Nonexistence of Compactly Supported Box Spline Prewavelets in Sobolev Spaces

R. A. Lorentz and P. Oswald

Abstract. Motivated by the use of multilevel Riesz basis preconditioners for elliptic problems in Sobolev spaces, we investigate compactly supported prewavelets with respect to homogeneous Sobolev norms for multiresolution analyses based on box splines in $\mathbb{R}^d, d \geq 2$. We show that, in contrast to the cases $L_2(\mathbb{R}^d)$ and $H^s(\mathbb{R}^1)$, prewavelets of compact support with respect to homogeneous Sobolev inner products do not exist for Sobolev exponents s with $s > 0$, $d \geq 2$. They do not even exist if we allow the Riesz basis property to be taken with respect to one Sobolev semi–norm, while the semi–orthogonality is determined with respect to another Sobolev semi–norm. The same is shown for tensor products of univariate multiresolution analyses.

§1. Introduction

The motivation for this work was the usefulness of Riesz bases of Sobolev spaces $H^s(\mathbb{R}^d)$ derived from multiresolution analyses in constructing multilevel preconditioners for the numerical solution of partial differential equations. For uniform partitions, these bases are typically of the form $\psi^\ell(2^j x - k)$, $\ell = 1, \ldots, 2^d - 1$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, where ψ^ℓ are compactly supported functions. Now while it is usually not so hard to obtain the Riesz property on one level, i.e., for the functions $\psi^\ell(2^j x - k)$ for a *fixed* j, it is a non–trivial task to prove the Riesz property for the complete system.

See [6] for a more complete discussion of this subject as well as for numerical methods to determine whether a proposed system is really an H^s-Riesz basis.

One relatively easy way out, if it works, would be to choose the functions ψ^ℓ from a wavelet space determined by a Sobolev inner product. Then the Riesz basis property on one level would immediately imply the Riesz basis property for the whole system.

This works quite well for B–splines in one dimension [7]. But up to now, no one has succeeded with such a construction for box splines in $\mathbb{R}^d, d \geq 2$.

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It is part of the folklore that this is not possible. We show that these doubts were justified.

Some other results connected with these investigations are those of [3,5,8], in which specific constructions of $L_2(\mathbb{R}^d)$-prewavelets for box splines are given. Also [4] have constructed biorthogonal prewavelets with respect to Sobolev norms. In [3], the same authors show that it is not possible to have an L_2-MRA, such that the generating functions are *strictly* orthogonal with respect to a Sobolev norm.

Our general approach is to find a necessary condition in order that translates of compactly supported functions ψ^ℓ form a Riesz basis for the wavelet space determined by the homogeneous Sobolev inner product for a multiresolution analysis generated by a function of compact support. This is given in Theorem 5. Then we show that the condition is satisfied neither in the box spline case nor in the tensor product case.

If the wavelet spaces are determined using the full Sobolev inner product, they are no longer dilates of each other. Thus a prewavelet basis on one level is no longer the dilate of a prewavelet basis on another level. In Section 3, we look at a useful choice of prewavelet bases in this context. But Theorem 1 there shows that such H^s-Riesz prewavelet bases with functions having a uniformly bounded number of coefficients in their symbols exist only if the corresponding homogeneous Sobolev wavelet spaces have Riesz bases of compact support. Thus for box splines and tensor product multiresolution analyses, useful prewavelet bases with respect to the full Sobolev norm also do not exist.

The sufficient condition given in Theorem 5 is quite explicit and although our original motivation was the box spline multiresolution analysis, Theorem 5 could potentially be used in a wide range of examples.

§2. Notation and Basic Facts

By $\widehat{f}$, we denote the Fourier transform of $f \in L_2(\mathbb{R}^d)$, $\widehat{f}(\theta) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\theta}dx$, by $\|f\|_{H^s}$, the Sobolev norm $\|f\|_{H^s} = \{(2\pi)^{-d}\int_{\mathbb{R}^d}(1+|\theta|^2)^s|\widehat{f}(\theta)|^2d\theta\}^{1/2}$ and by $\|f\|_{H^{s,h}}$, the homogeneous Sobolev norm $\|f\|_{H^{s,h}} = \{(2\pi)^{-d}\int_{\mathbb{R}^d}|\theta|^{2s}|\widehat{f}(\theta)|^2d\theta\}^{1/2}$.

