

- [20] R. Rannacher, S. Turek, Simple nonconforming quadrilateral Stokes element, *Numer. Meth. Partial Diff. Equations* 8, 1992, 97–111.
- [21] A. Ron, Z. Shen, The Sobolev regularity of refinable functions, CMS TSR #97-04, Univ. of Wisconsin, Madison, March 1997.
- [22] J. Xu, Iterative methods by space decomposition and subspace correction, *SIAM Review* 34, 1992, 581–613.
- [23] H. Yserentant, Old and new convergence proofs for multigrid methods, in: *Acta Numerica*, Cambr. Univ. Press, New York, 1993, 285–326.

References

- [1] J. Bramble, *Multigrid Methods*, Pitman Res. Notes. Math. Sci. vol. 294, Longman Sci.& Techn., Harlow, Essex, 1993.
- [2] J. Bramble, J. Pasciak, J. Xu, Parallel multilevel preconditioners, *Math. Comp.* 55, 1990, 1–22.
- [3] S. Brenner, Multigrid methods for nonconforming finite elements, in: *Proc. Fourth Copper Mountain Conf. on Multigrid Methods* (J. Mandel et al., eds.), SIAM, Philadelphia, 1989, 54–65.
- [4] A. S. Cavaretta, W. Dahmen, C. A. Micchelli, *Stationary Subdivision*, *Memoirs Amer. Math. Soc.* vol. 93, AMS, Providence, 1991.
- [5] Z. Chen, Uniform convergence for V-cycle nonconforming multigrid methods for second-order problems, *SMU Math Report 96–4*, South. Meth. Univ. Dallas, 1996.
- [6] Z. Chen, P. Oswald, Multigrid and multilevel methods for nonconforming rotated Q1 elements, *Math. Comp.* 1998 (to appear).
- [7] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, New York, 1978.
- [8] A. Cohen, I. Daubechies, G. Plonka, Regularity of refinable function vectors, Preprint, 1995.
- [9] T. N. T. Goodman, C. A. Micchelli, J. D. Ward, Spectral radius formulas for subdivision operators, in: *Recent Advances in Wavelet Analysis* (L. L. Schumaker, G. Webb, eds.), Academic Press, New York, 1994, 335–360. 1996.
- [10] W. Hackbusch, *Iterative solution of large sparse systems of equations*, *Appl. Math. Sci.* vol. 95, Springer, New York, 1994.
- [11] R.-Q. Jia, C. Micchelli, Using the refinement equation for the construction of pre-wavelets II: powers of two, in: *Curves and Surfaces* (P. J. Laurent, A. Le Mehaute, L. L. Schumaker eds.), Academic Press, New York, 1991, 209–246.
- [12] R.-Q. Jia, S. D. Riemenschneider, D.-X. Zhou, Smoothness of multiple refinable functions and multiple wavelets, Preprint, Univ. of Alberta, 1997.
- [13] Q. Jiang, On the regularity of matrix refinable functions, Preprint, Nat. Univ. of Singapore/Univ. of Beijing, November 1996.
- [14] R. Lorentz, P. Oswald, Criteria for hierarchical bases in Sobolev spaces, *GMD-Arbeitsbericht Nr. 1059*, GMD, Sankt Augustin, March 1997.
- [15] C. A. Micchelli, T. Sauer, Regularity of multiwavelets, Preprint, Univ. Nürnberg-Erlangen, 1996.
- [16] P. Oswald, On a hierarchical basis multilevel method with nonconforming P1 elements, *Numer. Math.* 62, 1992, 189–212.
- [17] P. Oswald, *Multilevel Finite Element Approximation: Theory and Applications*, Teubner, Stuttgart 1994.
- [18] P. Oswald, Multilevel preconditioners for discretizations of the biharmonic equation by rectangular finite elements, *J. Numer. Lin. Alg. Appl.* 2, 1995, 487–505.
- [19] P. Oswald, Intergrid transfer operators and multilevel preconditioners for nonconforming discretizations, *Appl. Numer. Math.* 23, 1997, 139–158.

$s = 2$					
λ	mult	rank	$m(b_0, \lambda)$	$\bar{m}(b_0, \lambda)$	cond
16.0000000000	1	1	0.0	0.0	4.8
4.0000000000	3	3	0.0	0.0	61.1
2.0000000000	2	2	0.0	0.0	4.0
1.0000000000	9	9	0.011962	0.0	1681.8
0.5000000000	9	9	0.005318	0.000406	1202.1
$s = 1$					
4.0000000000	1	1	0.0	0.0	4.8
1.0000000000	3	3	0.026816	0.0	61.1
0.5000000000	2	2	0.0	0.0	4.0
0.2500000000	9	9	0.001911	0.001708	1681.8
$s = 0$					
1.0000000000	1	1	0.073922	0.0	4.8
0.2500000000	3	3	0.034619	0.033489	61.1

Table 13: Zienkiewicz element: Leading eigenvalues of K .

For the discrete H^2 seminorm we have chosen

$$|v_0|_{0,2}^2 = \sum_{\alpha \in \mathbf{Z}^2} [(c_{\alpha+\epsilon^1}^1 - c_\alpha^1 - c_\alpha^2)^2 + (c_{\alpha+\epsilon^2}^1 - c_\alpha^1 - c_\alpha^3)^2 + (c_{\alpha+\epsilon^1}^2 - c_\alpha^2)^2 + (c_{\alpha+\epsilon^2}^2 - c_\alpha^2)^2 + (c_{\alpha+\epsilon^1}^3 - c_\alpha^3)^2 + (c_{\alpha+\epsilon^2}^3 - c_\alpha^3)^2] ,$$

the resulting B_0 is shown in Figure 14. From the computation of the good index set along the lines of subsection 3.1 we obtain $M_1 = 141$ and $N = 3$. The computations with the direct method shown in Table 12 give the values $\lambda_{\max} = 1.0$ and $\tilde{\lambda}_{\max} = 0.5 < 1$. This confirms the uniform estimates (10) resp. (9) as well as the numerical tests in [19] for this set of intergrid transfer operators for the Zienkiewicz element. In conclusion we mention that the iterative methods have not been tried so far for these elements, partly because of the complicated spaces Π and $\tilde{\Pi}$, partly because of solvability problems in (71).

$$B_0 : \left(\begin{array}{ccc} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix} \end{array} \right)$$

Figure 14. Zienkiewicz element: B_0 for $s = 2$.

$s = 2$					
λ	mult	rank	$m(b_0, \lambda)$	$m(\bar{b}_0, \lambda)$	cond
16.0000000000	1	1	0.0	0.0	4.0
4.0000000000	3	3	0.0	0.0	51.5
2.0000000000	2	2	0.0	0.0	2.9
1.0000000000	9	9	0.003517	0.0	2549.3
0.5000000000	8	8	0.0	0.0	327.7
0.2500000000	18	18	0.000456	0.003282	6744.3
$s = 1$					
4.0000000000	1	1	0.0	0.0	4.0
1.0000000000	3	3	0.029803	0.0	51.5
0.5000000000	2	2	0.0	0.0	2.9
0.2500000000	9	9	0.003548	0.004613	2549.3
$s = 0$					
1.0000000000	1	1	0.082479	0.0	4.0
0.2500000000	3	3	0.038475	0.038131	51.5

Table 12: Adini element: Leading eigenvalues of K .

To see this, take the usual definition of the local interpolation problem for the Zienkiewicz element (see [7]): Find the unique $p \in \Pi_3$ that interpolates function values and gradients at the three vertices P_1, P_2, P_3 of a triangle and additionally satisfies

$$p(S) = \frac{1}{3} \sum_{i=1}^3 p(P_i) + \frac{1}{6} \sum_{i=1}^3 \nabla(P_i) \cdot (S - P_i),$$

where $S = (P_1 + P_2 + P_3)/3$ is the center of the triangle. Now check that the above polynomials are linearly independent and satisfy the extra condition with respect to any triangle in \mathcal{T}_0 (it is enough to do this for the two triangles contained in the unit square).

Globally, we have $V_0 \subset C(\mathbb{R}^2)$ while the C^1 condition is violated for derivatives in normal direction across edges of \mathcal{T}_0 . The intergrid transfer operators used in the computations below are defined in full analogy to the Adini case. Also, the three functions ϕ^l correspond to the same sets of interpolation conditions (but with respect to the new V_0). Straightforward calculations lead to the $S(\theta)$ shown in Figure 13.

$$S : \frac{1}{16} \left(\begin{array}{ccc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 16 & 0 & 0 \\ 0 & 16 & 32 & 16 & 0 \\ 0 & 0 & 16 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 12 & -12 & -4 & 0 \\ 0 & 24 & 0 & -24 & 0 \\ 0 & 4 & 12 & -12 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 6 & -1 & 0 \\ 0 & -4 & 16 & -4 & 0 \\ 0 & -1 & 6 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -5 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -4 & 0 & 0 & 0 \\ 0 & -12 & -24 & -4 & 0 \\ 0 & 12 & 0 & -12 & 0 \\ 0 & 4 & 24 & 12 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & -1 & 0 \\ 0 & -5 & 0 & -5 & 0 \\ 0 & -1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -4 & -1 & 0 \\ 0 & 6 & 16 & 6 & 0 \\ 0 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} \end{array} \right)$$

Figure 13. Zienkiewicz element: S for $s = 2$.

$$S : \frac{1}{16} \left(\begin{array}{ccc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 16 & 8 & 0 \\ 0 & 16 & 32 & 16 & 0 \\ 0 & 8 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & -12 & 0 \\ 0 & 24 & 0 & -24 & 0 \\ 0 & 12 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 8 & -4 & 0 \\ 0 & -4 & 16 & -4 & 0 \\ 0 & -2 & 8 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -24 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 24 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -4 & -2 & 0 \\ 0 & 8 & 16 & 8 & 0 \\ 0 & -2 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right)$$

Figure 11. Adini element: S for $s = 2$.

A suitable discrete H^2 seminorm would be given, e.g., by

$$\begin{aligned} |v_0|_{0,2}^2 &= \sum_{\alpha \in \mathbf{Z}^2} [(c_{\alpha+e^3}^1 - c_{\alpha+e^2}^1 - c_{\alpha+e^1}^1 + c_{\alpha}^1)^2 + (c_{\alpha+e^1}^2 - c_{\alpha}^2)^2 \\ &\quad + (c_{\alpha+e^2}^2 - c_{\alpha}^2)^2 + (c_{\alpha+e^1}^2 + c_{\alpha}^2 - 2(c_{\alpha+e^1}^1 - c_{\alpha}^1))^2 \\ &\quad + (c_{\alpha+e^1}^3 - c_{\alpha}^3)^2 + (c_{\alpha+e^2}^3 - c_{\alpha}^3)^2 \\ &\quad + (c_{\alpha+e^2}^3 + c_{\alpha}^3 - 2(c_{\alpha+e^2}^3 - c_{\alpha}^3))^2] . \end{aligned}$$

$$B_0 : \left(\begin{array}{ccc} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -1 & 0 \\ 0 & 6 & 0 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ -1 & 6 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \right)$$

Figure 12. Adini element: B_0 for $s = 2$.

