{E}_{j,0}(v_j^*)_{L_2} &\leq \|v_j^* - \tilde{Q}_j v_j^*\|_{L_2} + (C 2^{-j} (|\tilde{Q}_j v_j^*(0)|^2 + |\tilde{Q}_j v_j^*(1)|^2))^{1/2} \\ &\leq C \|v_j^* - \tilde{Q}_j v_j^*\|_{L_2} = C \tilde{E}_j(v_j^*)_{L_2} . \end{aligned}$$

Here we can continue by applying the Jackson inequality for $\tilde{E}_j(\cdot)_{L_2}$ and the corresponding properties of the modulus of continuity:

$$\tilde{E}_{j,0}(v_j^*)_{L_2} \leq C \tilde{E}_j(v_j^*)_{L_2} \leq C \omega_1(2^{-j}, v_j^*) \leq C(\omega_1(2^{-j}, v_j) + \|v_j^0\|_{L_2} + \|v_j^1\|_{L_2}) .$$

Finally, note that

$$\tilde{E}_{j,0}(v_J)_{L_2} \leq \|v_J - v_j\|_{L_2} + \|v_j^{(0)}\|_{L_2} + \|v_j^{(1)}\|_{L_2} + \tilde{E}_{j,0}(v_j^*)_{L_2} , \quad j = 0, \ldots, J .$$

Substituting all this and taking into account also (15), we can conclude that

$$\begin{aligned} \inf_{v \in \tilde{H}^{1/2}(\Omega) \,:\, Q_J v = v_J} \|v\|_{\tilde{H}^{1/2}}^2 \leq C \Big(& \textstyle\sum_{j=0}^{J} (|v_j(0)|^2 + |v_j(1)|^2) \\ & + \textstyle\sum_{j=0}^{J} 2^j \|v_J - v_{j-1}\|_{L_2}^2 + \sum_{j=0}^{J} 2^j \omega_1(2^{-j}, v_j)_{L_2}^2 \Big) , \end{aligned} \tag{16}$$

where $v_{-1} = 0$.

The different parts of the expression in the right-hand side of (16) can be estimated as follows. Since

$$|v_j(0)|^2 = |\sum_{k=0}^{j} w_k(0)|^2 \leq (j+1) \sum_{k=0}^{j} |w_k(0)|^2 \leq C(J+1) \sum_{k=0}^{J} 2^k \|w_k\|_{L_2}^2 ,$$

analogously for $|v_j(1)|^2$, we get

$$\sum_{j=0}^{J}(|v_j(0)|^2+|v_j(1)|^2) \le C(J+1)^2 \sum_{k=0}^{J} 2^k \|w_k\|_{L_2}^2 \ . \qquad (17)$$

Next, fix some $0<\epsilon<1$, and use

$$\|v_J - v_{j-1}\|_{L_2}^2 \le (\sum_{k=j}^{J} \|w_k\|_{L_2})^2 \le C \sum_{k=j}^{J} 2^{(k-j)\epsilon} \|w_k\|_{L_2}^2$$

to obtain

$$\sum_{j=0}^{J} 2^j \|v_J - v_{j-1}\|_{L_2}^2 \le C \sum_{k=0}^{J} 2^{k\epsilon} \|w_k\|_{L_2}^2 \sum_{j=0}^{k} 2^{j(1-\epsilon)} \le C \sum_{k=0}^{J} 2^k \|w_k\|_{L_2}^2 \ . \qquad (18)$$

To estimate the last sum, we need the subadditivity of the modulus of continuity with respect to the function argument:

$$\omega_1(2^{-j}, v_j)_{L_2}^2 \le (\sum_{k=0}^{j} \omega_1(2^{-j}, w_k)_{L_2})^2 \le (J+1) \sum_{k=0}^{j} \omega_1(2^{-j}, w_k)_{L_2}^2 \ ,$$

and the Bernstein inequality for piecewise constant functions:

$$\omega_1(2^{-j}, w_k)_{L_2}^2 \le C 2^{k-j} \|w_k\|_{L_2}^2 \ , \quad w_k \in V_k \ , \ j \ge k \ .$$

