Scaling Function and Wavelet Preconditioners for Second Order Elliptic Problems

Jeonghwan Ko, Andrew J. Kurdila and Peter Oswald

Abstract. In this paper we present a theoretical framework and numerical comparisons for multilevel solution procedures associated with both scaling functions and wavelets of second order elliptic boundary value problems for a simple class of bounded domains. In particular, we consider a multiwavelet formulation using AFIF-elements. The advantage is in the simplicity of the boundary modification, and relatively small masks representing the differential operators, in contrast to other wavelet-based methods. A brief comparison to conventional finite element methodologies is included.

§1 Introduction

It is by now well-appreciated that wavelets and multiresolution analysis are useful as an analysis and discretization tool (see, *e.g.* , [31, 15, 5, 17]). Particular attention has been paid to the incorporation of wavelet and multiresolution analysis in the study of integral, differential, and pseudodifferential operators. For example, Beylkin, Coifman and Rokhlin in [3, 2] have derived estimates for the compression of pseudodifferential operators. Jaffard (see [24] and [25]) has obtained a number of interesting results applicable to elliptic boundary value problems including asymptotically optimal complexity results, local regularity, and refinement methodologies that retain optimal approximation order. Dahmen, Prößdorf and Schneider [12, 13, 14] have investigated the various approximation theoretic issues in the representation of pseudodifferential operators in terms of wavelets. More recently, Dahlke and DeVore [8] have utilized wavelet techniques to

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derive the regularity of solutions to boundary value problems in the scale of Besov spaces as a prelude to the investigation of adaptive algorithms.

One early result, derived, for example, by Jaffard in [24], that attracted the interest of many researchers active in computational mechanics and scientific computing, was the potential for asymptotically optimal complexity in wavelet Galerkin methods. Essentially, this result asserts that if an elliptic boundary value problem is cast in terms of a wavelet basis, there is a diagonal preconditioner that uniformly bounds the condition number of the matrix representation of the equations that must be solved. Some of the early papers in the use of wavelets for approximation of partial differential equations provided empirical verification of this fact for elliptic problems subject to periodic boundary conditions. Tables that illustrate that the condition number of the preconditioned matrix associated with elliptic equations does not increase with dimensionality can be found in [24] and [3]. In a philosophically similar result, Rieder et al. [34, 35, 36] show that multigrid methods can be derived using wavelets in which the contraction rate is independent of the number of levels of discretization.

At the same time, a number of researchers have concentrated on practical implementations of wavelet formulations in a growing number of applications. An important question, the discussion of which is neglected in many publications, is how one goes about actually computing with some of the "esoteric" wavelet bases that have been derived by analysts. Along these lines, the work of Latto, Tenenbaum, and Resnikoff [28], and the later, more comprehensive paper by Dahmen and Micchelli [11] give a good account of the calculations required to implement wavelets in approximation of differential equations. Similar results may also be found in [1] and [3]. Alternatively, fictitious domain techniques have been employed by Glowinski et al. in [19, 20] to enable a unified treatment of different boundary conditions.

In the present paper we focus on a class of domains which is still very close to the rectangular situation, and discuss multilevel preconditioners based on the use of scaling functions (ϕ-algorithms), and of wavelet functions (ψ-algorithms). In Section 2, we briefly state some general assumptions under which these multilevel algorithms lead to uniformly bounded condition number estimates, and, thus, to asymptotically optimal algorithms for second order elliptic problems. This material is essentially known, however, due to the assumption on the domains, no boundary modification is necessary. This also simplifies the implementation.

In Section 3, the particular example of a multiresolution analysis using so-called AFIF-elements is examined. These multiwavelets have been introduced by Hardin et al. [23, 18, 16], and are somehow intermediate to classical finite elements, and wavelet families such as Daubechies' compactly supported orthogonal wavelets. The application to the discretization of boundary value problems has previously been discussed in [27, 26].

Numerical comparisons, including AFIF-elements, Daubechies scaling functions, and low order finite elements, are reported on in Section 4 for typical H^1-problems in one and two dimensions. They confirm the theory outlined in Section 2 and show the asymptotical optimality of the preconditioners. In these experiments, the finite element solvers perform slightly better while the AFIF-element preconditioner is more robust if low-order terms are present.

§2 Multilevel framework

We start by introducing a multiresolution analysis on all of $\mathbb{R}^d$. Since we will use second order elliptic boundary value problems as our prototype for comparing multilevel preconditioners, we make the following assumptions:

(A1) The *scaling functions* $\phi^l \in L_2(\mathbb{R}^d)$, $l = 1, \ldots, L$, have compact support and belong to $H^t(\mathbb{R}^d)$ for some $t > 1$.

(A2) The set of integer shifts of the scaling functions has the property of *local linear independence*. That is, if Λ_0 denotes the set of all index pairs (l, α), $l = 1, \ldots, L$, $\alpha \in \mathbf{Z}^d$, such that

$$\operatorname{supp} \phi^l(\cdot - \alpha) \cap (0,1)^d \neq \emptyset ,$$

then we can conclude that

$$\| \sum_{(l,\alpha)\in\Lambda_0} c_{l,\alpha} \phi^l(\cdot - \alpha) \|_{L_2((0,1)^d))} = 0 \implies c_{l,\alpha} = 0 \quad \forall\, (l,\alpha) \in \Lambda_0 .$$

(A3) The space $V_0 = \operatorname{span}_{L_2}\{\phi^l(\cdot - \alpha)\}$ contains linear polynomials.

(A4) The scaling functions are refinable, *i.e.* ,

$$\phi^l(x) = \sum_{l'=1}^{L} \sum_{\alpha\in\mathbf{Z}^d} a_\alpha^{l,l'} \phi^{l'}(2x - \alpha)$$

for some finite sequences $(a_\alpha^{l,l'})$.

The scaling functions corresponding to Daubechies' family of orthonormal, compactly supported wavelets, Coifman's compactly supported wavelets, affine fractal wavelets, and standard finite elements on uniform partitions satisfy these hypotheses.

It is well-known that multiresolution analyses defined via sufficiently smooth scaling functions resp. wavelets provide explicit characterizations for a number of function spaces including the Sobolev and Besov scales on $\mathbb{R}^d$, see [31, 17]. For example, as was stated in [33, Sections 2 and 5.1], the system of functions

$$\phi^l_{j,\alpha}(x) = \phi^l(2^j x - \alpha) , \quad \alpha \in \mathbf{Z}^d , \ j = 0, 1, \ldots, \ l = 1, \ldots, L ,$$

generated from a set of scaling functions $\{\phi^l\}$ satisfying properties (A1) through (A4) yields a frame in $H^s(\mathbb{R}^d)$, $0 < s < \min(t, 2)$, after normalization. Criteria for the construction of wavelet and other multilevel Riesz bases in Besov-Sobolev spaces are discussed in [10]. Definitions for Riesz bases and frames, and their connection to multiresolution analyses and wavelets can be found in, *e.g.* , [15, 6, 5, 33].