This leads to the B_0 shown in Figure 12. The corresponding good index set (which we do not reproduce here) yields the values $M_1 = 124$ and $N = 3$ for this application. The computations with the direct method are shown in Table 12 and confirm our expectations: The computed values $\lambda_{\max} = 1.0$ and $\tilde{\lambda}_{\max} = 0.25 < 1$ indicate that (9) holds for this set of intergrid transfer operators. Note that the theory and computations for the Adini element given in [18] are based on a slightly different multilevel scheme.

The Zienkiewicz element will be considered on the same initial triangular partition \mathcal{T}_0 as the Morley element, compare Subsection 4.3. The local polynomial space $\Pi = \Pi^{Zienk}$ is 9-dimensional and intermediate to Π_2 and Π_3 . For the shift-invariant situation considered in this paper it is characterized by the following basis of homogeneous polynomials: $p_k(x)$, $k = 1, \dots, 6$, are the standard monomials in Π_2 , and

$$p_7(x) = x_1^3 - x_2^3, \quad p_8(x) = x_1 x_2 (x_1 - x_2), \quad p_9(x) = x_1^3 + x_2^3 + 2x_1 x_2 (x_1 + x_2).$$

j	no correction	P_0 -correction	\tilde{P}_0 -correction
5	2.07876718	2.07876718	2.22802805
10	2.24648689	2.24648690	2.25060800
15	2.25010304	2.25024865	2.25018179
20	4.17358244	2.25006921	2.25003511
25	16.00178001	2.25001345	2.25000658
30	16.00000010	2.25000253	2.25000123
35	16.00000000	2.25000047	2.25000023
40	16.00000000	2.25000009	2.25000004
45		2.25000002	2.25000001
50		2.25000000	2.25000000

Table 11: Morley element: Spectral radius bounds.

According to previous numerical experience [18, 19], for both of them we expect uniform boundedness for the standard iterated intergrid transfer operators.

Let us start with the Adini element. Here, the local polynomial space $\Pi = Q^{\text{Adini}}$ is given by

$$\Pi_3 \subset Q^{\text{Adini}} = \Pi_3 \cup \{x_1^3 x_2, x_1 x_2^3\} \subset \Pi_4,$$

equipped with a basis of monomials $\{p_k(x) : k = 1, \dots, 12\}$ as suggested by its definition. The partition \mathcal{T}_0 into unit squares is as for the rotated Q1 element, and the interpolation problem prescribes function values and gradients at all vertices of \mathcal{T}_0 . Obviously, the 12-dimensional local interpolation on each square is unsolvable with respect to Q^{Adini} , the global function $v_0 \in V_0$ is continuous along the edges of \mathcal{T}_0 , with possible discontinuities in the normal derivatives across edges. V_0 is spanned by three basis functions ϕ^k , each characterized by a single non-zero interpolation condition (at the origin):

$$\phi^1(0) = 1 \quad \text{resp.} \quad \frac{\partial \phi^2}{\partial x_1}(0) = 1 \quad \text{resp.} \quad \frac{\partial \phi^3}{\partial x_2}(0) = 1.$$

This gives $M = 3$. The operator I_1 is obtained by nodal value averaging, i.e., almost all interpolation conditions for $I_1 v_0$ are preserved from v_0 by continuity of the corresponding derivatives, with the exception of degrees of freedom corresponding to normal derivatives at midpoints of edges of \mathcal{T}_0 . For those, the average of the existing one-sided derivatives is taken. E.g.,

$$\frac{\partial I_1 v_0}{\partial x_2}\left(\frac{1}{2}, 0\right) = \frac{1}{2} \left(\frac{\partial v_0}{\partial x_2}\left(\frac{1}{2}, 0+\right) + \frac{\partial v_0}{\partial x_2}\left(\frac{1}{2}, 0-\right) \right).$$

The usual calculations lead to the matrix function $S(\theta)$ describing this intergrid transfer operator (the cell representation given in Figure 11 for $s = 2$ already includes the scaling factor 2).

$s = 2$					
λ	mult	rank	$m(b_0, \lambda)$	$m(\tilde{b}_0, \lambda)$	cond
16.0000000000	1	1	0.0	0.0	3.7
6.0000000000	1	1	0.0	0.0	2.8
4.0000000000	4	4	0.0	0.0	60.7
3.0000000000	2	2	0.0	0.0	54.9
2.2500000000	1	1	0.02261	0.028504	17.6
2.0000000000	3	3	0.0	0.0	27.4
1.6089411651	1	1	0.00281	0.001727	12.0
1.5000000000	4	4	0.0	0.0	51.8
$s = 1$					
4.0000000000	1	1	0.0	0.0	3.7
1.5000000000	1	1	0.0	0.0	2.8
1.0000000000	4	4	0.034017	0.0	60.717
0.7500000000	2	2	0.032903	0.046737	54.904
$s = 0$					
1.0000000000	1	1	0.064018	0.0	3.7
0.2500000000	5	5	0.0	0.0	2.8
0.1686821014	1	1	0.054871	0.057386	60.7

Table 10: Morley element: Leading eigenvalues of K .

For $s = 2$, the discrete H^2 -seminorm of our choice which leads to the cell representation of the B_0 used for this example, is as follows:

$$\begin{aligned}
|v_0|_{0,2}^2 = & \sum_{\alpha \in \mathbf{Z}^2} [(v_\alpha^1 - v_{\alpha+\epsilon^1}^1 - v_{\alpha+\epsilon^2}^1 + v_{\alpha+\epsilon^3}^1)^2 + (v_\alpha^4 + v_\alpha^1 - v_{\alpha+\epsilon^3}^1)^2 + (v_\alpha^4 - v_\alpha^2 - v_\alpha^3)^2 \\
& (v_\alpha^4 - v_{\alpha+\epsilon^2}^2 - v_{\alpha+\epsilon^1}^3)^2 + (v_{\alpha+\epsilon^2}^1 - v_{\alpha+\epsilon^1}^1 - v_\alpha^2 + v_\alpha^3)^2 \\
& + (v_{\alpha+\epsilon^2}^1 - v_{\alpha+\epsilon^1}^1 - v_{\alpha+\epsilon^2}^2 + v_{\alpha+\epsilon^1}^3)^2] .
\end{aligned}$$

The equivalence with $|v_0|_{H^2}^2$ can be derived by a local consideration (e.g., compute expressions for the constant second derivatives of v_0 on the two triangles contained in the unit square E_0).

The computations (with both the direct and P -corrected iterative methods, the Q -corrected iteration failed due to solvability problems in (71)) show that

$$\lambda_{\max} = \tilde{\lambda}_{\max} = 2.25 .$$

According to the theory in [19], this means that the additive preconditioner based on the standard set of intergrid transfer operators (as described in more detail in [19]) has an exponentially deteriorating preconditioning power: Condition numbers of a preconditioned linear system corresponding to a generic Kirchhoff plate problem will grow approximately at the same rate as $\approx h^{-\delta}$ where $\delta = \log_2 9/4$ is slightly larger than 1 and h denotes the element size of the final triangulation. This compares favorably with the condition number growth of $\approx h^{-4}$ of the stiffness matrix but is far away from the desired level-independent behavior of multilevel preconditioners. It is, however, not clear whether this negative result carries over to the multiplicative algorithm (or standard V-cycle multigrid method) with the same set of intergrid transfer operator.

4.4 Adini and Zienkiewicz Elements

In this concluding section we briefly examine two other nonconforming elements suitable for plate bending - the rectangular Adini element [7, Chapter 6.3] and the triangular Zienkiewicz element.

4.3 Morley element

The Morley element is the simplest nonconforming element suitable for plate bending. It is also the most ‘resistent’ element if it comes to multilevel theory in the nonconforming finite element case. Tests in [19] have shown that the standard intergrid transfer operators for this element (again obtained by averaged nodal interpolation) do not satisfy the uniform bounds (9). We will confirm these experimental findings by showing that $\lambda_{\max} = 2.25$.

Let us give some details. For the calculations below, the underlying shift-invariant initial triangulation \mathcal{T}_0 was formed by using e^1 , e^2 , and e^4 (not e^3 as in subsection 4.1) for the three edge directions. The Morley element space V_0 is characterized by the following requirements: A function $v_0 \in L_2(\mathbb{R}^2)$ belongs to V_0 if its restriction to any triangle of \mathcal{T}_0 is in Π_2 , if it is continuous at all vertices (i.e., at all \mathbb{Z}^2 -points), and if, for any edge e of \mathcal{T}_0 , it has continuous first derivative at the midpoint M_e in the direction normal to e . To be definite, we have chosen the edges of the master triangle Δ_0 (with the vertices $(0, 0), (1, 0), (0, 1)$) as e_0^1, e_0^2, e_0^3 and fixed

$$v_0(0, 0) = c_0^1, v_0(1, 0) = c_{e_1}^1, v_0(0, 1) = c_{e_2}^1,$$

$$\frac{\partial v_0}{\partial x_2}\left(\frac{1}{2}, 0\right) = c_0^2, \frac{\partial v_0}{\partial x_1}\left(0, \frac{1}{2}\right) = c_0^3, \frac{\partial v_0}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{\partial v_0}{\partial x_2}\left(\frac{1}{2}, \frac{1}{2}\right) = c_0^4,$$

as the local interpolation problem on Δ_0 . We hope that the reader can now derive the definition of the four nodal basis functions ϕ^l , $l = 1, \dots, 4$, such that their shifts form the Riesz basis \mathcal{N}_0 , and

$$v_0 = \sum_{l=1}^4 \sum_{\alpha \in \mathbb{Z}^2} c_\alpha^l \phi_\alpha^l.$$

We will not repeat how the definition of I_1 by nodal value averaging is turned into formal mathematics, see [19] for more details.

The entries of the cell representation of S (which include the factor $2^{s-d/2} = 2$ for $s = 2$) have been computed by a program, just to avoid errors which are very much likely if calculations on paper are performed with higher order polynomials and directional derivatives. The following Figure 10 gives the readers the chance to compare the result with their own findings.

$$S : \frac{1}{8} \left(\begin{array}{cccc} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & 7 & 4 & 0 \\ 0 & 7 & 16 & 7 & 0 \\ 0 & 4 & 7 & 4 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -2 & -6 & -4 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & 4 & 6 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & 6 & -1 & -4 & 0 \\ 0 & 4 & 1 & -6 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & -8 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right)$$

Figure 10. Morley element: S for $s = 2$.

The box sizes $N_0 = 2$ (for S) and $N = 3$ (determined from the good index set \mathcal{I}_{S, B_0}) are appropriate. We do not show the good index set itself, the resulting dimension of V_{S, B_0} is $M_1 = 169$.