Let V be an infinite–dimensional separable Hilbert space. Then $\{\varphi_k\}_{k\in\mathbb{Z}} \subset V$ is said to be a `Riesz system for V` if there exist constants $0 < A \le B < \infty$ such that for any sequence $(c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$,

$$A\sum_{k\in\mathbb{Z}}|c_k|^2\|\varphi_k\|_V^2 \le \|\sum_{k\in\mathbb{Z}}c_k\varphi\|_V^2 \le B\sum_{k\in\mathbb{Z}}|c_k|^2\|\varphi_k\|_V^2. \tag{1}$$

Note that we have included the norms of the basis functions in this definition. We will use the equivalence symbol $\asymp$ for such inequalities saying that $U \asymp U'$ if there exist constants c, C with $0 < c \le C < \infty$ such that $cU \le U' \le CU$. A Riesz system in V which is also a basis for V is called a `Riesz basis for V`.

Taking $V = H^s(\mathbb{R}^d)$ and $\varphi_k = \varphi(\cdot - k)$, $k \in \mathbb{Z}^d$, for some $\varphi \in H^s(\mathbb{R}^d)$, it follows that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is an $H^s(\mathbb{R}^d)$-Riesz system if and only if

$$F_s(\theta)^2 \asymp \|\varphi\|^2_{H^s},$$

where

$$F_s(\theta)^2 := \sum_{\alpha\in\mathbb{Z}^d} (1 + |\theta - 2\pi\alpha|^2)^s \ |\widehat{\varphi}(\theta - 2\pi\alpha)|^2. \tag{2}$$

Similarly, the translates of φ form an $H^{s,h}$–Riesz system if and only if

$$F_{s,h}(\theta)^2 \asymp \|\varphi\|^2_{H^{s,h}},$$

where

$$F_{s,h}(\theta)^2 := \sum_{\alpha\in\mathbb{Z}^d} |\theta - 2\pi\alpha|^{2s} \ |\widehat{\varphi}(\theta - 2\pi\alpha)|^2. \tag{3}$$

A dyadic multiresolution analysis (MRA) of $H^s(\mathbb{R}^d)$, $s \geq 0$, is an increasing scale of closed spaces $V_j \subset H^s(\mathbb{R}^d)$, $j \in \mathbb{Z}$, such that $v_j \in V_j \iff v_j(2\cdot) \in V_{j+1}$, $v_j \in V_j \iff v_j(\cdot - 2^{-j}k) \in V_j$ for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, $\overline{\cup V_j} = H^s(\mathbb{R}^d)$ and finally, that there exists a function $\varphi \in V_j$ such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is an H^s–Riesz basis for V_0. We then say that φ generates the MRA.

We look at two kinds of wavelet bases associated with an $H^s(\mathbb{R}^d)$–MRA:

$$W^s_{j+1} := V_{j+1} \ominus_{H^s} V_j \qquad ; \qquad W^{s,h}_{j+1} := V_{j+1} \ominus_{H^{s,h}} V_j,$$

i.e., wavelet spaces associated with the full norm, respectively the homogeneous norm. Note that the W^s_{j+1} are not dilation invariant, while the $W^{s,h}_{j+1}$ are.

If φ generates an H^s–MRA, then since $\varphi \in V_0 \subset V_1$, there is a sequence $(a_k)_{k\in\mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d)$ such that $\varphi(x) = \sum_{k\in\mathbb{Z}^d} a_k \varphi(2x - k)$ or, equivalently, $\widehat{\varphi}(\theta) = m(-\theta/2)\widehat{\varphi}(\theta/2)$ where $m(\theta) = 2^{-d}\sum_{k\in\mathbb{Z}^d} a_k e^{ik\cdot\theta}$ is the (scaling) symbol of φ. Similarly, if $\psi \in W^s_1$ or $\psi \in W^{s,h}_1$, then $\widehat{\psi}(\theta) = m_\psi(-\theta/2)\widehat{\varphi}(\theta/2)$. The symbols are all 2π–periodic functions belonging to $L_2(\mathcal{C})$, where $\mathcal{C} = [-\pi, \pi]^d$.

Let Λ be the set of vertices of the unit cube $[0,1]^d$ of $\mathbb{R}^d$ and $\Lambda' = \Lambda\backslash\{0\}$. Formulas for Riesz bounds (1) can be simplified by the use of symbols. For example, the system $\{\psi^\ell(\cdot - k)\}^L_{\ell=1,k\in\mathbb{Z}^d}$ is an $H^{s,h}$–Riesz basis for $W^{s,h}_1$ if and only if $L = 2^d - 1$, the system is an $H^{s,h}$–basis for $W^{s,h}_1$ and if

$$\sum_{\lambda\in\Lambda} F_{s,h}(\theta - \pi\lambda)^2 |\sum_{\mu\in\Lambda'} v_\mu m^\mu(-\theta + \pi\lambda)|^2 \asymp \sum_{\mu\in\Lambda'} |v_\mu|^2 \|\psi^\mu\|^2_{H^{s,h}} \tag{4}$$

a.e. in $\mathcal{C}$ for any numbers $v_\mu, \mu \in \Lambda'$. Here m^μ is the symbol of ψ^μ.