Extensions of such results to domains $\Omega \subset \mathbb{R}^d$, $d > 1$, usually require a sort of boundary modification in the system $\{\phi^l_{j,\alpha}\}$: for each $j \geq 0$, those of the $\phi^l_{j,\alpha}$ with support intersecting with a certain neighborhood of the boundary $\partial\Omega$ have to be replaced by their boundary-adapted counterparts $\phi^l_{j,\alpha,\Omega}$ while functions are simply dropped from the system if $\operatorname{supp} \phi^l_{j,\alpha} \cap \Omega = \emptyset$. See [7, 33] for examples in the case $d > 1$ where local support of the functions and local polynomial reproduction in the resulting subspaces are taken as the important features to be preserved. In other papers (such as [24, 25]), the boundary adaption is incorporated via a Schmidt orthogonalization process and may lead to functions with global support.

In this note, we considerably simplify the task by concentrating on the following class of domains. Let $Q = (0, 1)^d$ denote the open d-dimensional unit cube, and introduce the notation $Q_{j,\alpha} = 2^{-j}(Q - \alpha)$ for the *dyadic cubes of level* $j \geq 0$ ($\alpha \in \mathbf{Z}^d$). We call a bounded open domain $\Omega \subset \mathbb{R}^d$ *aligned with the cube structure* if for some $j_0 \geq 0$ there is an index set $\mathcal{I}_0 \subset \mathbf{Z}^d$ such that

$$\bar{\Omega} = \bigcup_{\alpha \in \mathcal{I}_0} \overline{Q_{j_0,\alpha}} .$$

In addition, we exclude the possibility of slits and cuts (what is actually needed is the extension property for $H^s(\Omega)$ ($s > 0$) to hold). The last assumption can be removed if the following construction is slightly changed, compare the discussion in [32] for the case of finite element multilevel schemes. After scaling, we may assume that $j_0 = 0$ which we will do throughout the paper.

For domains that are aligned with the cube structure, a boundary modification is not necessary, simply set

$$\Phi_{j,\Omega} = \{\phi^l_{j,\alpha}|_\Omega \, : \, \operatorname{supp} \phi^l_{j,\alpha} \cap \Omega \neq \emptyset\} , \quad j \geq 0 .$$

Then, $\Phi_{j,\Omega}$ is an algebraic basis in

$$V_{j,\Omega} = (\operatorname{span}\{\phi^l_{j,\alpha}\})|_\Omega , \quad j \geq 0 ,$$

which we call *nodal basis* in $V_{j,\Omega}$.

Theorem 1. *Suppose that $\{\phi^l\}$ satisfies (A1)–(A4), and that Ω is aligned with the cube structure ($j_0 = 0$). Then $\{V_{j,\Omega}\}$ is a multiresolution analysis with the following properties:*

(i) For any $g_j = \sum_{(l,\alpha)} c_{l,\alpha}\phi^l_{j,\alpha}|_\Omega \in V_{j,\Omega}$,

$$\|g_j\|^2_{L_2(\Omega)} \asymp \sum_{(l,\alpha)} 2^{-jd}|c_{l,\alpha}|^2 \tag{2.1}$$

(ii) For $0 < s < \min(t, 2)$, we have

$$\|f\|^2_{H^s(\Omega)} \asymp \inf_{g_j \in V_{j,\Omega}\,:\, f=\sum_{j=0}^\infty g_j} \sum_{j=0}^{\infty} 2^{2js}\|g_j\|^2_{L_2(\Omega)} \tag{2.2}$$

for all $f \in H^s(\Omega)$. In other words, the system

$$\tilde{\Phi}_\Omega = \bigcup_{j=0}^{\infty} \Phi_{j,\Omega} \tag{2.3}$$

is a frame in $H^s(\Omega)$ after normalization, i.e. ,

$$\|f\|^2_{H^s(\Omega)} \asymp \sum_{j=0}^{\infty}\sum_{(l,\alpha)} \frac{(f,\phi^l_{j,\alpha})^2_{H^s(\Omega)}}{2^{2js}\|\phi^l_{j,\alpha}\|^2_{L_2(\Omega)}} \qquad \forall\, f \in H^s(\Omega)\ . \tag{2.4}$$

In the above statements, $\asymp$ stands for a two-sided estimate, with positive constants that may only depend on Ω, on s, d, and on $\{\phi^l\}$. The summations $\sum_{(l,\alpha)}$ are with respect to index pairs representing the basis functions in $\Phi_{j,\Omega}$.

We sketch the proof of Theorem 1 for the sake of completeness only. By assumptions (A1)-(A2), and using the usual dilation arguments, we observe that for each cube $Q_{j,\beta} \subset \operatorname{supp}\phi^l_{j,\alpha}$ there is a function $\psi^l_{j,\alpha,\beta}$ with support in $Q_{j,\beta}$, such that

$$\|\psi^l_{j,\alpha,\beta}\|_{L_2(Q_{j,\beta})} \le C2^{jd/2}\ , \tag{2.5}$$

and

$$\int_{Q_{j,\beta}} \psi^l_{j,\alpha,\beta}(x)\phi^{l'}_{j,\alpha'}(x)\,dx = \delta_{l,l'}\delta_{\alpha,\alpha'} \tag{2.6}$$

for all $\alpha' \in \mathbf{Z}^d$, $l' = 1, \ldots, L$ (here, and in the following, C denotes a generic positive constant). With these *biorthogonal* functions at hand, various *quasi-interpolants* can be introduced. For the domains under consideration, for any $j \geq 0$ and any (l, α) corresponding to a nontrivial basis function $\phi^l_{j,\alpha}|_\Omega \in \Phi_{j,\Omega}$, we can fix the above cube $Q_{j,\beta}$ to be *inside* Ω (here, the specific structure of the domain is essentially used). Define

$$Q_j f = \sum_{(l,\alpha)} \int_{Q_{j,\beta}} \psi^l_{j,\alpha,\beta}(x) f(x)\, dx \cdot \phi^l_{j,\alpha}|_\Omega \ . \tag{2.7}$$

Obviously, by (2.6) we see that Q_j is a projection from $L_2(\Omega)$ onto $V_{j,\Omega}$, and the L_2-stability of the basis $\Phi_{j,\Omega}$ expressed by Theorem 1 (i) follows from the local support assumption (A1) and (2.5) in a standard way.

To prove assertion (ii) of Theorem 1, it is enough to verify Jackson-Bernstein inequalities for the sequence $\{V_{j,\Omega}\}$ as stated below, compare the approach of Dahmen [10], especially, Theorem 5.1.1 (i). Since (A3) impies at most reproduction of linear polynomials, the standard second-order L_2-modulus of continuity

$$\omega_2(\delta, f)_{L_2} = \sup_{0 \leq |h| \leq \delta} \|\triangle^2_h f\|_{L_2(\Omega)}$$

will be used, where the difference operator $\triangle^2_h$ is defined by

$$\triangle^2_h f(x) = \begin{cases} f(x+h) - 2f(x) + f(x-h) & \text{if } [x-h, x+h] \subset \Omega \\ 0 & \text{otherwise} \end{cases} \ .$$

We can prove a Jackson-type inequality

$$\|f - Q_j f\|_{L_2(\Omega)} \leq C\omega_2(2^{-j}, f)_{L_2} \ , \quad f \in L_2(\Omega) \ , \tag{2.8}$$

by first establishing it for smooth functions $f \in H^2(\Omega)$ in the form

$$\|f - Q_j f\|_{L_2(\Omega)} \leq C 2^{-2j} \|f\|_{H^2(\Omega)} \ , \tag{2.9}$$

and then using real interpolation (to this end, the L_2-boundedness of $Q_{j,\Omega}$ is needed which follows directly from the formula (2.7)). To show (2.9), the definition (2.7) of $Q_{j,\Omega}$, assumption (A3), and a Bramble-Hilbert argument have to be explored (here, the exclusion of slits is important for the proof, otherwise the union of the spaces $V_{j,\Omega}$ might not even be dense in $H^s(\Omega)$).