$s = 1$					
λ	mult	rank	$m(b_0, \lambda)$	$\tilde{m}(b_0, \lambda)$	cond
4.0000000000	1	1	0.0	0.0	2.0
1.0000000000	5	5	0.22678	0.0	101.2
0.5000000000	4	4	0.0	0.0	4.0
0.2500000000	10	10	0.21147	0.44198	62.5
$s = 0$					
1.0000000000	1	1	0.14003	0.0	2.0
0.5000000000	5	5	0.09759	0.09085	101.2

Table 8: Wilson element: Leading eigenvalues of K .

j	λ_{\max}		$\tilde{\lambda}_{\max}$	
	Q_0 -correction	P_0 -correction	\tilde{Q}_0 -correction	\tilde{P}_0 -correction
3	0.97593933	0.97593933	0.25053846	0.25053846
6	0.99962135	0.99962135	0.25006791	0.25006791
9	0.99999408	0.99999408	0.25000850	0.25000850
12	0.99999991	0.99999991	0.25000106	0.25000106
15	1.00000000	1.00000000	0.25000013	0.25000010
18	1.00000000	1.00000000	0.25000002	0.24999802
21	1.00000000	1.00000000	0.25000000	0.24987216
24	1.00000000	1.00000000	0.25000000	0.24204170
27	1.00000000	1.00000000	0.25000000	1.07151795
30	1.00000000	1.00000000	0.25000000	1.00718624
33	1.00000000	1.00000000	0.25000000	1.00011213
36	1.00000000	1.00000000	0.25000000	1.00000175
39	1.00000000	1.00000000	0.25000000	1.00000003
42	1.00000000	1.00000000	0.25000000	1.00000000

Table 9: Wilson element: Spectral radius bounds through iterative methods.

degrees of freedom of are determined by interpolation at the vertices and by prescribing values for the following two integrals of second derivatives:

$$\int_{E_\alpha} \frac{\partial^2 v_0}{\partial x_1^2}(x) dx, \quad \int_{E_\alpha} \frac{\partial^2 v_0}{\partial x_2^2}(x) dx.$$

Thus, we again have $M = 3$. The basis function ϕ^1 coincides with the usual hat function for bilinear conforming elements (centered at the origin) while ϕ^2, ϕ^3 are nonconforming quadratic bubble functions supported on the unit square E_0 and given by

$$\phi^2(x) = -x_1(1-x_1)/2, \quad \phi^3(x) = -x_2(1-x_2)/2, \quad x \in E_0.$$

Even though the solution of a discretized second order elliptic boundary value problem with respect to the Wilson element easily reduces to the solution of its bilinear part (the unknowns corresponding to the bubble functions can obviously be eliminated locally), we have introduced an intergrid transfer operator between the Wilson element spaces, again by inheriting resp. averaging of nodal values, and tested our methods. Obviously, we can use the continuity of ϕ^1 and set

$$\begin{aligned} (I_1 \phi^1)(x) &\equiv \phi^1(x) = \phi^1(2x) + \frac{1}{2} \sum_{j=1}^2 (\phi^1(2x - e^j) + \phi^1(2x + e^j)) \\ &\quad + \frac{1}{4} \left(\sum_{j=3}^4 (\phi^1(2x - e^j) + \phi^1(2x + e^j)) \right) \quad (e^4 = e^2 - e^1). \end{aligned}$$

Analogously, computing and averaging the values of ϕ^2 at $\frac{1}{2}\mathbb{Z}^2$ points, one is lead to define

$$(I_1 \phi^2)(x) = \frac{1}{4} \sum_{j=0}^3 \phi^2(2x - e^j) - \frac{1}{16} (\phi^1(2x - e^1) + 2\phi^1(2x - e^3) + \phi^1(2x - e^1 - 2e^2)).$$

Note that $Q_0 I_1$ is the identity on V_0 if the projection operator $Q_0 : V_1 \rightarrow V_0$ is defined as follows: $Q_0 v_1 \in V_0$ interpolates v_1 at the vertices of \mathcal{T}_0 , and the integrals of the second derivatives of $Q_0 v_1$ on a square E_α are prescribed by summation of the integrals of the corresponding second derivative of v_1 with respect to the four subsquares of E_α in \mathcal{T}_1 . With Q_0, I_1 , and their iterates at hand, the theory of [16] could be applied. We leave it as an exercise upon the reader to complete the elementary calculations for S . The non-zero entries of $S(\theta)$ are as follows (for brevity, we have set $z_1 = e^{-i\theta_1}, z_2 = e^{-i\theta_2}$):

$$\begin{aligned} S^{1,1}(\theta) &= \frac{(1+z_1)^2(1+z_2)^2}{4z_1z_2}, \quad S^{2,2}(\theta) = S^{3,3}(\theta) = \frac{1}{4}(1+z_1)(1+z_2), \\ S^{1,2}(\theta) &= -\frac{1}{16}z_1(1+z_2)^2, \quad S^{1,3}(\theta) = -\frac{1}{16}z_2(1+z_1)^2. \end{aligned}$$

A discrete H^1 seminorm for the Wilson element is given by

$$|v_0|_{0,1}^2 = \sum_{\alpha \in \mathbb{Z}^d} ((c_\alpha^1 - c_{\alpha+e^1}^1)^2 + (c_\alpha^1 - c_{\alpha+e^2}^1)^2 + (c_\alpha^2)^2 + (c_\alpha^3)^2).$$

The good index set is relatively small, leading to $N = 2$ and $M_1 = 34$. The computations for the eigenvalue problem associated with the norm behavior of the iterated intergrid transfer operators \tilde{I}_j^J are documented in Table 8 and 9, and lead to

$$\lambda_{\max} = 1, \quad \tilde{\lambda}_{\max} = 0.25.$$

For the iterative method based on Lemma 11 (the Q -correction method), the choices $\Pi = \Pi_1$ and $\tilde{\Pi} = \Pi_3$ were sufficient. The P -correction is again not able to produce the correct value for $\tilde{\lambda}_{\max}$.

j	no correction	P_0 -correction	\tilde{P}_0 -correction
5	1.11228520	1.11228520	0.67032241
10	1.01154154	1.01154154	0.67408183
15	1.00154316	1.00154316	0.67466854
20	1.00021008	1.00021410	0.67471473
25	0.99591509	1.00002989	0.67472248
30	6.32877271	1.00000418	0.67472536
35	4.00159174	1.00000058	0.67472679
40	4.00000155	1.00000008	0.67472763
45	4.00000000	1.00000001	0.67472870
50	4.00000000	1.00000000	0.67473365
55			0.67476760
60			0.67500943
70			0.68857511
80			0.90526181
90			0.99747004
100			0.99995018
110			0.99999903
120			0.99999998
130			1.00000000

Table 7: Nonconforming P1 element: Spectral radius bounds through iterative methods

which are represented by the vectors

$$b_1 = (1, 1, 1)^T, \quad b_2 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^T, \quad b_3 = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T.$$

Let $\tilde{\Pi} = \Pi_3$ be formed by the monomials \tilde{p}_k , $k = 1, \dots, 10$, where $\tilde{p}_k = p_k$, $k = 1, 2, 3$, $\tilde{p}_4(x) = x_1^2$, $\tilde{p}_5(x) = x_1x_2$, $\tilde{p}_6(x) = x_2^2$, and so on. From Figure 7 we compute

$$S(0) = \tilde{p}_1(\partial)S(0) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \tilde{p}_2(\partial)S(0) = \tilde{p}_4(\partial)S(0) = \begin{bmatrix} 1 & -1/2 & 1/2 \\ 1 & 0 & 1 \\ 1/2 & -1/2 & 1 \end{bmatrix}$$

$$\tilde{p}_3(\partial)S(0) = \tilde{p}_6(\partial)S(0) = \begin{bmatrix} 0 & 1 & 1 \\ -1/2 & 1 & 1/2 \\ -1/2 & 1/2 & 1 \end{bmatrix} \quad \tilde{p}_5(\partial)S(0) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1/2 & -1/2 & 1 \end{bmatrix},$$

and so on. Thus, if substituted into (71) for $k = 4, 5, 6$, we have $|k| = 2$ and obtain the following singular linear systems for the vectors \tilde{b}_k :

$$\begin{bmatrix} -4 & -4 & -4 \\ -4 & -4 & -4 \\ -4 & -4 & -4 \end{bmatrix} \tilde{b}_k = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix} \quad \text{resp.} \quad = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{resp.} \quad = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

for $k = 4$ resp. $k = 5$ resp. $k = 6$. Thus, only $k = 5$ gives rise to a useful left eigenmatrix E_k . Because of the recursive nature of the proof of Lemma 11, no conclusion can be drawn for $k \geq 7$. Modified restriction operators Q_0, \tilde{Q}_0 based on E_1, E_2, E_3 , and E_5 did not improve upon the results obtained by the P, \tilde{P} -correction methods. Evidently, the matter needs further study. The same problems have arisen for other triangular elements studied in this paper.

4.2 Wilson element

The Wilson element (see [7]) is a nonconforming enrichment of the usual conforming rectangular Q1 element. On each square E_α of the initial partition we have $v_0|_{E_\alpha} \in \Pi_2$, and the available 6

$s = 1$					
λ	mult	rank	$m(b_0, \lambda)$	$m(b_0, \lambda)$	cond
4.0000000000	1	1	0.0	0.0	8.1
1.0000000000	5	5	0.04594	0.0	317.0
0.6747284054	1	1	0.04073	0.008376	334.6
0.6347547417	2	2	0.02255	0.005206	102.2
$s = 0$					
1.0000000000	1	1	0.106	0.0	8.1
0.2500000000	5	5	0.0	0.0	317.0
0.1686821014	1	1	0.00048	0.00275	334.6

Table 6: Nonconforming P1 element: Leading eigenvalues of K .

$$B_0 : \begin{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{pmatrix}$$

Figure 8. Nonconforming P1 element: B_0 for $s = 1$.

$$\chi_{\mathcal{I}_{s,B_0}} : \begin{pmatrix} \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \end{bmatrix} \end{pmatrix}$$

Figure 9. Nonconforming P1 element: Good index set.

We have also tried to apply the improved iterative method based on Lemma 11. In contrast to the case of the rotated Q1 element, this approach failed due to solvability problems in (71) for $|k| = 2$. Indeed, the set $\Pi = \Pi_1$ is formed by the monomials $p_1(x) = 1, p_2(x) = x_1, p_3(x) = x_2$,

4 More Examples

4.1 Nonconforming P1 element

The investigation of the standard intergrid transfer operator for this element has been carried out in [16] (actually, this was the first paper where the uniform boundedness of the iterated intergrid transfer operators in the energy norm was noticed and rigorously established in the nonconforming case). The space V_0 is defined with respect to the triangulation \mathcal{T}_0 with vertex set \mathbb{Z}^2 and edges e_α^1, e_α^2 as before and diagonal edges e_α^3 in direction of $e^3 = e^1 + e^2$. A function v_0 belongs to V_0 if its restriction to any triangle is in Π_1 , the function is continuous across the midpoints M_e of all edges e in \mathcal{T}_0 and belongs to $L_2(\mathbb{R}^d)$ (since $v_0(M_e) = av(v_0, e)$, a definition by edge averages leads to exactly the same space). The basis \mathcal{N}_0 is generated by $M = 3$ functions ϕ^m associated with the 3 types of edges. The intergrid transfer operator I_1 is defined by nodal value averaging, in exactly the same manner as for the rotated Q1 element. For details we refer to [16], for the convenience of the reader we show the cell representation of the associated subdivision operator S in Figure 7.

For the discrete H^1 seminorm, we have chosen

$$|v_0|_{0,1}^2 = \sum_{\alpha \in \mathbb{Z}^2} ((c_\alpha^3 - c_\alpha^1)^2 + (c_\alpha^3 - c_\alpha^2)^2 + (c_\alpha^3 - c_{\alpha+e_2}^1)^2 + (c_\alpha^3 - c_{\alpha+e_1}^2)^2).$$

The resulting B_0 as well as the characteristic function of the good index set are given in Figure 8 and 9. We see that $N_0 = N = 2$ and $M_1 = 69$.