This yields (in analogy to the derivation of (17))

$$\sum_{j=0}^{J} 2^j \omega_1(2^{-j}, v_j)_{L_2}^2 \le C(J+1)^2 \sum_{k=0}^{J} 2^k \|w_k\|_{L_2}^2 \ . \qquad (19)$$

Since

$$\|v_J\|_{A^{1/2}}^2 = \sum_{k=0}^{J} 2^k \|w_k\|_{L_2}^2$$

by definition of the w_k, the estimates (17), (18), (19) substituted into (16) finally yield (14). This concludes the proof for $d=1$.

Essentially the same reasoning applies to the case $d \ge 2$. We sketch the changes. Fix some $J_0 \ge 1$ such that in the interior of any simplex $\Delta \in \mathcal{T}_J$ there is at least one vertex from $\mathcal{T}_{J+J_0}$. For $d=2,3$, $J_0=2$ will do. For given $v_J \in V_J$, define $\tilde{v}_{J+J_0} \in \tilde{V}_{J+J_0,0}$ by its nodal values $\tilde{v}_{J+J_0}(P)$ where P is in the vertex set of $\mathcal{T}_{J+J_0}$, by applying the following rules:

- Set $\tilde{v}_{J+J_0}(P)=0$ if $P \in \partial\Omega$.
- Define $\tilde{v}_{J+J_0}(P)$ as the average of the values $c_\Delta = v_J|_\Delta$ of v_J on those $\Delta \in \mathcal{T}_J$ for which P belongs to $\partial\Delta$.

- Each of the remaining vertices P from $\mathcal{T}_{J+J_0}$ belongs to the interior of some Δ (and in each Δ there is at least one such point!). For all P belonging to some Δ, set $\tilde{v}_{J+J_0}(P) = c^*_\Delta$, where the value c^*_Δ is chosen such that the resulting $\tilde{v}_{J+J_0}$ satisfies

$$\int_\Delta \tilde{v}_{J+J_0}\,dx = \int_\Delta v_J\,dx \qquad \forall\,\Delta \in \mathcal{T}_J\ .$$

This construction yields, on the one hand, $Q_J\tilde{v}_{J+J_0} = v_J$. On the other hand, since the c^*_Δ (except for those corresponding to boundary simplices) are as in the one-dimensional case local, convex combinations of $c_{\Delta'}$-values, (15) can be replaced by

$$\|v_J - \tilde{v}_{J+J_0}\|^2_{L_2} \le C2^{-Jd}\Big(\sum_{\Delta:\bar{\Delta}\cap\partial\Omega\neq\emptyset} c^2_\Delta + \sum_{\Delta,\Delta':\,\bar{\Delta}\cap\bar{\Delta}'\neq\emptyset} |c_\Delta - c_{\Delta'}|^2\Big)$$

$$\le C\Big(2^{-J}\int_{\partial\Omega}|v_J|^2\,dx + \omega_1(2^{-J},v_J)^2_{L_2}\Big)\ .$$

Again, the trick is to split the analogously defined v_j into boundary part $v^{\partial\Omega}_j \in V_j$ defined by

$$v^{\partial\Omega}_j|_\Delta = \begin{cases} c_\Delta & \text{if } \bar{\Delta}\cap\partial\Omega\neq\emptyset \\ 0 & \text{otherwise} \end{cases}\ , \qquad \Delta\in\mathcal{T}_j\ ,$$

and interior part $v^*_j = v_j - v^{\partial\Omega}_j$. With the indicated modifications, one can now repeat the above proof step by step. Clearly, the boundary terms $|v_j(0)|^2, |v_j(1)|^2$ have to be replaced by the integrals $\|v_j\|^2_{L_2(\partial\Omega)}$ which obviously satisfy

$$2^j\|v^{\partial\Omega}_j\|^2_{L_2(\Omega)} \asymp \|v_j\|^2_{L_2(\partial\Omega)} \le C2^j\|v_j\|^2_{L_2(\Omega)}\ ,\ j\ge 0\ .$$

Analogous estimates hold for $w^{\partial\Omega}_k$. With these remarks, Theorem 2 from Section 1 is established.