The Bernstein inequality

$$\omega_2(\delta, g_j)_{L_2} \leq C(\min(1, 2^j \delta))^s \|g_j\|_{L_2(\Omega)} \ , \quad g_j \in V_{j,\Omega} \ , \tag{2.10}$$

holds for all $0 < s < \min(t, 2)$, with a constant C depending on s but not on g_j and $j \geq 0$. The simple proof uses the definition of the modulus of continuity via differences and (A1). The case $\delta \geq 2^{-j}$ is trivial since

$\omega_2(\delta, f)_{L_2} \le 4\|f\|_{L_2(\Omega)}$ by the triangle inequality. For $\delta < 2^{-j}$, the local support property of the scaling functions leads to

$$\omega_2(\delta, g_j)_{L_2}^2 = \omega_2(\delta, \sum_{(l,\alpha)} c_{l,\alpha}\phi_{j,\alpha}^l)_{L_2}^2 \le C \sum_{(l,\alpha)} c_{l,\alpha}^2 \omega_2(\delta, \phi_{j,\alpha}^l)_{L_2}^2 \ .$$

Now using the continuity of the embedding $H^t(\Omega) \subset B_{2,\infty}^s(\Omega)$, where

$$\|f\|_{B_{2,\infty}^s(\Omega)} = \|f\|_{L_2(\Omega)} + \sup_{\delta>0} \delta^{-s}\omega_2(\delta, f)_{L_2} \ ,$$

and a dilation argument, we get

$$\omega_2(\delta, \phi_{j,\alpha}^l)_{L_2} \le C(2^j\delta)^s 2^{-jd/2}$$

for all index pairs (l, α) of interest. Substitution yields together with (2.1) the desired (2.10). For a more detailed exposition of these arguments, see [32]. Finally, the equivalence of (2.2) and (2.4) follows from (2.1) and general properties of frames (see [15, Section 3.2] or [33, Section 2]).

Norm equivalences as stated in Theorem 1 play a crucial role in deriving multilevel preconditioning methods for operator equations in $H^s(\Omega)$ (see [10, Section 5], [33, Section 2], and in a finite element context, [32, Section 4]). For the connection to multigrid theory, see [4, 22]. We quote some consequences of Theorem 1 for a generic H^1-elliptic boundary value problem, *i.e.* , we put $s = 1$ in the following. Let $a(\cdot,\cdot)$ be a symmetric, continuous, H^1-elliptic bilinear form, and $b(\cdot)$ a continuous linear functional on $H^1(\Omega)$. Denote by u the unique solution of the variational problem

$$a(u, v) = b(v) \qquad \forall\, v \in H^1(\Omega) \ , \tag{2.11}$$

and by $u_J \in V_{J,\Omega}$ the solution of its projection to $V_{J,\Omega}$:

$$a(u_J, v_J) = b(v_J) \qquad \forall\, v_J \in V_{J,\Omega} \ . \tag{2.12}$$

Introduce the following operator equation in $V_{J,\Omega}$:

$$\mathcal{P}_J u_J = \beta_J \ , \tag{2.13}$$

where

$$\mathcal{P}_J u_J = \sum_{j=0}^{J} \sum_{(l,\alpha)} \frac{a(u_J, \phi_{j,\alpha}^l|_\Omega)}{d_{j,\alpha}^l} \phi_{j,\alpha}^l|_\Omega \ , \quad \beta_J = \sum_{j=0}^{J} \sum_{(l,\alpha)} \frac{b(\phi_{j,\alpha}^l|_\Omega)}{d_{j,\alpha}^l} \phi_{j,\alpha}^l|_\Omega \ .$$

This equation is called *additive Schwarz formulation* for (2.12) associated with the subspace splitting

$$V_{J,\Omega} = \sum_{j=0}^{J} \sum_{(l,\alpha)} V_{j,\alpha,\Omega}^l$$

into one-dimensional subspaces spanned by the individual functions $\phi^l_{j,\alpha}|_\Omega$ from the finite section

$$\tilde{\Phi}_{J,\Omega} = \bigcup_{j=0}^{J} \Phi_{j,\Omega} \tag{2.14}$$

of the infinite system (2.3). The scaling factors in the definition of $\mathcal{P}_J$ and b_J are chosen such that

$$d^l_{j,\alpha} \asymp 2^{(2-d)j} \asymp 2^{2j} \|\phi^l_{j,\alpha}\|^2_{L_2(\Omega)} \asymp a(\phi^l_{j,\alpha}|_\Omega, \phi^l_{j,\alpha}|_\Omega) \ . \tag{2.15}$$

It is straightforward to check that (2.13) is equivalent to (2.12) and that $\mathcal{P}_J$ is symmetric positive definite with respect to $a(\cdot,\cdot)$ since

$$a(\mathcal{P}_J u_J, v_J) = \sum_{j=0}^{J} \sum_{(l,\alpha)} \frac{a(u_J, \phi^l_{j,\alpha}|_\Omega) a(v_J, \phi^l_{j,\alpha}|_\Omega)}{d^l_{j,\alpha}} \qquad \forall\, u_J, v_J \in V_{J,\Omega} \ ,$$

and $\tilde{\Phi}_{J,\Omega}$ is a generating system for $V_{J,\Omega}$. The important consequence of Theorem 1 (ii) is

Theorem 2. *Let the conditions of Theorem 1 and (2.15) be satisfied. The spectral condition number of $\mathcal{P}_J$ is uniformly bounded for $J \geq 0$. The bound depends on the constants in the two-sided estimate (2.4) (or, equivalently, on the constants in (2.1) and (2.2)), and in (2.15).*

The proof can be found in [32, Section 4.1]. What makes (2.13) interesting for solving (2.12) numerically is that in addition to the J-independent behavior of the condition numbers the action of $\mathcal{P}_J$ can be implemented as a sparse operation (see also [33, Section 2]). From the above definition of $\mathcal{P}_J$ it becomes clear that the matrix representation of (2.13) with respect to the nodal basis $\Phi_{J,\Omega}$ of $V_{J,\Omega}$ takes the form

$$C_J A_J = C_J b_J \ , \tag{2.16}$$

where

- A_J is the stiffness matrix with the entries $a(\phi^l_{J,\alpha}|_\Omega, \phi^{l'}_{J,\alpha'}|_\Omega)$ (the dimension n_J of this matrix coincides with the number of nontrivial basis functions $\phi^l_{J,\alpha}|_\Omega$ on level J),
- b_J is a vector with entries $b(\phi^l_{j,\alpha}|_\Omega)$,
- C_J is a symmetric multilevel preconditioner which possesses a recursive structure

$$C_0 = D_0 \ , \quad C_j = I_j C_{j-1} I_j^T + D_j \ , \ j = 1, \ldots, J \ , \tag{2.17}$$

where the matrices D_j are diagonal $n_j \times n_j$ matrices containing the scaling factors $(d^l_{j,\alpha})^{-1}$ on the diagonal, and the matrices I_j of dimension $n_j \times n_{j-1}$ contain the elements of the refinement equations (A4) and represent the exchange between neighboring levels. I_j^T is the transpose of I_j. Readers familiar with standard multigrid implementations [22] will recognize the analogy to a V-cycle with one smoothing step.