Table 6 and 7 show the results of the eigenvalue computations by the direct and iterative methods. As one can see, we have

$$\lambda_{\max} = 1.0, \quad \tilde{\lambda}_{\max} = 0.674728,$$

which confirms the known behavior of these intergrid transfer operators.

$$S : \frac{1}{4} \left(\begin{array}{ccc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right)$$

Figure 7. Nonconforming P1 element: S for $s = 1$.

j	edge averages		midpoint interpolation	
	Q_0 -correction	\tilde{Q}_0 -correction	Q_0 -correction	\tilde{Q}_0 -correction
3	1.06528879	0.53780300	1.03153189	0.49996536
6	1.00874606	0.52666389	1.00364429	0.47363953
9	1.00125316	0.52546350	1.00038201	0.46966883
12	1.00018134	0.52533376	1.00003951	0.46907033
15	1.00002628	0.52531977	1.00000408	0.46897974
18	1.00000381	0.52531827	1.00000042	0.46896601
21	1.00000055	0.52531811	1.00000004	0.46896393
24	1.00000008	0.52531809	1.00000000	0.46896361
27	1.00000001	0.52531809	1.00000000	0.46896357
30	1.00000000	0.52531809	1.00000000	0.46896356
33	1.00000000	0.52531809	1.00000000	0.46896356

Table 5: Rotated Q1 element: Spectral radius bounds using Lemma 11.

particular k (nonuniqueness does not hurt). Unfortunately, this situation happens very often. E.g., for our example of the rotated Q1 element, we have from Figure 1 that

$$S(0) = \begin{pmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{pmatrix}$$

which means that $|k| = 0$ and $|k| = 2$ may cause difficulties. The case $|k| = 0$ (i.e., $k = 1$) is covered by (62) while $|k| = 2$ occurs for $k = 4$ (again covered by (62)) and $k = 5, 6$. It turns out that (71) is still solvable for these two critical k which can be checked by direct computation. However, for some other elements (such as the nonconforming P1 element discussed in Subsection 4.1) the solvability of (71) was violated.

The practical importance of Lemma 11 is obvious: Having a large enough $\tilde{\Pi}$, we are able to form left eigenvectors for K by considering the restrictions \tilde{E}_k of the Hermitean left eigenmatrices $E_k + E_k^*$ to the good index set. If $\tilde{E}_k \neq 0$ and

$$\langle \tilde{E}_k(\theta), B_0(\theta) \rangle_{mat} = 0 \quad \text{resp.} \quad \langle \tilde{E}_k(\theta), \tilde{B}_0(\theta) \rangle_{mat} = 0, \quad (72)$$

then \tilde{E}_k is orthogonal (with respect to $\langle \cdot, \cdot \rangle_{mat}$) to the whole Krylov space $V(\mathcal{L}_S, B_0)$ resp. $V(\mathcal{L}_S, \tilde{B}_0)$. Note that the first relation in condition (72) can be transformed into the formula

$$\tilde{p}_k(\partial)(|T(\theta)\tilde{b}(\theta)|^2)|_{\theta=0} = 0, \quad (73)$$

where $\tilde{b}(\theta)$ is the vector function the existence of which was required in Lemma 11. The selection of a maximal subset of \tilde{E}_k such that (72) is satisfied defines projectors Q_0 resp. \tilde{Q}_0 which can be used in a modified power iteration to determine more accurate bounds for λ_{\max} and, more importantly, for $\tilde{\lambda}_{\max}$. We will not bore the reader with the obvious details. The present implementation requires the description of $\tilde{\Pi}$ and the submission of the vectors b_k , $k = 1, \dots, K_0$.

The following table shows the iteration results obtained from the new correction methods, for both variants of rotated Q1 element spaces (note that for the second version only the vector b_4 has to be changed). The additional application of P - resp. \tilde{P} -corrections was superfluous. In the present application, a correction that uses only the first 6 left eigenmatrices (corresponding to eigenvalues ≥ 1) would have led to the same result. Also, the additional efforts which were necessary for defining the maximal $\tilde{\Pi}$ could have been avoided. We see that now the bounds obtained by the iterative method are in full agreement with the direct method.

After applying (64),

$$\tilde{p}_k(\partial)(\tilde{a}^* X \tilde{a})(\pi e) = \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \tilde{p}_m(\partial) a(\pi e)^* \sum_{|r|+|s|=|n|} \tilde{c}_{r,s}^n \tilde{p}_r(\partial) X(\pi e) \tilde{p}_s(\partial) a(\pi e) ,$$

we see from (65) that for $e \neq 0$ all terms vanish: whenever $\tilde{c}_{m,n}^k \tilde{c}_{r,s}^n \neq 0$, either m or n (and therefore s) do not exceed K_0 which means that either $\tilde{p}_m(\partial) a(\pi e)$ or $\tilde{p}_s(\partial) a(\pi e)$ vanishes.

Thus, the above formula simplifies and can be processed further by using (67) and again (64). First, dropping all term swith $e \neq 0$ gives

$$\begin{aligned} \langle E_k(\theta), \mathcal{L}_S X(\theta) \rangle_{mat} &= 2^{-|k|-d} \tilde{p}_k(\partial)(\tilde{a}^* X \tilde{a})(0) \\ &= 2^{-|k|-d} \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \tilde{p}_m(\partial) a(0)^* \sum_{|r|+|s|=|n|} \tilde{c}_{r,s}^n \tilde{p}_r(\partial) X(0) \tilde{p}_s(\partial) a(0) . \end{aligned}$$

In the same way, as it was demonstrated above for $e \neq 0$ with (68), in the case $e = 0$ we can rewrite (67) in the form

$$\tilde{p}_k(\partial) a(0) = 2^{s+d/2} \tilde{b}_k = 2^{s+d/2} \tilde{p}_k(\partial) \tilde{b}(0) , \quad k = 1, \dots, K_1 .$$

Substitution yields

$$\begin{aligned} \langle E_k(\theta), \mathcal{L}_S X(\theta) \rangle_{mat} &= 2^{2s-|k|} \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \tilde{b}_m^* \sum_{|r|+|s|=|n|} \tilde{c}_{r,s}^n \tilde{p}_r(\partial) X(0) \tilde{b}_s \\ &= 2^{2s-|k|} \tilde{p}_k(\partial)(\tilde{b}^* X \tilde{b})(0) = 2^{2s-|k|} \langle E_k(\theta), X(\theta) \rangle_{mat} . \end{aligned}$$

This proves (70).

Since for Hermitean $X(\theta)$ and arbitrary $Y(\theta)$

$$\begin{aligned} \langle Y(\theta), X(\theta) \rangle_{mat} &= \sum_{r,s} (Y^{r,s}(\theta)^* X^{r,s}(\theta))_0 = \sum_{r,s} (Y^*)^{s,r}(\theta) (X^*)^{r,s}(\theta)_0 \\ &= \langle X^*(\theta), Y^*(\theta) \rangle_{mat} = \langle X(\theta), Y^*(\theta) \rangle_{mat} = \langle Y^*(\theta), X(\theta) \rangle_{mat}^* , \end{aligned}$$

we see that $E_k^*(\theta)$ is also a left eigenmatrix of \mathcal{L}_S associated with the eigenvalue λ_k (and so is the generalized Hermitean matrix function $E_k(\theta) + E_k^*(\theta)$).

A few words about the assumptions in Lemma 11. Given linearly independent \tilde{p}_k , the construction of a suitable $\tilde{b}(\theta)$ from the vectors \tilde{b}_k , $k = 1, \dots, K_1$, is straightforward (fix a space of trigonometric vector functions of sufficiently large degree, such that the linear system represented by (66) has maximal rank K_1). Thus, it suffices to find vectors \tilde{b}_k satisfying (67-68). According to (62), if we set

$$\tilde{b}_k = b_k = ((v_k)_0^1, \dots, (v_k)_0^M)^T , \quad k = 1, \dots, K_0 ,$$

we automatically satisfy (68) and partly (67), for $k \leq K_0$. The remaining \tilde{b}_k , $k = K_0 + 1, \dots, K_1$, should be found directly from (67). To this end, it is recommended to order the \tilde{p}_k such that the linear systems

$$(2^{s+d/2} \text{Id} - 2^{|k|} S(0)) \tilde{b}_k = \sum_{|m|+|n|=|k|, |m| \geq 1} 2^{|n|} \tilde{c}_{m,n}^k \tilde{p}_m(\partial) S(0) \tilde{b}_n , \quad k = K_0 + 1, \dots, K_1 , \quad (71)$$

can be solved one by one (in all examples below, this is achieved by ordering with respect to the degree of \tilde{p}_k). There is one difficulty with solving (71) which was already mentioned in [13]: If $2^{s+d/2-|k|}$ is an eigenvalue of $S(0)$ then it is a priori not clear whether a solution exists for this

Lemma 11 Suppose that there exists a trigonometric $M \times 1$ vector function $\tilde{b}(\theta)$ such that the vectors

$$\tilde{b}_k = \tilde{p}_k(\partial)\tilde{b}(0), \quad k = 1, \dots, K_1, \quad (66)$$

satisfy

$$2^{s+d/2}\tilde{b}_k = \sum_{|m|+|n|=|k|} 2^{|n|}\tilde{c}_{m,n}^k \tilde{p}_m(\partial)S(0)\tilde{b}_n, \quad k = 1, \dots, K_1, \quad (67)$$

and

$$0 = \sum_{|m|+|n|=|k|} 2^{|n|}c_{m,n}^k p_m(\partial)S(\pi e)\tilde{b}_n, \quad k = 1, \dots, K_0, \quad (68)$$

for all $e \neq 0$ from the set $\{0, 1\}^d$. For each $k = 1, \dots, K_1$, introduce the (generalized) matrix function $E_k(\theta)$ by the formula

$$E_k(\theta) = \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \sum_{|r|+|s|=|n|} \tilde{c}_{r,s}^n \tilde{b}_m \tilde{b}_s^* \Gamma_r(\theta). \quad (69)$$

Then

$$\langle E_k(\theta), (\mathcal{L}_S X)(\theta) \rangle_{mat} = 2^{2s-|k|} \langle E_k(\theta), X(\theta) \rangle_{mat}, \quad (70)$$

holds for arbitrary Hermitean $X(\theta)$ and all $k = 1, \dots, K_1$. I.e., $E_k(\theta)$ represents a left eigenmatrix for the matrix transfer operator \mathcal{L}_S corresponding to the eigenvalues $\lambda_k = 2^{2s-|k|}$. The conjugate generalized matrix function $E_k^*(\theta)$ also satisfies (70).