4 Discussion

The interest in the suboptimal result stated in Theorem 2 stems from the fact that L_2-stable bases, even Haar-like orthogonal systems, consisting of functions of small compact support can easily be constructed for the L_2-orthogonal complement spaces W_j. This is in contrast to smoother ansatz functions where the orthogonality constraint leads to theoretical and practical problems (e.g., see the proposals in [20, 21, 30, 13] to construct bases in $\tilde{W}_j$ for $d=2$). Thus, an implementation of a suboptimal but simpler preconditioner based on the piecewise constant case might be an alternative to some of the optimal but more complicated preconditioners. However, more careful testing is needed to make the correct judgements. Some theoretical and numerical comparisons within the framework of hp-methods can be found in recent papers of Stephan et al. [19, 25].

Good preconditioning and simplicity of implementation might be achieved if L_2-orthogonal splittings are replaced by biorthogonal splittings. A promising example in this direction is the three-point hierarchical basis preconditioner for linear finite element spaces investigated by Stevenson (see [29] for $d = 2$, [12] for numerical experiments in the context of boundary integral equations, and [23] for some theory covering also $H^{-1/2}$-applications). Compare also the recent paper [13]. Analogs for piecewise constant finite element spaces have also been considered (see [23]).

In order to evaluate the practical impact of different choices of multilevel preconditioners, we have run a few experiments for a model screen problem on the interval ($d = 1$)

$$Vu(x) \equiv -\int_0^1 u(y) \log|x-y|\, dy = f(x) \;, \quad x \in [0,1] \;, \tag{20}$$

and on the unit square ($d = 2$)

$$Vu(x) \equiv \int_{[0,1]^2} \frac{u(y)}{|x-y|}\, dy = f(x) \;, \quad x \in [0,1]^2 \;. \tag{21}$$

Discretization with piecewise constant functions on a dyadic square grid of meshsize 2^{-J} is used (it is obvious that the analog of Theorem 2 holds for these partition sequences as well). To make the computations feasible for large J, we have explored the fact that on uniform partitions the arising dense stiffness matrices A_J are Toeplitz matrices, and that the $\asymp 2^{Jd}$ different entries can easily be computed from available analytic formulae for the antiderivatives of the kernel functions $\log|x-y|$ for $d = 1$ and $1/|x-y|$ for $d = 2$(compare [19]). Thus, using a FFT-implementation of the matrix-vector multiplication with A_J, and taking into account that matrix-vector multiplications with any of the multilevel preconditioning matrices C_J described below take $\mathrm{O}(2^{Jd})$ arithmetical operations, we end up with a guaranteed upper bound of $\leq CJ2^{Jd}$ flops per pcg-step.

We start with the numerical results for $d = 1$ and equation (20. In Table 1, condition numbers for A_J and the preconditioned systems $C_J A_J$ arising from the Haar system, a 3-point and a 4-point system as proposed in [23, Section 4.3] (the latter two systems are modified near the boundary, see below). All these systems have the form $\Psi = \cup_{j\geq 0} \Psi_j$ with Ψ_0 consisting of the only function $\psi_{0,1}(x) = 1$, and

$$\Psi_j = \{\psi_{j,i} = \psi(2^{j-1}x - i)|_{[0,1]},\ i = 1, \ldots, 2^{j-1}\} \;, \quad j \geq 1 \;.$$