This allows us to solve (2.12) via the equivalent formulation (2.13) by a *preconditioned conjugate gradient (pcg)* method. According to Theorem 2, the number of pcg-iterations to reach a fixed error reduction is bounded by a constant independently of J. The complexity of one iteration step can be estimated by $\mathrm{O}(n_J)$ arithmetical operations, at least, for typical H^1-elliptic problems such as second order problems with Neumann or Robin boundary conditions. The corresponding algorithm is called *ϕ-algorithm* (associated with a *scaling function preconditioner*), in contrast to *ψ-algorithms* (associated with *wavelet preconditioners*) which will be briefly discussed next.

What we call wavelets associated with the sequence $\{V_{j,\Omega}\}$ are locally supported basis functions $\psi_{j,i}$, $i = 1, \ldots, m_j \equiv n_j - n_{j-1}$, for the L_2-orthogonal complement spaces

$$W_{j,\Omega} = V_{j,\Omega} \ominus_{L_2} V_{j-1,\Omega}\ , \quad j \geq 1\ , \tag{2.18}$$

leading to a L_2-stable Riesz basis in $W_{j,\Omega}$. Note that in other papers such $\psi_{j,i}$ are called *semiorthogonal wavelets* or *prewavelets*. Again, for domains aligned with the cube structure (and $j_0 = 0$), the construction of such functions can be reduced to the case of $\mathbb{R}^d$ and a boundary modification which is essentially one-dimensional. In some cases, like for the AFIF-elements discussed in Section 3, this boundary modification is trivial. What we gain in comparison with the frame concept, *i.e.* , the use of the scaling function system $\tilde{\Phi}_\Omega$, is the Riesz basis property of the system

$$\tilde{\Psi}_\Omega = \Phi_{0,\Omega} \cup \Psi_{1,\Omega} \cup \ldots \cup \Psi_{j,\Omega} \cup \ldots \quad (\Psi_{j,\Omega} \equiv \{\psi_{j,i}\})$$

in $H^s(\Omega)$ (resp. of the finite sections $\tilde{\Psi}_{J,\Omega}$ in $V_{J,\Omega}$), for a larger interval of Sobolev exponents, including $s = 0$, $s = \pm 1/2$, and $s = -1$. The following result is a direct consequence of Theorem 1 and duality arguments, it is a particular case of [10, Theorem 5.1.1 (ii)].

Theorem 3. *Let the assumptions of Theorem 1 be satisfied. Assume that the wavelet bases $\Psi_{j,\Omega}$ of the spaces $W_{j,\Omega}$ defined in (2.18) are uniformly L_2-stable, i.e.* ,

$$\| \sum_{i=1}^{m_j} c_i \psi_{j,i} \|^2_{L_2(\Omega)} \asymp \sum_{i=1}^{m_j} 2^{-jd} c_i^2 \tag{2.19}$$

for all coefficient choices and $j \geq 0$*. Then* $\tilde{\Psi}_\Omega$ *is, after normalization, a Riesz basis in* $H^s(\Omega)$ *for all* $-\min(t,2) < s < \min(t,2)$.

Again, the associated Schwarz formulation (with respect to sections of $\tilde{\Psi}_\Omega$) of symmetric H^s-elliptic variational problems will lead to uniformly well-conditioned linear systems, and to preconditioners which possess a recursive structure similar to (2.17):

$$\hat{C}_0 = D_0 \ , \quad \hat{C}_j = I_j \hat{C}_{j-1} I_j^T + \hat{I}_j \hat{D}_j \hat{I}_j^T \ , \ j = 1, \ldots, J \ . \tag{2.20}$$

The new matrices $\hat{I}_j$ are of size $n_j \times m_j$ and contain the wavelet mask coefficients given by the representations

$$\psi_{j,i} = \sum_{(l,\alpha)} \hat{a}^l_{j,\alpha,i} \phi^l_{j,\alpha}|_\Omega$$

as entries. The diagonal matrices $\hat{D}_j$ of dimension m_j contain scaling factors $(\hat{d}_{j,i})^{-1}$ which approximately equal the energy of the corresponding wavelet functions (replace the scaling functions by $\psi_{j,i}$ in the assumption (2.15)). At the first glance, applying a ψ-algorithm seems to be more costly, on the other hand, one expects a better robustness, especially, if lower order terms (like in a Helmholtz problem) are involved.

§3 AFIF elements

In this section we briefly describe a particular multiresolution analysis using so-called AFIF elements which is intermediate to classical finite element constructions and the Daubechies wavelets and is amenable to the multilevel theory presented in Section 2. The underlying scaling functions resp. multiwavelets have been introduced and investigated in [23, 18, 16] and are constructed using fractal interpolation functions. As for finite element examples, they

- have local support,
- allow for reproduction of low-order algebraic polynomials, and
- are interpolatory at nodes.

Just as importantly, the AFIF scaling functions $\phi^l_{j,\alpha}$ depart from conventional finite element nodal basis functions in that

- they form an L_2-orthogonal basis for $V_{j,\Omega}$ if Ω is aligned with the cube structure, and
- they have highly unusual local smoothness characteristics (in fact, they are "fractal").

Recall some definitions from the above-mentioned papers (see also [29]). An *iterated function system* is a complete metric space (X,d) and a collection of n strict contractions

$$w_i \; : \; X \to X \; , \quad i = 1, \ldots, n \; .$$

Associated with these mappings, we define the set-valued map

$$W(A) \equiv \bigcup_{k=1,\ldots,n} w_i(A)$$

acting on the complete metric space $(H(X), d_H)$, where $H(X)$ is the collection of all compact subsets of X, and d_H is the Hausdorff distance between sets

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y), \right\} \; .$$

The following theorem (see [29]) establishes an important property of W:

Theorem 4. *Let $\{X, w_k : k = 1 \ldots n\}$ be a (strictly contractive) iterated function system. Then the set-valued map W is a contraction on $(H(X), d_H)$ with contraction rate < 1. There is a unique fixed point of the equation $G = W(G)$ given by*

$$G = \lim_{i\to\infty} W^i(A) \qquad \forall A \in H(X) \; .$$

There is a lot of research on how this fixed point equation can be used to visualize strange attractors that arise in chaotic dynamical systems. What is of interest for us is that there exists a standard machinery for utilizing this theorem to generate attractors that are actually the graphs of functions on X. Consider the case

$$X_{[0,1]} \equiv [0,1] \times \mathbb{R} \; ,$$

and define the mappings

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \left\{ \begin{array}{c} u_i(x) \\ v_i(x,y) \end{array} \right\} , \qquad i = 1, 2 \; ,$$

by

$$\begin{aligned} w_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{5} \end{bmatrix} \left\{ \begin{array}{c} x \\ y \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \\ w_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & -\frac{1}{5} \end{bmatrix} \left\{ \begin{array}{c} x \\ y \end{array} \right\} + \left\{ \begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right\} \end{aligned}$$