Proof. Using the above formulas (especially (60), (66), and (64)), we first verify that

$$\begin{aligned} \langle E_k(\theta), X(\theta) \rangle_{mat} &= \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \sum_{|r|+|s|=|n|} \tilde{c}_{r,s}^n \langle \tilde{b}_m \tilde{b}_s^* \tilde{\Gamma}_r(\theta), X(\theta) \rangle_{mat} \\ &= \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \sum_{|r|+|s|=|n|} \tilde{c}_{r,s}^n \tilde{b}_m^* \tilde{p}_r(\partial) X(0) \tilde{b}_s \\ &= \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \sum_{|r|+|s|=|n|} \tilde{c}_{r,s}^n \tilde{p}_m(\partial) \tilde{b}(0)^* \tilde{p}_r(\partial) X(0) \tilde{p}_s(\partial) \tilde{b}(0) \\ &= \tilde{p}_k(\partial) (\tilde{b}^* X \tilde{b})(0). \end{aligned}$$

This can be used immediately to transform

$$\begin{aligned} 2^{|k|} \langle E_k(\theta), \mathcal{L}_S X(\theta) \rangle_{mat} &= \tilde{p}_k(\partial) (\tilde{b}(2\cdot) (\mathcal{L}_S X)(2\cdot) \tilde{b}(2\cdot))(0) \\ &= 2^{-d} \sum_{e \in \{0,1\}^d} \tilde{p}_k(\partial) ((S(\cdot + \pi e) \tilde{b}(2\cdot))^* X(\cdot + \pi e) S(\cdot + \pi e) \tilde{b}(2\cdot))(0) \\ &= 2^{-d} \sum_{e \in \{0,1\}^d} \tilde{p}_k(\partial) (\tilde{a}^* X \tilde{a})(\pi e), \end{aligned}$$

where this time we have set $\tilde{a}(\theta) = S(\theta)\tilde{b}(2\theta)$.

Now we use (68) in conjunction with (64) or, what is the same, with (58) for $k = 1, \dots, K_0$. This gives

$$\begin{aligned} \tilde{p}_k(\partial) a(\pi e) &= \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \tilde{p}_m(\partial) S(\pi e) 2^{|n|} \tilde{p}_n(\partial) \tilde{b}(2\pi e) \\ &= \sum_{|m|+|n|=|k|} c_{m,n}^k p_m(\partial) S(\pi e) 2^{|n|} \tilde{b}_n = 0, \quad e \neq 0. \end{aligned}$$

where we have temporarily introduced the row vector functions

$$A_n(\theta) = \sum_{|r|+|s|=|n|} c_{r,s}^n p_r(\partial) a(\theta)^* p_s(\partial) S(\theta) .$$

Since the trigonometric vector function $a(\theta)$ is arbitrary, the constant vectors $p_r(\partial) a(\pi e)$ can be chosen arbitrarily, too. Thus, comparing coefficients in front of $a(\pi e) = p_1(\partial) a(\pi e)$ in the above expressions for both sides of (61), we see that (61) implies

$$\sum_{|m|+|n|=|k|} 2^{|m|} c_{m,n}^k p_n(\partial) S(\pi e) b_m = \begin{cases} 2^{s+d/2} b_k & , \quad e = 0 \\ 0 & , \quad e \neq 0 \end{cases} , \quad (62)$$

for all $k = 1, \dots, K_0$ (recall that setting $r = 1$ implies $s = n$ for the other index, and $c_{r,s}^n = 1$ for the corresponding coefficient in the last expression, and that $|k| - |n| = |m|$).

Let

$$\tilde{\Pi} = \text{span}\{\tilde{p}_k : k = 1, \dots, K_1\} \quad (K_1 \geq K_0)$$

denote a space of polynomials generated by another set of linearly independent homogeneous polynomials \tilde{p}_k which is subject to the following conditions. First, we assume that $\Pi \subset \tilde{\Pi}$. Thus, without loss of generality, we may assume that $\tilde{p}_k = p_k$, $k = 1, \dots, K_0$. Secondly, (52) will be replaced by requiring that $\tilde{\Pi}$ is invariant with respect to \mathbb{Z}^d -shifts (together with the homogeneity of the \tilde{p}_k , this implies invariance with respect to general \mathbb{R}^d -shifts).

The last requirement is a bit technical. From the shift-invariance, we clearly have extensions of (54) and (58) to the larger set $\tilde{\Pi}$:

$$\tilde{p}_k(x+y) = \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \tilde{p}_m(x) \tilde{p}_n(y) \quad \forall x, y \in \mathbb{R}^d , \quad (63)$$

$$\tilde{p}_k(\partial)(ab)(\theta) = \sum_{|m|+|n|=|k|} \tilde{c}_{m,n}^k \tilde{p}_m(\partial) a(\theta) \tilde{p}_n(\partial) b(\theta) , \quad (64)$$

for all $k = 1, \dots, K_1$. What we need to assume is that

$$c_{m,n}^k \neq 0 \implies \min(m, n) \leq K_0 . \quad (65)$$

Let us illustrate this condition first for the case $\Pi = \Pi_r$. Then $\tilde{\Pi} = \Pi_{2r+1}$ will be appropriate. Indeed, if $|\gamma| \leq 2r+1$ then $c_{\alpha+\beta}^\gamma \neq 0$ implies $\gamma = \alpha + \beta$ and $|\alpha| + |\beta| = |\gamma| \leq 2r+1$ from which $\min(|\alpha|, |\beta|) \leq r$. Thus, at least one of the two monomials x^α, x^β belongs to $\Pi = \Pi_r$. Note that this case has been considered in [13] (and many other papers). If $\Pi \neq \Pi_r$, the answer might be nontrivial. For the rotated Q1 element, if we take $\Pi = Q_1^{\text{rot}}$ with the p_k , $k = 1, \dots, 4$, as introduced above, an appropriate $\tilde{\Pi}$ is spanned by Π_3 (i.e., by all cubic polynomials, with the basis functions denoted by \tilde{p}_k , $k = 1, \dots, 10$, where we set $\tilde{p}_k = p_k$, $k = 1, \dots, 4$, and

$$\tilde{p}_5(x) = x_1^2 + x_2^2, \tilde{p}_6(x) = x_1 x_2, \tilde{p}_7(x) = x_1^3, \tilde{p}_8(x) = x_1^2 x_2, \tilde{p}_9(x) = x_1 x_2^2, \tilde{p}_{10}(x) = x_2^3$$

and by two more functions of degree 4:

$$\tilde{p}_{11} = x_1^4 - x_2^4, \quad \tilde{p}_{12} = x_1 x_2 (x_1^2 - x_2^2) .$$

The inclusion of Π_3 is obvious since $\Pi_1 \subset Q_1^{\text{rot}}$. For \tilde{p}_{11} , we compute

$$\begin{aligned} \tilde{p}_{11}(x+y) &= \tilde{p}_{11}(x) p_1(y) + p_1(x) \tilde{p}_{11}(y) + 6(x_1^2 y_1^2 - x_2^2 y_2^2) \\ &\quad + 4(\tilde{p}_7(x) p_2(y) + p_2(x) \tilde{p}_7(y) - \tilde{p}_{10}(x) p_3(y) - p_3(x) \tilde{p}_{10}(y)) . \end{aligned}$$

Since $2x_1^2 = \tilde{p}_5(x) + p_4(x)$, $2x_2^2 = \tilde{p}_5(x) - p_4(x)$, we obtain

$$6(x_1^2 y_1^2 - x_2^2 y_2^2) = 3(\tilde{p}_5(x) p_4(y) + p_4(x) \tilde{p}_5(y)) ,$$

which shows that in each term of the representation (63) for $\tilde{p}_{11}(x+y)$ there is one factor p_k with $k \leq K_0 = 4$. Thus, (65) is satisfied for \tilde{p}_{11} , the verification in the case of \tilde{p}_{12} is left upon the reader (it can also be shown that there is no possibility to further enlarge $\tilde{\Pi}$ while satisfying (65)).

It will play a crucial role in the transformations below. The scalar product

$$\langle a(\theta), b(\theta) \rangle \equiv (a^* b)_0 = \sum_{\alpha} a_{\alpha}^* b_{\alpha}$$

makes sense if at least one of the two functions is a trigonometric polynomial. E.g.,

$$\langle \Gamma_k(\theta), a(\theta) \rangle = \sum_{\alpha} p_k(\alpha) a_{\alpha} = p_k(\partial) a(0) . \quad (59)$$

These formulas easily generalize to vector and matrix functions. For instance, if we introduce the scalar product for $M \times M$ matrices as before by setting

$$\langle X(\theta), Y(\theta) \rangle_{mat} = \sum_{r,s=1}^M \langle X^{r,s}(\theta), Y^{r,s}(\theta) \rangle ,$$

then taking $X(\theta) = a^*(\theta) b(\theta) \Gamma_k(\theta)$ where a, b are trigonometric $M \times 1$ vector functions, we have for any trigonometric $M \times M$ matrix $Y(\theta)$ the formula

$$\begin{aligned} \langle b^*(\theta) a(\theta)^* \Gamma_k(\theta), Y(\theta) \rangle_{mat} &= \sum_{r,s} \langle \Gamma_k(\theta), a^s(\theta) b^r(\theta)^* Y^{r,s}(\theta) \rangle \\ &= \langle \Gamma_k(\theta), b^*(\theta) y(\theta) a(\theta) \rangle = p_k(\partial) (b^* Y a)(0) . \end{aligned} \quad (60)$$

The scalar product of $M \times 1$ vector functions is denoted by

$$\langle a(\theta), b(\theta) \rangle_{vec} = \sum_{r=1}^M \langle a^r(\theta), b^r(\theta) \rangle .$$

We come to the consequences of (53). By definition of the subdivision operator S associated with I_1 and the fact that p_k is a homogeneous polynomial of degree $|k|$, we see that (53) implies

$$S(\theta) v_k(2\theta) = 2^{s-d/2} 2^{-|k|} v_k(\theta) , \quad k = 1, \dots, K_0 ,$$

(in the sense of generalized functions), or, equivalently,

$$\langle a(\theta), S(\theta) v_k(2\theta) \rangle_{vec} = 2^{s-d/2-|k|} \langle a(\theta), v_k(\theta) \rangle_{vec} \quad (61)$$

for all trigonometric vector functions $a(\theta)$. Recall that the factor $2^{s-d/2}$ is due to the normalization factor introduced at the very beginning. Substitution of (57) and transformation (using the expression of the dual of the subdivision operator and (59)) yield

$$\begin{aligned} \langle a(\theta), v_k(\theta) \rangle_{vec} &= \sum_{|m|+|n|=|k|} c_{m,n}^k \sum_{r=1}^M b_m^r \langle \Gamma_n(\theta), a^r(\theta)^* \rangle \\ &= \sum_{|m|+|n|=|k|} c_{m,n}^k p_n(\partial) a(0)^* b_m . \end{aligned}$$

and

$$\begin{aligned} \langle a(\theta), S(\theta) v_k(2\theta) \rangle_{vec} &= \underbrace{\langle 2^{-d} \sum_{e \in \{0,1\}^d} S^* \left(\frac{\theta}{2} + \pi e \right) a \left(\frac{\theta}{2} + \pi e \right), v_k(\theta) \rangle_{vec}}_{\tilde{a}(\theta)} \\ &= 2^{-d} \sum_{|m|+|n|=|k|} c_{m,n}^k p_n(\partial) \tilde{a}(0)^* b_m \\ &= 2^{-d} \sum_{|m|+|n|=|k|} c_{m,n}^k 2^{-|n|} p_n(\partial) (\tilde{a}(2 \cdot))(0)^* b_m \\ &= 2^{-d} \sum_{e \in \{0,1\}^d} \sum_{|m|+|n|=|k|} 2^{-|n|} c_{m,n}^k A_n(\pi e) b_m , \end{aligned}$$

and have

$$\begin{aligned}
p_1(x+y) &= p_1(x)p_1(y) , \\
p_2(x+y) &= p_2(x)p_1(y) + p_1(x)p_2(y) , \\
p_3(x+y) &= p_3(x)p_1(y) + p_1(x)p_3(y) , \\
p_4(x+y) &= p_4(x)p_1(y) + 2p_2(x)p_2(y) - 2p_3(x)p_3(y) + p_1(x)p_4(y) .
\end{aligned}$$

This example also shows that it is natural to include situations where Π does not coincide with any of the spaces Π_r (only the latter case is covered by [13]).