Figure 2: Generating functions ψ for $d = 1$

The generating functions ψ for the above mentioned choices are depicted in Figure 2. We made the following modifications for the 3-point and 4-point scheme which slightly improved the condition numbers: For levels $j \leq 2$, the Ψ_j were defined by the Haar function (Figure 2 a)) and not by the functions in Figure 2 b) and c). Additionally, in case of the 3-point scheme, in each Ψ_j, $j > 2$, the function $\psi_{j,2^j-1}$ located at the right end of $[0,1]$ was replaced by the Haar function of level j with the same support (the idea was to preserve at least orthogonality with respect to constants, i.e., zero moments of order one, for all functions in the 3-point system). For computing the extremal eigenvalues of A_J and $C_J A_J$, we have used the Matlab routine *eigs*, called with a routine for matrix-vector multiplications. In all cases, the matrix $C_J A_J$ is given as the representation of the corresponding multilevel Schwarz operator

$$\mathcal{P}_J u_J = \sum_{j=0}^{J} \sum_i 2^j \|\psi_{j,i}\|_{L_2}^{-2} a(u_J, \psi_{j,i}) \tag{22}$$

with respect to the basis of box functions in V_J. The symmetric bilinear form is generated by the operator V for $d = 1$, see (20). For a survey of the additive Schwarz theory and the connection with hierarchically defined multilevel systems we refer to [27, Section 4.1-2] and [23]. The situation of Theorem 2 is implemented by the Haar system: the corresponding Ψ_j form orthogonal bases in the orthogonal complement spaces W_j for all $j \geq 0$.

J	$\kappa(A_J)$ no prec.	$\kappa(C_J A_J)$ Haar	3-point	4-point
4	28.38	4.86	7.02	2.84
6	114.85	8.57	10.91	3.77
8	459	13.11	13.95	4.85
10	1836	18.41	16.19	6.10
12		24.50	17.82	7.52
14		31.37	19.02	9.12

Table 1. Condition numbers for the 1d screen problem

Table 2 contains the iteration count to compute the solution of (20) with $f \equiv 1$ by the pcg-method. The iteration was started with the zero vector and stopped when the residual of the preconditioned linear system was reduced by 10^{-6}.

J	no prec.	Haar	3-point	4-point
4	8	7	14	7
6	17	14	20	8
8	31	19	23	8
10	55	23	25	8
12	96	27	26	8
14	160	30	27	8
16		34	28	8

Table 2. Iteration count for the 1d screen problem

The numerical results are in full agreement with the asymptotic behaviour of the bounds in Theorem 2: The additive Schwarz theory predicts $\kappa(C_J A_J) = \mathrm{O}(J^2)$ for the Haar case which is supported by the linear growth in J of the iteration count of the pcg-method. Below, we will confirm this by providing a theoretical proof of the sharpness of Theorem 2. In view of the results in [23] which let us expect an O(1) bound for $\kappa(C_J A_J)$ in the other two cases, the stabilization of the iteration count for the 3-point and 4-point preconditioner is not a surprise. The latter should also be compared with the results in [15].

We ran a set of analogous experiments for the problem (21). The multilevel preconditioners considered stem from the 2-dimensional analogs of the corresponding one-dimensional systems considered above. Clearly, the sets Ψ_j are now spanned by the shifts of three different ψ^l, $l = 1, 2, 3$, associated with the horizontal, vertical, and diagonal directions of a two-dimensional grid. Figure 3 a)-c) shows the supports and values of the generating ψ^l in all three cases. In addition, we have considered the 'true' two-dimensional orthogonal Haar system (Figure 3 d)) which has been used in the literature, see [25]. For the 3-point and 4-point cases, the same modifications for levels $j \leq 2$ (where the

Figure 3: Generating functions ψ^l for $d = 2$

replacement is with the 'true' Haar system) and boundary functions have been applied. All other specifications have been preserved. The computations of condition numbers and iteration counts are shown in Tables 3 and 4.