In the nonhomogeneous case, the corresponding equation depends on j. Let $\psi^{\lambda,j+1} \in W^s_{j+1}$ have symbol $m^{\lambda,j+1}(\theta) = 2^{-d}\sum_{k\in\mathbb{Z}^d} b_k e^{ik\cdot\theta}$, i.e.,

$$\widehat{\varphi}^{\lambda,j+1}(\theta) = 2^{-jd} m^{\lambda,j+1}(-2^{-j}\theta/2)\widehat{\varphi}(2^{-j}\theta/2) \tag{5}$$

for $\lambda \in \Lambda'$. Then the system $\{\psi^{\lambda,j+1}(\cdot - 2^{-j}k)\}_{\lambda\in\Lambda',k\in\mathbb{Z}^d}$ is an H^s–Riesz system if and only if (4) holds with $F_{s,h}$ replaced by $2^{-jd/2}F_s^{j+1}$ where

$$F_s^{j+1}(\theta)^2 := \sum_{\alpha\in\mathbb{Z}^d} (1+|2^{j+1}(\theta-2\pi\alpha)|^2)^s|\widehat{\varphi}(\theta-2\pi\alpha)|^2. \tag{6}$$

All of these formulas can be derived exactly as in the L_2 case. If $F_{s,h}$, respectively F_s, and the m^μ, respectively $m^{\mu,j+1}$, are continuous, then (4), respectively the inhomogeneous analog, holds uniformly for $\theta \in \mathcal{C}$. This is, for example, the case when φ and ψ^λ, respectively $\psi^{\lambda,j+1}$, are of compact support as will be shown in Lemma 2.

The orthogonality condition that $\psi \in W_1^{s,h}$ can also be written in terms of symbols. If m is the scaling symbol and if m_ψ is the symbol of $\psi \in V_1$, then $\psi \in W_1^{s,h}$ if and only if

$$\sum_{\lambda\in\Lambda} F_{s,h}(\theta-\pi\lambda)^2 m(-\theta+\pi\lambda)\overline{m_\psi(-\theta+\pi\lambda)} = 0. \tag{7}$$

Also if $\psi^{j+1} \in V_{j+1}$, then $\psi^{j+1} \in W_{j+1}^s$ if and only if (7) holds with $F_{s,h}$ replaced by F_s^{j+1}.

§3. The Nonhomogeneous Case

Given an $H^s(\mathbb{R}^d)$–MRA generated by a compactly supported $\varphi \in V_0$, we want to find a Riesz basis built up of Riesz bases of the $W_{j+1}^s, j \in \mathbb{Z}$. As mentioned in Section 2, it is not possible to do this by taking dilations of a Riesz basis of W_1^s. The best one could do is to look for bases of the following form: for each $j \in \mathbb{Z}$, $\{\psi^{\lambda,j+1}(\cdot - 2^{-j}k)\}_{\lambda\in\Lambda',k\in\mathbb{Z}^d}$ is a Riesz basis of W_{j+1}^s such that a) the $\psi^{\lambda,j+1}$ are of uniformly bounded support in the sense that if $\psi^{\lambda,j+1}(x) = \sum_{k\in\mathbb{Z}^d} b_k^{\lambda,j+1}\varphi(2^{j+1}x-k)$, then there is a J such that $b_k^{\lambda,j+1} = 0$ for all $|k| > J$, b) the Riesz bounds of these bases are uniformly bounded in j for all $j \in \mathbb{Z}^+$.

The following theorem shows that if we have such a basis, then there is a Riesz basis of functions of compact support for $W_1^{s,h}$, i.e. for the homogeneous case. But, as will be shown in Section 4, there are no Riesz bases of compact support for $W_1^{s,h}$ for MRA's constructed from box splines and from tensor products in $\mathbb{R}^d, d \geq 2$. Consequently, it will follow from this theorem that there are no bases for these MRA's satisfying a) and b).