Clearly, each mapping is a strict contraction on $X_{[0,1]}$ so that by Theorem 4 there is a unique attractor $G^1 \in H(X_{[0,1]})$. It is easy to see that G^1 is actually the graph of a continuous function $\phi^1 : [0,1] \to \mathbb{R}$ which can be extended by zero to a continuous function on $\mathbb{R}$ with support in $[0,1]$. Similarly, the choice

$$\begin{aligned} w_1\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{10} & -\frac{1}{5} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ w_2\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{1}{5} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2} \\ -\frac{3}{10} \end{Bmatrix} \end{aligned}$$

leads to another function ϕ^2 on $[0,1]$ which can be extended by setting

$$\phi^2(x) = \begin{cases} \phi^2(2-x) & \text{if } x \in (1,2] \\ 0 & \text{if } x \notin [0,2] \end{cases}$$

to a continuous function on the axis, with support on $[0,2]$.

A detailed investigation of the above choice of scaling functions ϕ^l, $l = 1,2$ (and of a whole family of similar AFIF functions) can be found in [18, 16, 30], we summarize the important properties in the following theorem:

Theorem 5. *The AFIF scaling functions ϕ^1, ϕ^2 as defined above satisfy properties (A1)-(A4) of Section 2 ($d = 1$, $L = 2$). In particular, $\phi^l \in H^t(\mathbb{R})$ for all $t < 3/2$. Moreover, the functions $\{\phi^l_{j,\alpha}\}$ form an orthogonal basis in V_j, $j \geq 0$. With the usual tensor-product construction, the results generalize to $d > 1$.*

The refinement equations for (A4) which are important for the evaluation of the values of functions from V_j at intermediate points and for the implementation of the multilevel algorithms (see the definition of the matrices I_j entering (2.17)) can be found in [16, Section III]. Figure 1 depicts the classical linear, Lagrangian finite element (a), the finite element created from AFIF scaling functions (b), and the classical quadratic Lagrangian finite element (c) (more precisely, for (b) the graphs of the nontrivial AFIF basis functions $\phi^2_{0,-1}, \phi^1_{0,0}, \phi^2_{0,0}$ on $[0,1]$ are shown, analogously for (a) and (c)). The linear finite element and AFIF element have similar approximation properties, both contain piecewise linear (but not quadratic) functions within their span. On the other hand, the AFIF element and quadratic finite element have similar cardinality. That is, both have three basis functions that intersect a single element in one dimension. This fact should be kept in mind when we compare the performance of algorithms based on the different choices in terms of numbers of iterations for a fixed error reduction or in terms of condition numbers of the multilevel preconditioned systems (see the next section).

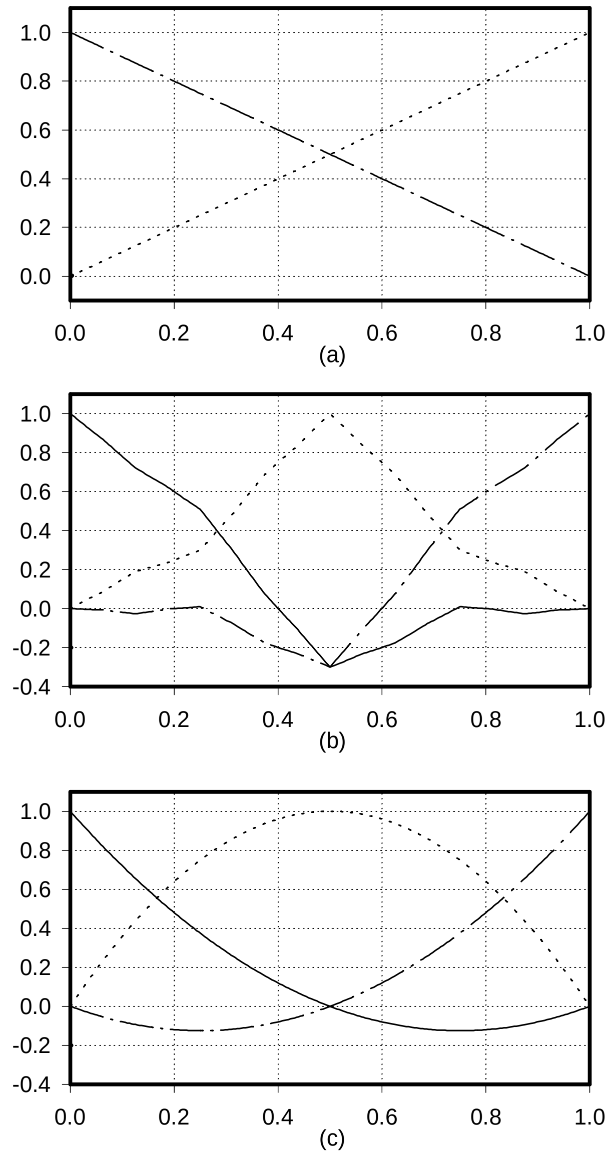

Figure 1. Elemental Functions in 1D : (a) Linear FEM, (b) AFIF, (c) Quadratic FEM