	$\kappa(A_J)$	$\kappa(C_J A_J)$			
J	no prec.	'true' Haar	2-point	3-point	4-point
3	21.9	4.55	11.5	16.9	16.5
4	45.0	6.53	16.5	25.9	27.6
5	90.6	8.78	22.2	35.1	42.6
6	181.5	11.32	28.5	44.3	60.9
7		14.14	35.4	52.6	82.4
8		17.20			

Table 3. Condition numbers for the 2d screen problem

J	no prec.	'true' Haar		2-point		3-point	4-point
		add.	mult.	add.	mult.		
3	9	9	5	17	10	22	21
4	13	12	6	23	11	30	28
5	19	15	7	27	13	34	33
6	27	17	8	31	14	39	36
7	38	20	9	35	15	43	40
8	50	22	9	39	16	44	45

Table 4. Iteration count for the 2d screen problem

Although both Haar preconditioners are covered by the estimate $\kappa(C_J A_J) = \mathrm{O}(J^2)$, $J \to \infty$, which follows from Theorem 2, the 'true' Haar system gave better results. To our surprise, the 3-point and 4-point preconditioners did not lead to an improvement within the range $J \leq 8$, even in comparison with no preconditioning (Table 4 also indicates that preconditioning might not be the most important issue in the case $d = 2$). This should be contrasted with the results of [23], where a theoretical O(1) behaviour of condition numbers has been found for the case of $H^{-1/2}(\mathbb{R}^2)$. Seemingly, the boundary modification (i.e., the way of restricting multilevel systems defined on $\mathbb{R}^2$ to a bounded domain such as a square screen) needs further elaboration.

To conclude our discussion of the practical aspects, let us mention that multiplicative Schwarz algorithms and other multigrid preconditioners based on the splittings discussed in this paper may lead to a further improvement. Compare [6], [24] for recent research on such methods in connection with elliptic problems in Sobolev spaces of negative order. Note that according to a general result on additive and multiplicative Schwarz methods (see [27, p. 79–81]), the above norm equivalences automatically guarantee convergence results for the multiplicative Schwarz methods which are of almost the same order (in our applications, the possible change would consist in an additional factor $\log J$ in the corresponding bounds for the condition numbers which is negligible in practice). For comparison, we have implemented a standard symmetric multiplicative Schwarz method associated with the multilevel splittings for the two Haar systems as a preconditioner in a cg-iteration (this method is essentially equivalent to a V-cycle multigrid preconditioner with one pre- and one post-smoothing step using the extrapolated Jacobi method as a smoother). The resulting iteration count is shown in Table 4, next to the results for the additive preconditioners corresponding to these two cases (clearly, the same starting vector and stopping criteria have been used). The relaxation parameters for the Jacobi smoothers were $\omega = 1.2$ and $\omega = 1.3$, respectively, and had been determined experimentally. The experiments document what is expected from the experience in similar applications - the iteration count can be approximately halved if a multiplicative method is used.

There are other aspects, one needs to take into account in connection with solving boundary integral equations such as (21). First of all, as can be seen from the iteration counts in Table 4, one is not so much confronted with the problem of constructing good preconditioners but with accelerating matrix-vector multiplications involving the dense stiffness matrices A_J. The use of FFT-techniques is restricted to very specific domains and uniform, non-adapted grids. One way of overcoming storage and time limitations is to replace A_J by sparse approximations. The required matrix compression can, for instance, be achieved if higher order zero moment conditions are satisfied for the multilevel bases. In this respect, the approach via biorthogonality offers more flexibility as has been stressed in [11]. However, it has some difficulties with the $H^{-1/2}$ case discussed in this paper, especially if the splittings are transplanted from polyhedral domains/manifolds to general manifolds of interest. Compare [11] and the papers cited therein.