According to (52), there are unique decompositions

$$p_k(x) = \sum_{l=1}^M \sum_{\alpha} (v_k)_\alpha^l \phi^l(x - \alpha) , \quad k = 1, \dots, K_0 .$$

Using (52) and (54), we obtain

$$\begin{aligned}
p_k(x + \beta) &= \sum_{l=1}^M \sum_{\alpha} (v_k)_\alpha^l \phi^l(x + \beta - \alpha) \\
&= \sum_{l=1}^M \sum_{\gamma} (v_k)_{\gamma+\beta}^l \phi^l(x - \gamma) \\
&= \sum_{|m|+|n|=|k|} c_{m,n}^k p_n(\beta) \sum_{l=1}^M \sum_{\gamma} (v_m)_\gamma^l \phi^l(x - \gamma) \\
&= \sum_{l=1}^M \sum_{\gamma} \left(\sum_{|m|+|n|=|k|} c_{m,n}^k p_n(\beta) (v_m)_\gamma^l \right) \phi^l(x - \gamma) , \quad \beta \in \mathbb{Z}^d .
\end{aligned}$$

Comparing coefficients for $\gamma = 0$, we see that

$$(v_k)_\beta^l = \sum_{|m|+|n|=|k|} c_{m,n}^k p_n(\beta) (v_m)_0^l \quad (55)$$

for all $l = 1, \dots, M, k = 1, \dots, K_0$. Thus, if we introduce the generalized functions

$$\Gamma_k(\theta) = \sum_{\alpha} p_k(\alpha) e^{-i\alpha\theta} , \quad k = 1, \dots, K_0 , \quad (56)$$

then the (generalized) vector functions $v_k(\theta) = ((v_k)^1(\theta), \dots, (v_k)^M(\theta))^T$ associated with these coefficient sequences can be represented by

$$v_k(\theta) = \sum_{|m|+|n|=|k|} c_{m,n}^k b_m \Gamma_n(\theta) , \quad b_m = ((v_m)_0^1, \dots, (v_m)_0^M)^T . \quad (57)$$

In order to state and prove the following lemma, let us denote by $p_k(\partial)$ ($\partial \equiv -iD_\theta$) the differential operators acting according to

$$p_k(\partial) : a(\theta) = \sum_{\alpha} a_{\alpha} e^{-i\alpha\theta} \mapsto p_k(\partial)a(\theta) = \sum_{\alpha} p_k(\alpha) a_{\alpha} e^{-i\alpha\theta} .$$

Let us list some useful identities. A consequence of (54) is the Leibniz formula

$$p_k(\partial)(ab)(\theta) = \sum_{|m|+|n|=|k|} c_{m,n}^k p_m(\partial)a(\theta) p_n(\partial)b(\theta) . \quad (58)$$

If $\lambda\nu^*$ is real then the same procedure shows that the choice $y(\theta)z^*(\theta) + z(\theta)y^*(\theta)$ would lead to a Hermitean eigenmatrix.

To apply this idea, recall that in our applications $A^*(\theta) = B(\theta) = S(\theta)$. Thus, all what we need are suitable eigenfunctions of the subdivision operator associated with S . For practical reasons, we again restrict our attention to the generalized eigenfunctions $\{v_k : k = 1, \dots, L\}$ of the subdivision operator S associated with the invariant polynomial space Π_S considered before. Following the above recipe, we would try to introduce Hermitean eigenmatrices $E(k, k') = v_k(\theta)v_{k'}^*(\theta) + v_{k'}(\theta)v_k^*(\theta)$ of \mathcal{L}_S^* (non-Hermitean eigenfunctions are not of interest for obvious reasons). Unfortunately, since both v_k and $v_{k'}$ have infinite Fourier spectrum, the above products do not have a meaningful interpretation.

However, one can try to repair the above definition. E.g., let us take for simplicity the v_1 from Figure 6. Denote by $v_1|_{\mathcal{I}_N}(\theta)$ the trigonometric vector function which we obtain by restricting the Fourier spectrum of all entries to the box $[-N, N]^2$, and define

$$X = \lim_{N \rightarrow \infty} \frac{1}{4N^2} v_1|_{\mathcal{I}_N} (v_1|_{\mathcal{I}_N})^* ,$$

in the sense of generalized (matrix) functions resp. cell representations. The reader may check that the limit exists. One easily verifies that X is a Hermitian 2×2 cell with all entries being delta functions (i.e., the 4 infinite coefficient arrays coincide with those in the representation of v_1) which is indeed a generalized eigenmatrix for \mathcal{L}_S^* for the eigenvalue 4. Following this recipe, one could try to find other candidates for generalized eigenmatrices. Their restrictions to the good index set can be used to define alternative projections for stabilizing the power iteration (recall that we are specifically interested in X associated with eigenvalue 1).

A more systematic approach can be deduced from the material in [13]. Since this paper is hard to read and is not general enough to cover some of the natural applications (e.g., of those arising from the problems discussed in this paper), we give an independent representation. Let us start with a finite-dimensional space Π of algebraic polynomials with real coefficients defined on \mathbb{R}^d such that

$$\Pi \subset \text{span } \mathcal{N}_0 \quad (52)$$

and

$$I_1 p = p \quad \forall p \in \Pi . \quad (53)$$

Again, (52) and (53) are meaningful in an algebraic sense. Let $\{p_k : k = 1, \dots, K_0\}$ be a basis in Π consisting of homogeneous polynomials (for simplicity, we will write $|k|$ for the degree of p_k). Moreover, we require constants to be in Π . Without loss of generality, we will always assume that $p_1(x) \equiv 1$. From these assumptions, it already follows that we have formulas

$$p_k(x+y) = \sum_{|m|+|n|=|k|} c_{m,n}^k p_m(x) p_n(y) \quad \forall x, y \in \mathbb{R}^d , \quad (54)$$

for all $k = 1, \dots, K_0$, with real coefficients $c_{m,n}^k$ satisfying $c_{m,n}^k = c_{n,m}^k$ and $c_{1,n}^n = c_{n,1}^n = 1$. If $\Pi = \Pi_r$ for some r , following the tradition, we will equip Π with the standard monomial basis $p_\gamma(x) = x^\gamma = x_1^{\gamma_1} \dots x_d^{\gamma_d}$, $|\gamma| = \gamma_1 + \dots + \gamma_d \leq r$. Then (54) simplifies to

$$p_\gamma(x+y) = \sum_{\alpha+\beta=\gamma} \binom{\alpha+\beta}{\alpha} p_\alpha(x) p_\beta(y) ,$$

giving explicit expressions for the constants $c_{m,n}^k (\equiv c_{\alpha,\beta}^\gamma)$.

In the case of the rotated Q1 element, the choices $\Pi = \Pi_1$ or $\Pi = \Pi_s = Q_1^{\text{rot}}$ would be advisable (the first one for reasons of simplicity, the second one because it is the maximal possible Π satisfying (53)). In the latter case, we would take $p_2(x) = x_1$, $p_3(x) = x_2$, and $p_4(x) = x_1^2 - x_2^2$ as before,

However, the results in the third column show that there is still a deficit. The convergence history of the \tilde{P}_0 -corrected power iteration with starting vector \tilde{b}_0 clearly indicates that the approach via Lemma 10 only partly removes the eigenspace corresponding to $\lambda = 1$ which is irrelevant for the representation \tilde{b}_0 in the basis of eigenvectors. Again, after about 30 iterations, perturbations take over and destroy the almost correctly found value $\tilde{\lambda}_{\max}$ (compare with the results documented in subsection 3.1). The same limit as for the P_0 -corrected iteration is obtained, since in both versions $P_0 = \tilde{P}_0$.

In order to improve the present iterative method, we will make use of results from [13], where the equivalence of \mathcal{L}_S with the adjoint of a certain new subdivision operator in a larger space is explored. We reproduce the purely algebraic result for the convenience of the reader. Let us consider the generalized transfer operator from (45). Since, by definition of $\mathcal{L}_{A,B}$ and by (27),

$$\begin{aligned} (\mathcal{L}_{A,B}X)_\alpha^{l,m} &= \sum_{k=1}^{M_r} \sum_{\beta}^{M_c} \sum_{n=1}^{M_c} \sum_{\gamma} A_{2\alpha-\beta}^{l,k} X_{\beta-\gamma}^{k,n} B_\gamma^{n,m} \\ &= \sum_{k=1}^{M_r} \sum_{n=1}^{M_c} \sum_{\delta} (\sum_{\gamma} A_{2\alpha-\delta-\gamma}^{l,k} B_\gamma^{n,m}) X_\delta^{k,n} \\ &\equiv \sum_{(k,n)} \sum_{\delta} C_{2\alpha-\delta}^{(l,m),(k,n)} X_\delta^{k,n} , \end{aligned}$$

we will introduce the $M_r M_c \times M_r M_c$ matrix function \mathcal{C} with entries

$$C^{(l,m),(k,n)}(\theta) = A^{l,k}(\theta) B^{n,m}(\theta) , \quad l, k = 1, \dots, M_r, \quad m, n = 1, \dots, M_c ,$$

and identify $M_r \times M_c$ matrix functions $X(\theta)$ with vector functions

$$\mathcal{X}(\theta) = (X^{(l,m)}(\theta) \equiv X^{l,m}(\theta) : (l,m) \in \{1, \dots, M_r\} \times \{1, \dots, M_c\})$$

of length $M_r M_c$. Then the operation $\mathcal{L}_{A,B}X$ is equivalent to

$$(\mathcal{L}_{\mathcal{C}, \text{Id}_{M_r M_c}} \mathcal{X})(\theta) = 2^{-d} \sum_{\epsilon \in \{0,1\}^d} \mathcal{C}\left(\frac{\theta}{2} + \pi\epsilon\right) \mathcal{X}\left(\frac{\theta}{2} + \pi\epsilon\right) . \quad (51)$$

The dual operator is the subdivision operator generated by \mathcal{C}^* :

$$(\mathcal{L}_{\mathcal{C}^*, \text{Id}_{M_r M_c}} \mathcal{X})(\theta) = \mathcal{C}^*(\theta) \mathcal{X}(2\theta) .$$

The important observation is that (generalized) eigenfunctions of $\mathcal{L}_{\mathcal{C}, \text{Id}_{M_r M_c}}^*$ can be found from the (generalized) eigenfunctions of $\mathcal{L}_{A, \text{Id}_{M_r}}^*$ and $\mathcal{L}_{B^*, \text{Id}_{M_c}}^*$. The following argument is heuristic (and contains a flaw) but will help to see the basic idea.