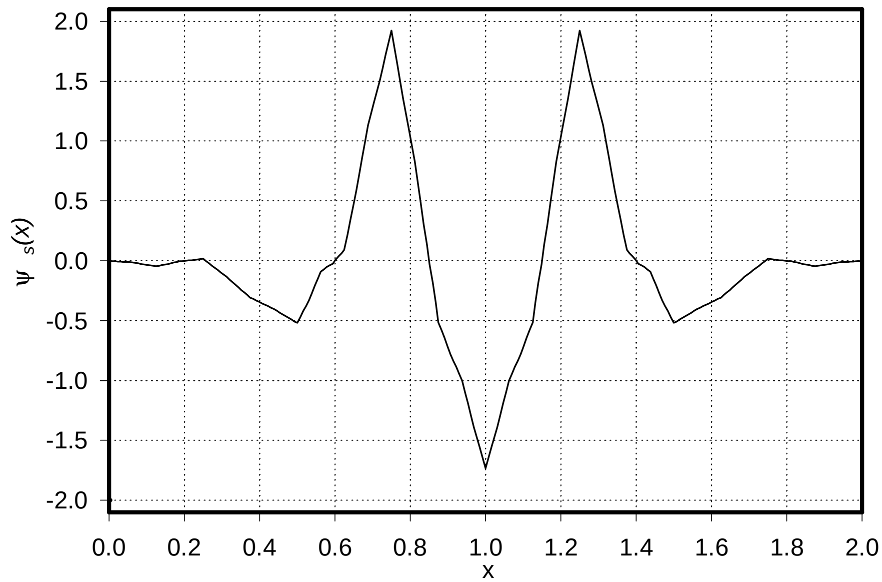

Figure 2. Symmetric AFIF Wavelet

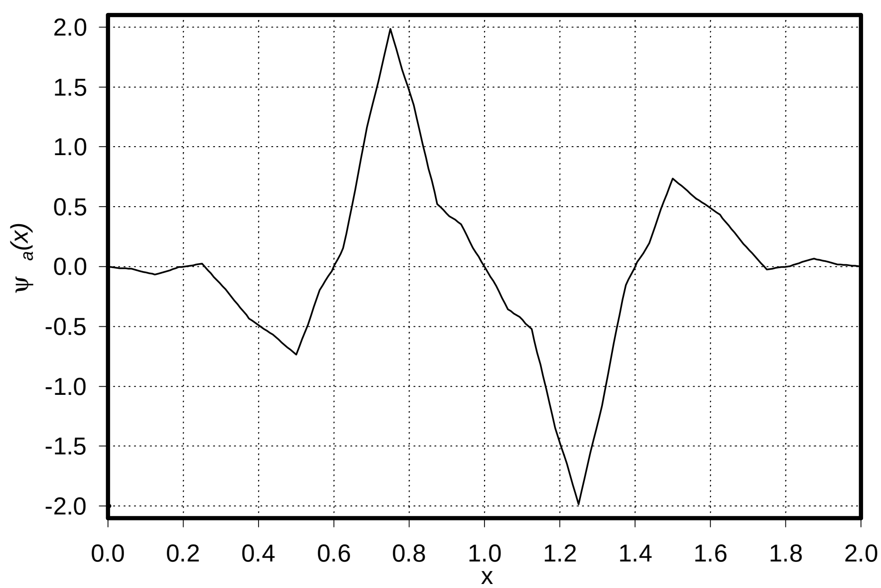

Figure 3. Antisymmetric AFIF Wavelet

Though the AFIF functions do not have explicit expressions, and are only implicitly defined via recursions involving the refinement equation, the accurate calculation of terms that typically arise, *e.g.* , such as

$$\int_\Omega \phi^l_{j,\alpha}\phi^{l'}_{j,\alpha'}\,dx \quad \text{or} \quad \int_\Omega \nabla\phi^l_{j,\alpha}\cdot\nabla\phi^{l'}_{j,\alpha'}\,dx\ ,$$

can be performed efficiently (see [23] or, in general, [11]). For example, the elemental mass and stiffness matrices for the above AFIF case are given by

$$[M_e] = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{25}{96} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

and

$$[K_e] = \begin{bmatrix} \frac{85}{21} & -\frac{80}{21} & \frac{43}{21} \\ -\frac{80}{21} & \frac{100}{21} & -\frac{80}{21} \\ \frac{43}{21} & -\frac{80}{21} & \frac{85}{21} \end{bmatrix}$$

respectively (see [27], the entries correspond to $\Omega = e = [0,1]$, and to the three functions $\phi^2_{0,-1}|_{[0,1]}$, $\phi^1_{0,0}|_{[0,1]}$, and $\phi^2_{0,0}|_{[0,1]}$ shown in Figure 1 (b)).

According to Section 2, Theorem 5 implies the optimal condition number estimates for the ϕ-algorithm (or scaling function preconditioner) corresponding to the AFIF discretization of second order elliptic boundary value problems on domains Ω which are aligned with the cube structure. It is easy to see that, due to the support and symmetry properties of the AFIF scaling functions, the restricted functions $\phi^l_{j,\alpha}|_\Omega$, $j \geq j_0$, preserve the L_2-orthogonality property for such domains in any dimension $d \geq 1$. Even though we did not explicitly deal with the case of essential boundary conditions in Section 2, zero Dirichlet boundary conditions can be incorporated in a similar way (by keeping only those $\phi^l_{j,\alpha}$ with support in Ω in the basis). This has been utilized in the study of multigrid methods in [27].

We finish this section with a short description of multiwavelets $\{\psi^1, \psi^2\}$ associated with the AFIF multiresolution analysis. As derived in [16], the AFIF wavelet functions are constructed to be interpolants at the quarter integer points and orthonormal to each other and the AFIF scaling functions. The function space W_0 generated by the two wavelet functions is the orthogonal complement of V_0. Several choices are discussed in [16], the one which we present here as most convenient for our purposes consists of a *symmetric* $\psi^s \equiv \psi^1$, and an *antisymmetric* $\psi^a \equiv \psi^2$. Both functions are supported on $[0,2]$, and are depicted in Figures 2 and 3, resp.. This particular choice of wavelet functions makes the analysis on a finite domain much easier. The expressions for the wavelet mask coefficients which enter the matrices $\hat{I}_j$ from (2.20) can be found in [16, section IV] for the 1D case (one needs to be careful with the normalization factors introduced in [16]).

Wavelet systems restricted to the interval $[0,1]$ have been described in [16, Theorem 4.4]. For our case, the choice of a basis for the complement spaces $W_{j,[0,1]}$ as defined by (2.18) consists of all nontrivial restrictions of the symmetric $\psi^s_{j,i}|_{[0,1]}$ and only those antisymmetric $\psi^a_{j,i}$ with support completely contained in $[0,1]$. It should be emphasized that the resulting restricted function spaces $V_{j,[0,1]}$ and $W_{j,[0,1]}$ are still orthogonal complements. This fact generalizes immediately to $d>1$ and rectangular domains via tensor-product arguments. As in [15, Section 10.1], the multivariate wavelet spaces are defined as spans of tensor-products of univariate scaling functions from V_{j-1} and univariate wavelets from W_j (except for pure ϕ-products). For a 2D-rectangle, this means to take the span of all

$$\phi_{j-1}(x_1)\psi_j(x_2),\ \psi_j(x_1)\phi_{j-1}(x_2),\ \psi_j(x_1)\psi_j(x_2),$$

where ϕ_{j-1}, ψ_j denote the generic basis functions from the above defined V_{j-1} and W_j on the respective 1D-intervals. This construction automatically ensures L_2-orthogonality of the multivariate wavelet spaces for different j for rectangular domains. Domains Ω with re-entrant corners such as L-shaped domains need more care (compared with the above construction for the rectangle it is enough to modify a few functions in the vincinity of the re-entrant corner to preserve full orthogonality as required in Theorem 3, we leave this as an exercise to the reader). In any case, elementary rules guarantee the applicability of Theorem 3 for the AFIF elements, and justify the optimality of the corresponding ψ-algorithm as described at the end of Section 2. Since the underlying system $\tilde{\Psi}_\Omega$ is an orthogonal basis in $L_2(\Omega)$, this ψ-algorithm should perform extremely well for problems with dominating L_2-elliptic part. For the corresponding numerical experiments, see Section 4.

§4 Numerical examples

An important goal of this paper is to assess the numerical performance of the class of multilevel ϕ- and ψ-preconditioning methods introduced above. In particular, we include standard multilevel finite element solvers (linear and quadratic elements) into the comparison. A motivation is that until now relatively few empirical results on wavelet preconditioning methods are documented, and fewer still make a serious attempt to calibrate performance to standard finite element formulations. Careful studies of the numerical performance of these algorithms are critical to establish viable, worthwhile directions for future research. Our numerical studies are still preliminary, and concern 1D and 2D Neumann boundary value problems for the Poisson equation.