On the other hand, solutions of (21) typically exhibit boundary singularities which require a treatment by adaptive refinement. Note that isotropic h-refinement can be achieved by considering subspaces built from *subsystems* of the above multilevel systems. Thus, the condition number bounds for the Haar system (orthogonal splittings) as well as for the other systems are also bounds for preconditioned discretizations in the adaptive case (clearly, then J corresponds to the highest refinement level involved). This further explains our interest in results covering the range of large J. However, due to dominating edge singularities occuring in the solutions of (21), there is definitely need in anisotropic refinement near the edges of Ω. This case is not covered by the results of the present paper. Another problem which has not been addressed in our paper (and which is possibly out of the reach of geometric multilevel methods) is to deal with very complicated surfaces arising, e.g., in connection with capacitance extraction for circuit design.

In the remainder of this section we will show that the factor $(J+1)^2$ in (4) and (14) cannot be improved asymptotically. We give the argument for the asymptotical sharpness of the estimate (14) for the case $d=1$ and with the $\tilde{H}^{1/2}$-norm replaced by the $H^{1/2}$-norm (this gives the sharpness of the upper estimate in (4) in the important $\tilde{H}^{-1/2}$-case). Let $\Omega=(0,1)$ be equipped with uniform partitions $\mathcal{T}_j$ of stepsize 2^{-j}. Consider

$$v_J = -H_2 + \sum_{k=1}^{J-1} (H_{3\cdot 2^{k-1}+1} - H_{3\cdot 2^{k-1}}) \in V_J \ ,$$

where H_j are the Haar functions with normalization $\|H_j\|_{L_\infty}=1$, i.e.,

$$H_{2^k+l}(x) = \begin{cases} 1 \ , & x \in ((l-1)2^{-k},(2l-1)2^{-k-1}) \\ -1 \ , & x \in ((2l-1)2^{-k-1}, l2^{-k}) \\ 0 \ , & \text{otherwise} \end{cases} \ ,$$

for all $l=1,\ldots,2^k$, $k=0,1,\ldots$. The function v_J is depicted in Figure 4 for $J=4$, it is antisymmetric about $x=\frac{1}{2}$. Since $H_2 \in W_1$, and $H_{3\cdot 2^{k-1}+1} - H_{3\cdot 2^{k-1}} \in W_{k+1}$, $k=1,\ldots,J-1$, it follows that

$$\|v_J\|^2_{A^{1/2}} = 2 + \sum_{k=2}^{J} 2^k 2^{-k+2} = 2(2J-1) \asymp J+1 \ . \qquad (23)$$

On the other hand, consider any $v \in H^{1/2}(\Omega)$ such that $Q_J v = v_J$, and let $\tilde{w}_j = \tilde{R}_j v \in \tilde{W}_j$, $j \geq 0$. Without loss of generality, v (and all $\tilde{w}_j$) can be

Figure 4: The extremal function v_J, $J = 4$.

assumed anti-symmetric about $x = \frac{1}{2}$. Thus, $\tilde{w}_0 = \tilde{w}_1 = 0$, and $\tilde{w}_j(\frac{1}{2}) = 0$ for all j. In addition, think of v as belonging to $\cup_{k\geq 0}\tilde{V}_k$ which is a dense subspace in $H^{1/2}(\Omega)$. Then all summations below are finite since $\tilde{w}_j$ vanishes for large enough j. Denote

$$\alpha_{k,j} = 2^j \int_{1/2}^{1/2+2^{-j}} \tilde{w}_k \, dx \ , \quad j \geq 1 \, , \ k \geq 2 \, .$$