Theorem 1. *Let φ be a function of compact support which generates an $H^s(\mathbb{R}^d)$–MRA. Suppose that there exist Riesz bases of the W_{j+1}^s satisfying a) and b) as above. Then there exist functions $\psi^\lambda \in W_1^{s,h}$ of compact support such that $\{\psi^\lambda(\cdot - k)\}_{\lambda\in\Lambda',k\in\mathbb{Z}^d}$ is an $H^{s,h}$–Riesz basis for $W_1^{s,h}$.*

Proof For each $j \in \mathbb{Z}$, let the translates of $\{\psi^{\lambda,j+1}\}_{\lambda\in\Lambda}$ be a normalized H^s-Riesz basis of V^{j+1} satisfying the assumptions of the theorem. Let $m^{\lambda,j+1}$

be the symbol of $\psi^{\lambda,j+1}$ as in (5). It then follows that

$$\sum_{k\in\mathbb{Z}^d} |b_k^{\lambda,j+1}|^2 \asymp 2^{(j+1)(d-2s)} \int\limits_{\mathbb{R}^d} (2^{-2(j+1)} + |\theta|^2)^s |\widehat{\psi}^{\lambda,j+1}(\theta)|^2 d\theta.$$

The normalization of the $\psi^{\lambda,j+1}$ implies that $\sum_{k\in\mathbb{Z}^d} |b_k^{\lambda,j+1}|^2 \asymp 2^{(j+1)(d-2s)}$ as $j\to\infty$.

Thus the sequences $2^{-(j+1)(d-2s)/2} b_k^{\lambda,j+1}$ are uniformly bounded in j for each $\lambda\in\Lambda'$ and $k\in\mathbb{Z}^d$. From assumption a), only a finite number of these sequences are nonzero so that there is a subsequence $(j_r)_{r\in\mathbb{Z}}$ and there are coefficients b_k^λ such that $2^{-(j_r+1)(d-2s)/2} b_k^{\lambda,j_r+1}$ converges to b_k^λ as $r\to\infty$ for all k with $|k|\le J$ and all $\lambda\in\Lambda'$.

For each $\lambda\in\Lambda'$, let $\psi^\lambda(x) = \sum_{k\in\mathbb{Z}^d} b_k^\lambda \varphi(x-k)$, where $b_k^\lambda = 0$ for $|k|>J$. Clearly, the ψ^λ are of compact support. We claim that $\{\psi^\lambda(x-k)\}_{\lambda\in\Lambda, k\in\mathbb{Z}^d}$ is an $H^{s,h}$–Riesz basis for $W_1^{s,h}$.

Since the translates of $\psi^{\lambda,j+1}$ form an H^s–Riesz basis for W_{j+1}^s, (4) holds with $F_{s,h}$ replaced by $2^{-jd/2} F_s^{j+1}$.

The above bounds on $b_k^{\lambda,j+1}$ imply that $2^{(j_r+1)(2s-d)} m^{\lambda,j_r+1}$ converges uniformly to m^λ (the symbol of ψ^λ), as $r\to\infty$. Thus, in the limit, we obtain

$$\sum_{\lambda\in\Lambda} F_{s,h}(\theta)^2 |\sum_{\mu\in\Lambda'} v_\mu m^\mu(-\theta+\pi\lambda)|^2 \asymp \sum_{\mu\in\Lambda'} |v_\mu|^2,$$

i.e., (4) holds and the translates of ψ^λ, $\lambda\in\Lambda'$ form an $H^{s,h}$–Riesz system.

To show that the $\psi^\lambda\in W_1^{s,h}$, we use the nonhomogeneous orthogonality condition (7) with $F_{s,h}$ and ψ replaced by F_s^{j+1} and $\psi^{\lambda,j+1}$ for some $\lambda\in\Lambda'$. Multiplying through by $s^{-(j+1)(d-2s)/2}$, taking $j=j_r$ and letting $r\to\infty$, we see that m^λ satisfies the homogeneous orthogonality condition (7). It remains to show that the span of $\{\psi^\lambda(\cdot - k)\}_{\lambda\in\Lambda, k\in\mathbb{Z}^d}$ is dense in $W_1^{s,h}$. That is, we would like to show that any $v_1\in V_1$ can be written as

$$v_1(x) = v_0(x) + \sum_{\lambda\in\Lambda'}\sum_{k\in\mathbb{Z}^d} a_k^\lambda \psi^\lambda(x-k) \tag{8}$$

for some $v_0\in V_0$. Since the span of $\{\psi^{\lambda,j+1}(\cdot - 2^{-j}k)\}_{\lambda\in\Lambda', k\in\mathbb{Z}^d}$ is H^s–dense in W_{j+1}^s, there exist $a_k^{j,\lambda}$ and $v_0^{j+1}\in V_0$ such that

$$v_1(2^jx) = v_0^{j+1}(2^jx) + \sum_{\lambda\in\Lambda'}\sum_{k\in\mathbb{Z}^d} a_k^{\lambda,j+1}\psi^{\lambda,j+1}(x-2^{-j}k),$$

which we write as

$$\begin{aligned} &v_1(x) = \\ &\quad v_0^{j+1}(x) + \sum_{\lambda\in\Lambda'}\sum_{k\in\mathbb{Z}^d} 2^{(j+1)(d-2s)/2} a_k^{\lambda,j+1} 2^{-(j+1)(d-2s)/2} \psi^{\lambda,j+1}(2^{-j}(x-k)). \end{aligned}$$