Let $y(\theta)$ and $z(\theta)$ be eigenfunctions of $\mathcal{L}_{A, \text{Id}_{M_r}}^*$ and $\mathcal{L}_{B^*, \text{Id}_{M_c}}^*$ with eigenvalues λ and μ , respectively. I.e., we assume that

$$A^*(\theta)y(2\theta) = \lambda y(\theta) , \quad B(\theta)z(2\theta) = \mu z(\theta) .$$

Consider now the vector function $\mathcal{X}(\theta)$ of length $M_r M_c$ which corresponds to the $M_r \times M_c$ matrix function $y(\theta)z^*(\theta)$. Since

$$(C^*)^{(l,m),(k,n)}(\theta) = C^{(k,n),(l,m)}(\theta)^* = A^{k,l}(\theta)^* B^{m,n}(\theta)^* ,$$

we formally have

$$\begin{aligned} (\mathcal{L}_{\mathcal{C}^*, \text{Id}_{M_r M_c}} \mathcal{X})^{(l,m)}(\theta) &= \sum_{(k,n)} A^{(k,l)}(\theta)^* B^{(m,n)}(\theta)^* y^k(2\theta)(z^n(2\theta))^* \\ &= \left(\sum_{k=1}^{M_r} (A^*)^{l,k}(\theta) y^k(2\theta) \right) \left(\sum_{n=1}^{M_c} B^{m,n}(\theta) z^n(2\theta) \right)^* \\ &= \lambda y^l(\theta) (\mu z^m(\theta))^* = \lambda \mu^* \mathcal{X}^{(l,m)}(\theta) . \end{aligned}$$

j	no correction	P_0 -correction	\tilde{P}_0 -correction
5	1.01685082	1.01685082	0.52815081
10	1.00065766	1.00065766	0.52538729
15	1.00002630	1.00002628	0.52531977
20	1.00001488	1.00000105	0.52531812
25	1.01415476	1.00000004	0.52531801
30	3.59552480	1.00000000	0.52531611
35	3.99955718	1.00000000	0.52526870
40	3.99999957		0.52408109
45	4.00000000		0.49271896
50	4.00000000		1.48807892
55			1.01077314
60			1.00042254
65			1.00001689
70			1.00000068
75			1.00000003
80			1.00000000

Table 3: Rotated Q1 element (edge averages): Spectral radius bounds.

j	no correction	P_0 -correction	\tilde{P}_0 -correction
5	1.00764557	1.00764557	0.47777126
10	1.00017941	1.00017941	0.46933939
15	1.00000409	1.00000408	0.46897974
20	1.00001655	1.00000009	0.46896428
25	1.01684967	1.00000000	0.46896441
30	3.65812206	1.00000000	0.46899986
35	3.99963331	1.00000000	0.47055956
40	3.99999964		0.53207637
45	4.00000000		0.92754676
50	4.00000000		0.99812057
55			0.99995723
60			0.99999903
65			0.99999998
70			1.00000000

Table 4: Rotated Q1 element (midpoint interpolation): Spectral radius bounds.

with some constants $c_{k,k',\alpha}$ for all k and α , we immediately get

$$(v_k)_{\alpha+\gamma}^m = \sum_{k'=1}^L c_{k,k',\alpha} (v_{k'})_{\gamma}^m$$

and

$$\begin{aligned} (Xv_k)_{\alpha}^l &= \sum_{m=1}^M \sum_{\beta} X_{\alpha-\beta}^{l,m} (v_k)_{\beta}^m = \sum_{m=1}^M \sum_{\gamma} X_{-\gamma}^{l,m} (v_k)_{\alpha+\gamma}^m \\ &= \sum_{k'=1}^L c_{k,k',\alpha} \sum_{m=1}^M \sum_{\gamma} X_{-\gamma}^{l,m} (v_{k'})_{\gamma}^m = \sum_{k'=1}^L c_{k,k',\alpha} (Xv_{k'})_0^l. \end{aligned}$$

Thus, (48) implies $Xv_k = 0$ for $k = 1, \dots, L$ vice versa. Since all matrices $X(\theta)$ under consideration have support in the finite self-conjugate index set \mathcal{I}_{S,B_0} , only a limited number of entries of the v_k is needed to implement (48). If we represent $X \in V_{S,B_0}$ in the basis \mathcal{N}_{S,B_0} ,

$$X = \sum_{(k,n,\beta)} x(k,n,\beta) X(k,n,\beta),$$

we see that (48) is equivalent to

$$x = (x(k,n,\beta)) \perp X_L = \text{span}(x^{l,k'} : l = 1, \dots, M, k' = 1, \dots, L), \quad (49)$$

where

$$x^{l,k'}(k,n,\beta) = \sum_{m=1}^M \sum_{\gamma} X(k,n,\beta)_{-\gamma}^{l,m} (v_{k'})_{\gamma}^m, \quad (k,n,\beta) \in \mathcal{I}_{S,B_0}.$$

Note that the dimension of X_L may be smaller than LM .

By testing with the vectors b_0 and \tilde{b}_0 , the values for L_0 and \tilde{L}_0 are determined, and orthogonal bases $(x^i : i = 1, \dots, \dim X_{L_0})$ and $(\tilde{x}^i : i = 1, \dots, \dim X_{\tilde{L}_0})$ are chosen in X_{L_0} and $X_{\tilde{L}_0}$, respectively. For this purpose, the routine *orth* is used. Finally, due to this construction the projector

$$P_0 : x \mapsto x - \sum_{i=1}^{\dim V_{L_0}} ((x^i)^* x) x^i \quad (50)$$

induces a projector $\mathcal{P}_0 : V_{S,B_0} \rightarrow V_{S,B_0,W_0}$. Analogously, \tilde{P}_0 and $\tilde{\mathcal{P}}_0$ are introduced. Thus, if we run power iteration (or any other iterative method of Krylov space type) with \mathcal{L}_S replaced by $\mathcal{P}_0 \mathcal{L}_S$ resp. by $\tilde{\mathcal{P}}_0 \mathcal{L}_S$ and with starting matrix B_0 resp. \tilde{B}_0 , we shall get better upper bounds for λ_{\max} resp. $\tilde{\lambda}_{\max}$. In practice, the implementation is with the $M_1 \times M_1$ matrices $P_0 K$ resp. $\tilde{P}_0 K$ replacing K and starting vectors b_0 resp. \tilde{b}_0 .

The calculations for the intergrid transfer operators for the two versions of nonconforming rotated Q1 finite elements are given in Tables 3 and 4. The three columns show approximations to leading eigenvalues of K , $P_0 K$, $\tilde{P}_0 K$ obtained by power iterations from starting vectors b_0 , b_0 , \tilde{b}_0 , respectively (the normalization was with respect to the maximum norm, and the iteration was stopped if successive approximations differed by $< 10^{-8}$). In the first case we observed $L_0 = \tilde{L}_0 = 3$, and $\dim X_{L_0} = \dim X_{\tilde{L}_0} = 4$ while in the second case these values changed to 4 resp. 6.

Let us comment on the numerical results. The first two columns show what we would have expected from the theory. While in both cases, according to the theory, the limit should equal λ_{\max} , without the correction numerical perturbations take over after about 20 iterations which leads to a serious deterioration from the desired result. The P_0 -correction is sufficient to give an accurate value of λ_{\max} .

Proof. It suffices to establish that (46) implies

$$\langle (\mathcal{L}_S X_0)(\theta) v_k(\theta), u(\theta) \rangle_{((\mathcal{D}(\mathbf{T}^d))^M)' \times (\mathcal{D}(\mathbf{T}^d))^M} = 0 \quad \forall u(\theta) \in (\mathcal{D}(\mathbf{T}^d))^M, \quad k = 1, \dots, L.$$

One easily verifies that

$$\begin{aligned} & \langle (\mathcal{L}_S X_0)(\theta) v_k(\theta), u(\theta) \rangle_{((\mathcal{D}(\mathbf{T}^d))^M)' \times (\mathcal{D}(\mathbf{T}^d))^M} \\ &= \langle (\mathcal{L}_{\{S \star X_0 S, \text{Id}_1\}} v_k(2 \cdot))(\theta), u(\theta) \rangle_{((\mathcal{D}(\mathbf{T}^d))^M)' \times (\mathcal{D}(\mathbf{T}^d))^M} \\ &= \langle S(\theta) v_k(2\theta), X_0(\theta) S(\theta) u(2\theta) \rangle_{((\mathcal{D}(\mathbf{T}^d))^M)' \times (\mathcal{D}(\mathbf{T}^d))^M} \\ &= \lambda_k \langle B_0(\theta) v_k(\theta), \underbrace{S(\theta) u(\theta)}_{\equiv \tilde{u}(\theta)} \rangle_{((\mathcal{D}(\mathbf{T}^d))^M)' \times (\mathcal{D}(\mathbf{T}^d))^M} = 0, \end{aligned}$$

where in the last step (46) and $\tilde{u} \in (\mathcal{D}(\mathbf{T}^d))^M$ have been used. All other steps are just algebraic manipulations which hold true since the functions appearing in the second argument have always finite Fourier spectrum.

To apply Lemma 10, assume that a system of eigenvectors $\{v_k : k = 1, \dots, L\}$ of the subdivision operator S is given, such that its section $\{v_k : k = 1, \dots, L_0\}$ satisfies (46) for $X_0 = B_0$, and $\{v_k : k = 1, \dots, \tilde{L}_0\}$ satisfies (46) for $X_0 = \tilde{B}_0$ (without loss of generality, $0 \leq L_0 \leq \tilde{L}_0 \leq L$). The corresponding subspaces W are denoted by W_0 resp. \tilde{W}_0 ($W_0 \subset \tilde{W}_0$). Then we have

$$V(\mathcal{L}_S, B_0) \subset V_{S, B_0, W_0} \equiv V_{S, B_0} \cap V_{W_0}$$

and

$$V(\mathcal{L}_S, \tilde{B}_0) \subset V_{S, B_0, \tilde{W}_0} \equiv V_{S, B_0} \cap V_{\tilde{W}_0}$$

for the Krylov spaces of interest. Since

$$V_{S, B_0, \tilde{W}_0} \subset V_{S, B_0, W_0} \subset V_{S, B_0},$$

power iteration with the operator \mathcal{L}_S , in combination with projection into V_{W_0} resp. $V_{\tilde{W}_0}$ should give much better bounds for λ_{\max} and $\tilde{\lambda}_{\max}$ than without such a projection step.

To implement these ideas, we need to determine suitable eigenvector systems $\{v_k : k = 1, \dots, L\}$. We restrict our attention to those v_k that correspond to certain low-order homogeneous polynomials which are preserved under the action of the intergrid transfer operators I_j under consideration. To be precise, let Π_S be the (largest) space of polynomials which is both invariant under the action of I_1 and arbitrary \mathbb{Z}^d -shifts. In our above example of the rotated Q1 element, we can take $\Pi_S = Q_1^{\text{rot}} = \text{span}\{1, x_1, x_2, x_1^2 - x_2^2\}$ (note that due to the assumption of finite masks in the cell representation of S , the definition of I_1 can easily be extended to arbitrary polynomials from Q_1^{rot}). Considering the coefficients in the basis \mathcal{N}_0 , we observe that the homogeneous polynomials $p_1(x) = 1$, $p_2(x) = x_1$, $p_3(x) = x_2$, $p_4(x) = x_1^2 - x_2^2$ in Q_1^{rot} generate generalized eigenvectors v_1, v_2, v_3, v_4 of the subdivision operator S corresponding to the eigenvalues $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 1/2$, $\lambda_4 = 1/4$. The cell representations of these eigenvector functions v_k are given in Figure 6 for the spaces based on edge averages (later, we will derive more compact representations for these coefficient arrays, compare (57)). We do not show v_3 since it can easily be obtained from v_2 by symmetry. For the case of midpoint interpolation, only v_4 has to be changed accordingly.

s = 1					
λ	mult	rank	$m(b_0, \lambda)$	$m(\tilde{b}_0, \lambda)$	cond
4.0000000000	1	1	0.0	0.0	6.6
1.0000000000	4	4	0.00655	0.0	690.3
0.5000000000	4	4	0.0	0.0	9351.1
0.4689635611	1	1	0.00027	0.000931	1430.1
s = 0					
1.0000000000	1	1	0.08944	0.0	6.6
0.2500000000	4	4	0.00231	0.004906	690.3

Table 2: Rotated Q1 element (midpoint interpolation): Leading eigenvalues of K .