We have

$$\alpha_{k,j} = 2^{k-j}\alpha_{k,k} \ , \quad j \geq k \geq 2 \ , \tag{24}$$

since $\tilde{w}_k \in \tilde{W}_k$ is linear on $[\frac{1}{2}, \frac{1}{2} + 2^{-k}]$ and vanishes at $\frac{1}{2}$. Moreover, for $j \leq k$, we have by Jensen's inequality

$$2^{k-j}|\alpha_{k,j}|^2 \leq 2^{k-j}(2^j \int_{1/2}^{1/2+2^{-j}} |\tilde{w}_k| \, dx)^2 \leq 2^{k-j}(2^j \int_{1/2}^{1/2+2^{-j}} \tilde{w}_k^2 \, dx) \leq 2^k \|\tilde{w}_k\|_{L_2}^2 .$$

This yields

$$2^k \|\tilde{w}_k\|_{L_2}^2 \geq m_k^2 \equiv \max_{j=1,\dots,k} 2^{k-j}|\alpha_{k,j}|^2 \ , \quad k \geq 2 \ . \tag{25}$$

Finally, since $Q_J v = v_J$, we have also $Q_j v = Q_j v_J$, $j = 0, \dots, J$, and by the definition of orthogonal projections into spaces of piecewise constant functions we obtain

$$\sum_{k=2}^{\infty} \alpha_{k,j} = (Q_j v_J)(\frac{1}{2}+) = -H_2(\frac{1}{2}+) + \sum_{k=1}^{j-1} H_{3\cdot 2^{k-1}+1}(\frac{1}{2}+) = j \ , \quad j = 1, \dots, J \, . \tag{26}$$

Altogether, using Theorem 4 for $s = 1/2$, we arrive at

$$\inf_{v\in H^{1/2}:\, Q_J v = v_J} \|v\|_{H^{1/2}}^2 \geq C \inf_{v\in H^{1/2}:\, Q_J v = v_J} \sum_{k=0}^{\infty} 2^k \|\tilde{w}_k\|_{L_2}^2 \geq C \inf \sum_{k=0}^{\infty} m_k^2 \equiv C \cdot I^* \ , \tag{27}$$

where the last infimum (which we have denoted by I^*) is taken with respect to all sequences $\{\alpha_{k,j}\}_{k\geq 2}$, $j=1,\ldots,J$, that satisfy (24), (26), and $m_k \geq 0$ is given in (25). Thus, there exists a sequence $\{\bar\alpha_{k,j}\}_{k\geq 2}$, $j=1,\ldots,J$, such that (24), (26) hold, and

$$\sum_{k=2}^{\infty} \bar m_k^2 \leq 2I^* , \quad \bar m_k^2 = \max_{j=1,\ldots,k} 2^{k-j}|\bar\alpha_{k,j}|^2 .$$

Observe that by definition of $\bar m_k \geq 0$ and (24) we have

$$|\bar\alpha_{k,j}| \begin{cases} = 2^{k-j}|\bar\alpha_{k,k}| \leq 2^{k-j}\bar m_k & , \quad k \leq j \\ \leq 2^{(j-k)/2}\bar m_k & , \quad k > j \end{cases} .$$

Thus, if we introduce the sequence

$$\beta_j = \sum_{k=2}^{j} 2^{k-j}\bar m_k + \sum_{k=j+1}^{\infty} 2^{(j-k)/2}\bar m_k , \quad j \geq 1 ,$$

then by (26)

$$\beta_j \geq \sum_{k=2}^{\infty} |\bar\alpha_{k,j}| \geq j , \quad j=1,\ldots,J ,$$

and $\beta_j \geq 0$ for $j > J$. Since the above transformation matrix is exponentially decaying away from the diagonal, it is also bounded as a linear transformation acting on l_2, and the l_2-sequence norm of $\{\beta_j\}$ is bounded by that of $\{\bar m_k\}$. This gives

$$J^3/3 \leq \|\{\beta_j\}\|_{l_2}^2 \leq C\|\{\bar m_k\}\|_{l_2}^2 \leq CI^* .$$

Together with (27) and (23), this proves our claim. We leave it upon the reader to derive from this result the asymptotical sharpness of the upper estimate in (4) and to adapt the example to the case $d \geq 2$.

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