Assume, without loss of generality, that v_1 is of compact support. Then there is an L such that $a_k^{\lambda,j+1} = 0$ for $|k| > L$. Since $2^{(j+1)(d-2s)/2} a_k^{\lambda,j+1}$ are uniformly bounded and a subsequence of $2^{-(j+1)(d-2s)/2}\psi^{\lambda,j+1}(2^{-j}\cdot)$ converges to ψ^{λ}, we can find another subsequence j_ℓ such that $2^{(j_\ell+1)(d-2s)/2}a_k^{\lambda,j_\ell+1}$ converges to some a_k^{λ} for each $|k| \leq L$ and $\lambda \in \Lambda'$ and such that $2^{-(j_\ell+1)(d-2s)/2}\psi^{\lambda,j_\ell+1}(2^{-j_\ell}\cdot)$ converges to ψ as $\ell \to \infty$, since all of these functions are of compact support. But then also $v_0^{j_\ell+1}$ converges uniformly to some $v_0 \in V_0$ and we have the representation (8) as desired. ◇

§3. The Homogeneous Case

In all of this section, we will assume that, for some $s > 0$, the compactly supported scaling function φ generates an $H^s(\mathbb{R}^d)$–MRA and for some t with $s < t$ also an $H^t(\mathbb{R}^d)$-MRA. The main theorem of this section will be a necessary condition in order that the integer translates of some compactly supported functions $\psi^{\lambda} \in V_1$, $\lambda \in \Lambda'$ be an $H^{s,h}$–Riesz basis of $W_1^{s,h}$.

We first need some technical lemmas

Lemma 2. *Suppose that φ generates an $H^s(\mathbb{R}^d)$–MRA and an $H^t(\mathbb{R}^d)$–MRA for some s, t with $0 \leq s < t$. Then the function $F_{s,h}(\theta)^2$ given by (3) is continuous.*

Proof Since φ is of compact support, $\widehat{\varphi}$ is continuous. We have

$$\sup_{\theta\in\mathcal{C}} \sum_{|\alpha|\geq N} |\theta - 2\pi\alpha|^{2s}|\widehat{\varphi}(\theta - 2\pi\alpha)|^2 \leq B \sup_{\substack{\theta\in\mathcal{C}\\ |\alpha|\geq N}} |\theta - 2\pi\alpha|^{-2(t-s)} \leq BCN^{-2(t-s)}$$

for some B, $C > 0$ since $\hat{\varphi}$ generates an $H^t(\mathbb{R}^d)$–MRA.

From this inequality and the fact that the function g_N given by

$$g_N(\theta) := \sum_{|\alpha|\leq N} |\theta - 2\pi\alpha|^{2s}|\widehat{\varphi}(\theta - 2\pi\alpha)|^2$$

is uniformly continuous, the lemma follows. ◇

If, starting from the definition (3) of $F_{s,h}$, we replace $\widehat{\varphi}(\theta)$ by $m(\theta/2)\widehat{\varphi}(\theta/2)$ and split the sum according to the elements of $\mathbb{Z}^d/(2\mathbb{Z})^d$, we obtain

$$F_{s,h}(\theta)^2 = 2^{2s}\sum_{\lambda\in\Lambda} |m(-\frac{\theta}{2} + \pi\lambda)|^2 F_{s,h}(\frac{\theta}{2} - \pi\lambda)^2. \tag{9}$$

For the following lemma, we choose the normalization $\widehat{\varphi}(0) = 1$, so that $m(0) = 1$. In addition, we use the fact that $\widehat{\varphi}(\theta) = \Pi_{j=1}^{\infty} m(2^{-j}\theta)$, the convergence being uniform on compact subsets of $\mathbb{R}^d$.

Lemma 3. *Let the compactly supported scaling function φ generate an $H^s(\mathbb{R}^d)$–MRA and an $H^t(\mathbb{R}^d)$–MRA for some $0 < s < t$. Then $F_{s,h}(\theta)^2$ vanishes only at $\theta = 0$ and, everywhere in a neighborhood of $\theta = 0$,*

$$F_{s,h}(\theta)^2 \asymp |\theta|^{2s}.$$

Proof Let $g(\theta) = F_{s,h}(\theta)^2$, then $g(\theta) \geq |\widehat{\varphi}(\theta)|^2|\theta|^{2s}$. Since g, according to Lemma 2, and $\widehat{\varphi}$ are continuous and $\widehat{\varphi}(0) = 1$, there is a $\delta_1 > 0$ and an $A > 0$ such that $A|\theta|^{2s} \leq g(\theta)$ for all θ with $|\theta| < \delta_1$.