Most studies of the numerical performance of wavelet and scaling function preconditioners for Galerkin formulations have utilized a model equation in one dimension [3, 1, 9, 19]. While studies of one-dimensional boundary value problems are seldom of interest in applications per se, there remain important conclusions that can be drawn from this class of problems. In particular, the numerical examples in one dimension set precedents in optimal complexity that are realized in some more general problems over classes of domains in higher dimensions. This is the reason we included them here. The two-dimensional tests concentrate on simple domains like the unit square and L-shaped domains. Finally, robustness with respect to singular perturbations caused by a zero-order Helmholtz term is investigated. Some related experiments comparing ϕ- and ψ-algorithms for linear finite elements on square domains can be found in [21].

Our numerical experiments show that (i) the multilevel preconditioning techniques presented in this paper are amenable to both scaling function and wavelet constructions, (ii) all selections (AFIF scaling and wavelet functions, Daubechies scaling functions, linear FEM and quadratic FEM) achieve *asymptotic optimal complexity* without tailoring the underlying function systems to the domain if the latter is well-aligned with the cube structure. Also, we see that for the standard elliptic problems, multilevel finite element preconditioning of BPX-type yields as a rule a better performance than wavelet-based methods. This supports our opinion that the ongoing development of wavelet-like solution methods for PDEs should include a thorough testing and comparison with conventional methods for partial differential equations.

4.1 1D tests

We consider the weak formulation in $H^1(0,1)$ of the two-point boundary value problem

$$\begin{aligned} -\frac{d^2u}{dx^2} + u &= -2x^3 + 3x^2 + 12x - 6 \ , \qquad x \in (0,1) \ , \\ u'(0) &= u'(1) = 0 \ , \end{aligned}$$

which has the exact solution $u(x) = 3x^2 - 2x^3$. For the discretizations and solvers, we employ (i) linear finite elements, (ii) quadratic finite elements, (iii) Daubechies scaling functions, and (iv) AFIF scaling functions as defined in Section 3. We exclusively use the ϕ-algorithm (*i.e.* , the pcg-method with the multilevel preconditioner based on the scaling functions) discussed in Section 2. As scaling factors (see (2.15)) we choose

$$d^l_{j,\alpha} = a(\phi^l_{j,\alpha}, \phi^l_{j,\alpha}) \ ,$$

which corresponds to multilevel diagonal (or Jacobi) scaling, and seems to be the most reliable choice on the average. An analogous choice is made for the scaling factors $\hat{d}_{j,i}$ in the ψ-algorithms below.

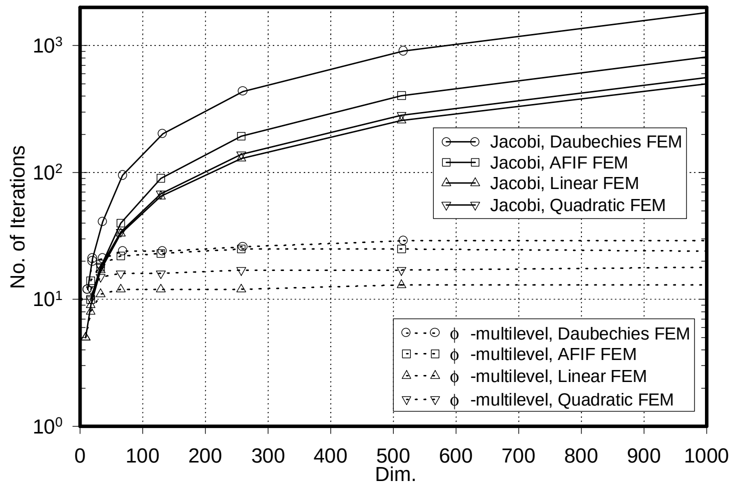

Figure 4. Number of Iterations (Error Reduction by 10^{-6}), 1D, log-Plot

Table 1. Number of Iterations to Decrease Residual by 10^{-6}, 1D

	Daubechies		AFIF		Linear		Quadratic	
J	n_J	iter	n_J	iter	n_J	iter	n_J	iter
3	12	12	17	14	9	5	17	12
4	20	20	33	19	17	8	33	15
5	36	21	65	22	33	11	65	16
6	68	24	129	23	65	12	129	16
7	132	24	257	25	129	12	257	17
8	260	26	513	25	257	12	513	17
9	516	29	1025	24	513	13	1025	18
10	1028	29	2049	26	1025	13	2049	17

The results of the numerical study are summarized in Table 1 and depicted graphically in Figure 4. A first interesting conclusion in considering the results of Table 1 is that the multilevel ϕ-preconditioner for both the AFIF and Daubechies scaling functions yield nearly identical results. Both methods require just under 30 iterations to reduce the residual to a value of 10^{-6} of its starting value. This fact is quite counter-intuitive in light of the fact that many authors have noted the extremely poor conditioning of the Gramian matrix associated with truncating the Daubechies wavelets to the interval. Because the performance of these two formulations are so

Table 2. Number of Iterations to Decrease Residual by 10^{-6}, 2D, Square Domain

	AFIF		Linear		Quadratic	
J	n_J	iter	n_J	iter	n_J	iter
3	289	28	81	9	289	19
4	1089	29	289	11	1089	20
5	4225	30	1089	12	4225	20
6	16641	31	4225	13	16641	20
7	66049	31	16641	12	66049	21

similar, in the remaining numerical examples, we will only consider scaling functions and wavelets associated with the AFIF basis. In comparison, the classical linear finite element basis and classical quadratic elements require 13 and 17 iteration, respectively, to achieve the same decrease in the residual.

Finally, as noted earlier, these numerical experiments should be evaluated keeping in mind the cardinality of the masks representing the elemental mass and stiffness operators. In Section 3 we showed that the AFIF elemental mass and stiffness matrices have $3 \times 3 = 9$ entries. Of course, it is well-known that for the classical linear and quadratic finite elements we have $2 \times 2 = 4$ and $3 \times 3 = 9$ entries, respectively. In contrast, it is shown in [26] that in the order 3 Daubechies case these elemental matrices have $5 \times 5 = 25$ entries. In higher space dimensions, the differences are even more significant.

4.2 2D tests

In this section, we report on similar performance results for the model problem

$$\begin{aligned} -\Delta u + u = f \quad & \text{in} \quad \Omega\,, \\ \frac{\partial u}{\partial n} = 0 \quad & \text{on} \quad \partial\Omega\,. \end{aligned}$$

Table 2 summarizes the number of iterations required to reduce the initial residual by a factor of 10^{-6} using the ϕ-algorithm in the case $\Omega = [0,1]^2$. For this particular problem, the number of iterations approaches a value of approximately 10, 20, and 30, for the linear, quadratic, and AFIF elements, respectively. The performance of the ϕ-preconditioner in comparison to the Jacobi preconditioner (*i.e.* , when diagonal scaling of the discretization matrix A_J is used as preconditioner) is depicted in Figure 5. Similar results are summarized in Table 3 and Figure 6 for an L-shaped domain.