Suppose now that there is no $\delta_2 > 0$ and no $B > 0$ such that $g(\theta) \leq B|\theta|^{2s}$ for all θ with $|\theta| < \delta_2$. Then for each $k = 1, 2, \ldots$, there is a θ_k with $|\theta_k| < k^{-1}$ such that $k|\theta_k|^{2s} \leq g(\theta_k)$. From (9), it follows that $g(\theta) \geq 2^{2s}|m(-\theta/2)|^2 g(\theta/2)$ and thus, for any $\ell = 1, 2, \ldots$,

$$g(\theta) \geq 2^{2\ell s} \prod_{j=1}^{\ell} |m(-2^{-j}\theta)|^2 g(2^{-\ell}\theta). \tag{10}$$

For each k, choose $\ell(k)$ such that $\delta_3/2 \leq |2^{\ell(k)}\theta_k| < \delta_3$, where δ_3 is chosen so that $|\widehat{\varphi}(\theta)|^2 \geq B$ for $|\theta| \leq \delta_3$. Then $\ell(k) \to \infty$ as $k \to \infty$. Using (10),

$$\begin{aligned} g(2^{\ell(k)}\theta_k) &\geq k 2^{2\ell(k)s}|\theta_k|^{2s} \prod_{j=1}^{\ell(k)} |m(-2^{-j}2^{\ell(k)}\theta_k)|^2 \\ &\geq k\left(\frac{\delta_3}{2}\right)^{2s} \prod_{j=1}^{\ell(k)} |m(-2^{j}2^{\ell(k)}\theta_k)|^2. \end{aligned}$$

Since the product $\Pi_{j=1}^{\ell} m(2^j\theta)$ converges to $\widehat{\varphi}(\theta)$ uniformly on compact subsets, it follows that for ℓ sufficiently large $\Pi_{j=1}^{\ell}|m(2^{-j}\theta)|^2 \geq C$ for some $C > 0$ and all θ with $|\theta| < \delta_3$. Consequently, since $\ell(k) \to \infty$ and $|2^{\ell(k)}\theta_k| \leq \delta_3$,

$$g(2^{\ell(k)}\theta_k) \geq k\left(\frac{\delta_3}{2}\right)^{2s} C$$

for k large enough, which contradicts the boundedness of $g(\theta)$ on the disk $|\theta| \leq \delta_3$.

To show that $F_{s,h}(\theta)^2$ vanishes only at $\theta = 0$, we use the fact that $(1 + |\theta - 2\pi\alpha|^2)^s \sum_{\alpha \in \mathbb{Z}^d} |\widehat{\varphi}(\theta - 2\pi\alpha)|^2 > A > 0$. Thus, for each θ, there is an $\alpha(\theta) \in \mathbb{Z}^d$, for which $\widehat{\varphi}(\theta - 2\pi\alpha(\theta)) \neq 0$. For each $\theta \in \mathcal{C}\backslash\{0\}$, $|\theta - 2\pi\alpha(\theta)|^{2s}|\widehat{\varphi}(\theta - 2\pi\alpha(\theta)|^2 \neq 0$ and hence $F_{s,h}(\theta) \neq 0$. ♢

The last tool we need is a factorization theorem for complex trigonometric polynomials vanishing on hyperplanes.

Lemma 4. *Let T be a complex 2π–periodic trigonometric polynomial on $\mathbb{R}^d$. Let $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ and $a = \gcd(x_1, \ldots, x_d)$. If $T(\theta) = 0$ for all θ with $x \cdot \theta = 0$, then $1 - e^{-ix\cdot\theta/a}$ divides $T(\theta)$, i.e., the quotient is a trigonometric polynomial.*

Proof The special case that $x = e_1 = (1, 0, \ldots, 0)$ is easy since this is essentially the univariate case. For the general case, one uses the fact that if $x \in \mathbb{Z}^d$ has $\gcd(x_1, \ldots. x_d) = 1$, then there exists an automorphism on $\mathbb{Z}^d$ mapping x to e_1 to reduce the general case to the special case. ♢