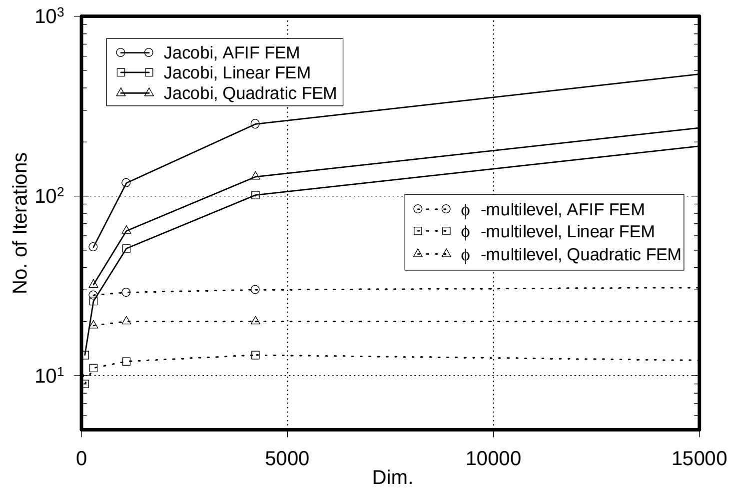

Figure 5. Number of Iterations (Error Reduction by 10^{-6}), 2D, Square Domain, log-Plot

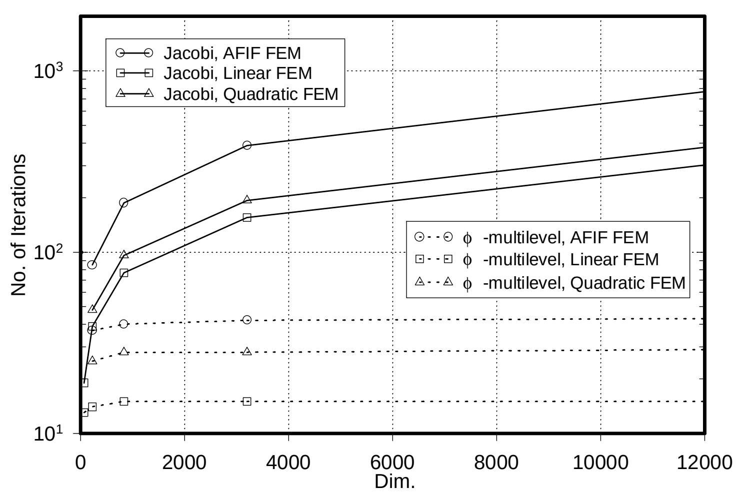

Figure 6. Number of Iterations (Error Reduction by 10^{-6}), 2D, L-Shaped Domain, log-Plot

Table 3. Number of Iterations to Decrease Residual by 10^{-6}, 2D, L-Shaped Domain

	AFIF		Linear		Quadratic	
J	n_J	iter	n_J	iter	n_J	iter
3	225	37	65	13	225	25
4	833	40	225	14	833	28
5	3201	42	833	15	3201	28
6	12545	43	3201	15	12545	29
7	49665	43	12545	15	49665	29

Again, all calculations are carried out using the ϕ-multilevel algorithm, and each case asymptotically achieves the expected level-independent iteration count. For the L-shaped domain, however, the number of iterations to reduce the original residual by 10^{-6} is about 50% greater than in the corresponding simulations for the square domain. This decrease in efficiency is expected, and is attributed to the re-entrant corner.

4.3 Robustness of ϕ- and ψ-algorithms

In the following set of numerical experiments, we vary the scalar magnitude of the source term. First, we consider a one-dimensional problem.

$$
\begin{aligned}
-\frac{d^2u}{dx^2} + qu = f \ , \qquad & x \in [0,1] \ , \\
u'(0) = u'(1) = 0 \ . &
\end{aligned}
$$

We vary the positive constant q between 10^2 and 10^{10}, and have tabulated the condition numbers of the *preconditioned* matrix representations $C_J A_J$ resp. $\hat{C}_J A_J$ with $J = 10$ corresponding to linear finite element nodal basis functions, to AFIF scaling functions resp. to AFIF wavelet functions. *I.e.*, in the first two cases, we employ again the ϕ-algorithm while in the last case the ψ-preconditioner discussed at the end of Section 2 is used. The results of the numerical comparisons are summarized in Table 4. Clearly, the multilevel preconditioner based on AFIF scaling function basis yields condition numbers that are about 10 and nearly constant. The condition numbers of the wavelet preconditioner applied to the AFIF case actually *decrease* with the strength of the source term, from a maximum near 10 to a minimum of 1 which is expected by the theory. The BPX-preconditioning of the classical finite element method exhibits an increase of the condition numbers from 5 to about 30 over the same range of q (note, however, that an inexpensive modification of the BPX-preconditioner would resolve this defect, see [32, Section 4.2.3]). As for any ϕ-algorithm, the condition numbers will grow as $\sim J$ in the limiting case $q \to \infty$. The better behavior for the AFIF-element (compared to the BPX-preconditioner for the linear elements) is due to the L_2-orthogonality of the AFIF-scaling functions within each level.

Table 4. Condition Numbers, 1D, Varying Source Strength

q	Linear FEM	ϕ AFIF	ψ AFIF
10^2	4.7	12.	10.6
10^3	6.7	11.4	10.2
10^4	9.23	10.2	9.20
10^5	11.5	11.0	7.45
10^6	14.0	11.4	4.88
10^7	18.3	9.6	2.24
10^8	29.4	10.6	1.20
10^9	31.9	10.9	1.023
10^{10}	32.2	11.0	1.011

Table 5. Condition Numbers, 2D, Varying Source Strength

q	Linear FEM	ϕ AFIF	ψ AFIF
10^2	7.15	14.	2.06
10^3	11.3	11.1	1.81
10^4	15.7	9.57	1.87
10^5	36.6	8.40	1.55
10^6	56.5	7.13	1.11
10^7	60.4	7.0	1.00
10^8	60.8	7.0	1.00
10^9	60.8	7.0	1.00
10^{10}	60.8	7.0	1.00

In Table 5, we summarize completely analogous results for the Neumann problem in two dimensions

$$-\Delta u + qu = f \quad \text{in } \Omega = [0,1]^2 , \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega .$$

Here, the final discretization level is $J = 6$. As shown in Table 5, the ϕ-multilevel preconditioner of the AFIF scaling function basis yields condition numbers that vary from 15 to 7, while for the ψ-multilevel preconditioning of the AFIF wavelet basis the condition numbers are again reduced if q grows. In contrast, the condition numbers for the BPX-preconditioning of the linear finite element discretization increase from 7 to 60 as the strength of the source term is increased.

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Jeonghwan Ko
Department of Aerospace Engineering
Texas A&M University
College Station, TX 77843
ko@aero.tamu.edu

Andrew J. Kurdila
Department of Aerospace Engineering
Department of Mathematics
Texas A&M University
College Station, TX 77843
kurdila@discovery.tamu.edu

Peter Oswald
Institute for Algorithms and Scientific Computing
GMD - German National Research Center for Information Technology
D-53754 Sankt Augustin, Germany
peter.oswald@gmd.de