Before we state the main theorem of this section, we look at the matrix $M_0(\theta) = \overline{(m^\mu(-\theta+\pi\lambda))}_{\lambda,\mu\in\Lambda'}$, where m^μ are the symbols of $\psi^\mu \in W_1^{s,h}$. If $\{\psi^\mu(x-d)\}_{\mu\in\Lambda',\alpha\in\mathbb{Z}^d}$ is to be a Riesz basis of $W_1^{s,h}$, then (4) must be satisfied. Since φ, ψ^μ are of compact support, $F_{s,h}$ and m^μ are continuous, (4) holds everywhere. Taking $\theta = 0$, we obtain

$$\sum_{\lambda\in\Lambda'} F_{s,h}(-\pi\lambda)|\sum_{\mu\in\Lambda'} v_\mu m^\mu(\pi\lambda)|^2 \asymp \sum_{\mu\in\Lambda} |v_\mu|^2 \cdot \|\psi^\mu\|^2_{H^{s,h}}.$$

Since $F_{s,h}(-\lambda\pi) \neq 0$ for $\lambda \in \Lambda', M_0(\theta)$ is invertible at $\theta = 0$ and since its components are continuous, it is invertible in a neighborhood of $\theta = 0$.

Theorem 5. *Let the compactly supported φ generate an $H^s(\mathbb{R}^d)$–MRA and an $H^t(\mathbb{R}^d)$–MRA for some s, t with $0 \le s < t$. If the integer translates of ψ^λ, $\lambda \in \Lambda'$ form a Riesz basis of $W_1^{s,h}$, then*

$$|\theta|^{2s}T(\theta) \asymp m(-\theta+\pi\lambda)| \tag{11}$$

in a neighborhood of $\theta = 0$ for each $\lambda \in \Lambda'$. Here $T(\theta)$ is the numerator of the rational trigonometric polynomial $M_0^{-1}(\theta)C(\theta)$ with $C(\theta) = \overline{(m^\lambda(-\theta))}_{\lambda\in\Lambda'}$.

Proof The orthogonality conditions (7) for the ψ^λ can be written as

$$m(-\theta)F_{s,h}(\theta)^2 C(\theta) = -M_0(\theta)D(\theta), \tag{12}$$

where $C(\theta)$, $M_0(\theta)$ are as above and $D(\theta) = (m(-\theta+\pi\lambda)F_{s,h}(\theta-\pi\lambda)^2)_{\lambda\in\Lambda'}$.

In a neighborhood of $\theta = 0$, $m(-\theta) \asymp 1$, $F_{s,h}(\theta)^2 \asymp |\theta|^{2s}$, $M_0(\theta)$ is invertible and $m(-\theta+\pi\lambda)F_{s,h}(\theta-\pi\lambda)^2 \asymp m(-\theta+\pi\lambda)$ since $F_{s,h}(-\pi\lambda) \neq 0$ for $\lambda \in \Lambda'$. Inverting the matrix in (12) and observing that $M_0(\theta)^{-1}C(\theta)$ is a rational trigonometric polynomial which does not vanish at $\theta = 0$, we obtain (11). ♢

§4. Application to Box Splines and Tensor Products

In this section we will show that (11) does not hold for box splines and tensor product MRA's in $\mathbb{R}^d$ with $d \ge 2$.

The box spline B_X in $\mathbb{R}^d$ associated with directions $X = (x^1, \dots, x^n)$ with $x^k \in \mathbb{Z}^d$ is defined by

$$\widehat{B}_X(\theta) = \prod_{k=1}^{n} \frac{\sin(x^k \cdot \theta/2)}{x^k \cdot \theta/2} e^{-i(x^1+\ldots+x^n)\cdot\theta/2}.$$

In order that B_X be a function, we must assume that $x^1, \dots, x^n$ span $\mathbb{R}^d$. B_X is of compact support if the matrix formed by any set of d directions in X is unimodular. Then B_X generates an $H^s(\mathbb{R}^d)$–MRA for all s with $0 \le s < r_X - 1/2$, where r_X is the minimum number of columns one can delete from X so that the remaining columns no longer span $\mathbb{R}^d$. As an example, take $x^k = e_k$, $k = 1, \dots, d$ and $x^{d+1} = (1, \dots, 1)$. These are linear box splines. In this case, $r_X = 2$ so that B_X generates an $H^s(\mathbb{R}^d)$–MRA for all $0 \le s < 3/2$.

Theorem 6. *Let $d \geq 2$ and X be a set of directions as above with $r_X \geq 1$. Let $0 < s < r_X - 1/2$. Then for the $H^s(\mathbb{R}^d)$–MRA generated by B_X, there is no $H^{s,h}$–Riesz basis of $W_1^{s,h}$ of the form $\{\psi^\lambda(\cdot - k)\}_{\lambda \in \Lambda', k \in \mathbb{Z}^d}$ with compactly supported functions ψ^λ.*

Proof All of the hypotheses of Theorem 5 are satisfied. The scaling function of B_X is

$$m_X(\theta) = \prod_{k=1}^{n} \cos \frac{x^k \cdot \theta}{2} e^{-i(x^1 + \ldots + x^d)\cdot\theta/2}.$$

Thus, near $\theta = 0$, $m_X(-\theta + \pi\lambda) \asymp \prod_{k \in J_\lambda} |\sin(x^k \cdot \theta/2)|$ where $J_\lambda = \{k | x^k \cdot \lambda \text{ is odd}\}$ and $\lambda \in \Lambda'$. From the assumptions on X, it follows that J_λ contains at least r_X elements. Then (11) becomes

$$|\theta|^{2s} T(\theta) \asymp \prod_{k \in J_\lambda} |\sin \frac{x^k \cdot \theta}{2}| \tag{13}$$

everywhere in a neighborhood of $\theta = 0$ for each $\lambda \in \Lambda'$. We fix one $\lambda \in \Lambda'$ and take an $x^\ell \in J_\lambda$. Then T must vanish on the whole hyperplane $x^\ell \cdot \theta = 0$ since otherwise there would be a $\theta_\ell \neq 0$ with $x^\ell \cdot \theta_\ell = 0$ lying in the neighborhood for which (11) holds and for which $T(\theta_\ell) \neq 0$. But this is not possible since then the left hand side of (13) does not vanish, while the right hand side does. Now use Lemma 4 to conclude that $T(\theta) = (1 - e^{-ix^\ell \cdot \theta / a_\ell}) R(\theta)$, where $a_\ell = \gcd(x_1^\ell, \ldots, x_d^\ell)$. Dividing both sides of (13) by $1 - e^{-ix^\ell \cdot \theta/a_\ell}$, we obtain

$$|\theta|^{2s} R(\theta) \asymp \prod_{k \in J_\lambda \setminus \{x^\ell\}} |\sin \frac{x^k \cdot \theta}{2}|$$

since $|(1 - e^{-ix^\ell \cdot \theta/a_\ell})^{-1} \sin(x^\ell \cdot \theta/2)| \asymp 1$ in a neighborhood of $\theta = 0$.

We continue the procedure with the other elements of J_λ to finally obtain

$$|\theta|^{2s} S(\theta) \asymp 1,$$

where S is some trigonometric polynomial. But now we are left with a contradiction since $|\theta|^{2s} = 0$ for $\theta = 0$ and so (13) is not satisfied for $\theta = 0$. $\diamondsuit$

Using Theorem 1, we immediately have

Corollary 7. *Let the box spline B_X satisfy the assumptions of Theorem 6. Then there are no compactly supported H^s–Riesz bases of the W_{j+1}^s of the form $\{\psi^{\lambda,j+1}(\cdot - 2^{-j}k)\}_{\lambda \in \Lambda', k \in \mathbb{Z}^d}$ with compactly supported $\psi^{\lambda,j+1}$ satisfying a) and b) of Section 3.*

Corollary 8. *Let the box spline B_X satisfy the assumptions of Theorem 6. Let $0 < s_1 \neq s_2 < r_X - 1/2$. Then there is no compactly supported*

$H^{s_1,h}$–Riesz basis of $W_1^{s_2,h}$ of the form $\{\psi^\lambda(\cdot - k)\}_{\lambda\in\Lambda', k\in\mathbb{Z}^d}$ with compactly supported ψ^λ.

Proof The proof is exactly as in Theorem 6. That the basis is an $H^{s_1,h}$–Riesz basis is used to conclude that $M_0(\theta)$ is invertible in a neighborhood of $\theta = 0$. The orthogonality condition results in (11) with $s = s_2$. ♢

We remark that the proof of Theorem 6 would break down for $d = 1$ at the place where one looks at the hyperplane perpendicular to a direction x^ℓ.

Theorem 9. *Let φ_1 and φ_2 generate $H^s(\mathbb{R}^1)$–MRA's for some s with $0 < s$ and be of compact support. Then the $H^s(\mathbb{R}^2)$–MRA generated by $\varphi_1(x)\cdot\varphi_2(y)$ has no $H^{s,h}$–Riesz basis of $W_1^{s,h}$ consisting of translates of compactly supported functions $\psi^\lambda, \lambda\in\Lambda'$.*

Proof This follows as in the proof of Theorem 6 since $m(\theta_1,\theta_2) = m_{\varphi_1}(\theta_1)m_{\varphi_2}(\theta_2)$ so that the zeros of m are the coordinates axes.

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Rudolph A. Lorentz
GMD
Schloß Birlinghoven
D-53754 St. Augustin
Germany
`lorentz@gmd.de`

Peter Oswald
GMD
Schloß Birlinghoven
D-53754 St. Augustin
Germany
`oswald@gmd.de`
http://www.gmd.de/SCAI/people/oswald.html