\tilde{B}_0 since $\lambda_{\max} = \rho(\mathcal{L}_S|_{V(\mathcal{L}_S, B_0)})$ resp. $\tilde{\lambda}_{\max} = \rho(\mathcal{L}_S|_{V(\mathcal{L}_S, \tilde{B}_0)})$. However, special attention should be paid to rounding errors which usually make the iterates leave the subspace of interest very soon, and lead to uninteresting upper bounds rather than to correct estimations for these values. Compare the calculations presented below.

Our first attempt to correct this situation is based on the following observations. Note that due to the finite support of the Fourier spectrum of $S(\theta)$ and $T(\theta)$, all operators considered so far have natural extensions to the space $((\mathcal{D}(\mathbb{T}^d))^M)'$ of generalized vector functions (since we are only interested in purely algebraic statements, the reader may think of $((\mathcal{D}(\mathbb{T}^d))^M)'$ as the collection of all $M \times 1$ cells whose entries are coefficient arrays of infinite size ($N = \infty$) (i.e., formal d -dimensional Fourier series)). E.g., $v \in ((\mathcal{D}(\mathbb{T}^d))^M)'$ is called eigenvector of the subdivision operator S corresponding to eigenvalue λ if

$$\begin{aligned} & \langle S(\theta)v(2\theta) - \lambda v(\theta), u(\theta) \rangle_{((\mathcal{D}(\mathbb{T}^d))^M)' \times (\mathcal{D}(\mathbb{T}^d))^M} \\ &= \langle v(\theta), (\mathcal{L}_{\{S^*, \text{Id}_1\}} u)(\theta) - \lambda^* u(\theta) \rangle_{((\mathcal{D}(\mathbb{T}^d))^M)' \times (\mathcal{D}(\mathbb{T}^d))^M} = 0 \quad \forall u(\theta) \in (\mathcal{D}(\mathbb{T}^d))^M. \end{aligned} \quad (44)$$

For our convenience, we have introduced the notation

$$(\mathcal{L}_{\{A, B\}} X)(\theta) = 2^{-d} \sum_{e \in \{0, 1\}^d} A\left(\frac{\theta}{2} + e\pi\right) X\left(\frac{\theta}{2} + e\pi\right) B\left(\frac{\theta}{2} + e\pi\right) \quad (45)$$

for a general transfer operator characterized by a $M_r \times M_r$ matrix function $A(\theta)$ and a $M_c \times M_c$ matrix function $B(\theta)$ with trigonometric polynomials as entries which acts on $M_r \times M_c$ matrix functions $X(\theta)$. Thus, $\mathcal{L}_S \equiv \mathcal{L}_{\{S^*, S\}}$. The practical computation (in terms of masks of Fourier coefficients) with $\mathcal{L}_{\{A, B\}}$ is based on (27) and analogous to what was explained for \mathcal{L}_S . Furthermore, Id_M stands for the identity matrix function of size $M \times M$. The duality pairing in (44) is to be understood as the $(\ell^2(\mathbb{Z}^d))^M$ scalar product of the corresponding sequences of Fourier coefficients. It makes sense since the second argument has finite Fourier spectrum.

Lemma 10 *Let $\{v_k : k = 1, \dots, L\} \subset ((\mathcal{D}(\mathbb{T}^d))^M)'$ be a finite set of eigenvectors of the subdivision operator S corresponding to the set of eigenvalues $\{\lambda_k : k = 1, \dots, L\}$. Let $X_0(\theta)$ be any Hermitean $M \times M$ matrix function whose entries are trigonometric polynomials. If*

$$W \equiv \text{span} \{v_k : k = 1, \dots, L\} \subset \ker X_0,$$

i.e., if

$$X_0(\theta)v_k(\theta) = 0, \quad k = 1, \dots, L, \quad (46)$$

in the sense of $((\mathcal{D}(\mathbb{T}^d))^M)'$, then

$$V(\mathcal{L}_S, X_0) \subset V_W \equiv \{X(\theta) : W \subset \ker X\}. \quad (47)$$

In other words, \mathcal{L}_S leaves V_W invariant (the matrix functions $X(\theta)$ are assumed to be of size $M \times M$ and to have trigonometric entries).

$$S : \frac{1}{32} \left(\begin{array}{c} \left[\begin{array}{ccccc} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 6 & 6 & 0 \\ 0 & 0 & 30 & 30 & 0 \\ 0 & 0 & 6 & 6 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array} \right] \left[\begin{array}{ccccc} -3 & 5 & 5 & -3 & 0 \\ 2 & 18 & 18 & 2 & 0 \\ -3 & 5 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & -3 \\ 0 & 0 & 5 & 18 & 5 \\ 0 & 0 & 5 & 18 & 5 \\ 0 & 0 & -3 & 2 & -3 \end{array} \right] \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ -1 & 6 & 30 & 6 & -1 \\ -1 & 6 & 30 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array} \right)$$

Figure 4. Rotated Q1 element (midpoint interpolation): S for $s = 1$.

While the cell representation of B_0 essentially does not change, the good index set is slightly larger resulting in $N = 3$ and $M_1 = 64$:

$$\chi_{\mathcal{I}_{S,B_0}} : \left(\begin{array}{c} \left[\begin{array}{cccccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array} \right] \left[\begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} \right] \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{array} \right] \end{array} \right)$$

Figure 5. Rotated Q1 element (midpoint interpolation): Good index set \mathcal{I}_{S,B_0} .

We have again computed the upper part of the spectrum of the matrix K associated with the restriction of \mathcal{L}_S to the subspace $V_{S,B_0} = V_{\mathcal{I}_{S,B_0}}$ defined by the above good index set. The numerical results are represented in Table 2, leading to the values

$$\lambda_{\max} = 1, \quad \tilde{\lambda}_{\max} = 0.468964.$$

Thus, the energy norms of the iterated intergrid transfer operators \tilde{I}_j^J (defined by nodal value averaging as explained before) for this version of rotated Q1 element spaces are uniformly bounded. This corrects the statement to the contrary contained in [19] (seemingly, a mistake has occurred in the computations leading to column a) of [19, Table 1]). From the theory of [19], we expect an suboptimal condition number growth for the associated additive Schwarz preconditioner. Details are left to the interested reader.

3.2 The rotated Q1 element: Iterative methods

Now we present results on the iterative techniques mentioned at the end of section 2. As is clear from the derivations in section 2 and subsection 3.1, the values of λ_{\max} resp. $\tilde{\lambda}_{\max}$ can theoretically be obtained by power iteration using the operator \mathcal{L}_S with starting matrix B_0 resp.

edge midpoints rather than on integral averages along edges (note that both versions lead to the same finite element space if triangular P1 elements are considered). To be precise, replace the second condition in the above definition of V_0 by the requirement that

$$(v_0|_{E_\alpha})(M_{e_\alpha^1}) = (v_0|_{E_{\alpha-e_2}})(M_{e_\alpha^1}) , \quad (v_0|_{E_\alpha})(M_{e_\alpha^2}) = (v_0|_{E_{\alpha-e_1}})(M_{e_\alpha^1}) \quad \forall \alpha \in \mathbb{Z}^2 .$$

Here, M_e denotes the midpoint of the edge e . Thus, functions from the new nonconforming rotated Q1 space V_0 are continuous across midpoints of all edges in \mathcal{T}_0 . Consequently, the Riesz basis in V_0 can be generated from two functions ϕ^1, ϕ^2 defined by

$$\phi^m|_{E_\alpha} \in Q_1^{\text{rot}} \quad \forall \alpha \in \mathbb{Z}^2 , \quad \phi^m(M_{e_\alpha^k}) = \begin{cases} 1 & , \quad m = k, \alpha = 0 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

The coefficients in the basis representation

$$v_0 = \sum_{m=1}^2 \sum_{\alpha \in \mathbb{Z}^2} c_\alpha^m \phi_{0,\alpha}^m$$

are given by $c_\alpha^m = v_0(M_{e_\alpha^m})$.

The intergrid transfer operator of interest will be defined by nodal value averaging. I.e., for a given $v_0 \in V_0$, the resulting $v_1 = I_1 v_0 \in V_1$ satisfies $v_1(M_e) = v_0(M_e)$ for all edges e from \mathcal{T}_1 which are interior to some square E_α of \mathcal{T}_0 , and

$$v_1(M_e) = \frac{1}{2}((v_0|_E)(M_e) + (v_0|_{E'})(M_e))$$

if e belongs to the common edge $e' = \partial E \cap \partial E'$ of two neighboring squares E, E' ($E \neq E'$) in \mathcal{T}_0 . To produce the entries for S , it is enough to look at the representation of v_0 on a generic square (say, on E_0):

$$p_0(x) \equiv (v_0|_{E_0})(x) = A + Bx_1 + Cx_2 + D(x_1^2 - x_2^2) ,$$

where one computes

$$\begin{aligned} A &= \frac{1}{4}(3c_0^1 + 3c_0^2 - c_{e_2}^1 - c_{e_1}^2) , & B &= 2c_{e_2}^1 + c_0^1 - 2c_0^2 , \\ C &= c_{e_1}^2 + c_0^2 - 2c_0^1 , & D &= c_{e_1}^2 + c_0^2 - c_{e_2}^1 - c_0^1 , \end{aligned}$$

for the coefficients. Now it remains to evaluate the polynomial p_0 at all points of the form $(\frac{1}{2} \pm \frac{1}{4}, \frac{j}{2})$ resp. $(\frac{j}{2}, \frac{1}{2} \pm \frac{1}{4})$ ($j = 0, 1, 2$), to derive similar formulae for other $p_\alpha(x) \equiv (v_0|_{E_\alpha})(x - \alpha)$ by index shifting, and to apply the above definition of I_1 . E.g., let $v_0 = \phi^1$. By the above formulae, we have

$$p_0(x) = \frac{3}{4} + x_1 - 2x_2 - (x_1^2 - x_2^2) , \quad p_{-e_2}(x) = -\frac{1}{4} + x_1 - (x_1^2 - x_2^2) ,$$

(all other p_α vanish). This yields (compare (23))

$$\begin{aligned} \tilde{s}_0^{1,1} &= (I_1 \phi^1)\left(\frac{1}{4}, 0\right) = \frac{1}{2}(p_0\left(\frac{1}{4}, 0\right) + p_{-e_2}\left(\frac{1}{4}, 1\right)) = \frac{15}{16} , \\ \tilde{s}_{e_2}^{1,1} &= (I_1 \phi^1)\left(\frac{1}{4}, \frac{1}{2}\right) = p_0\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{3}{16} , \\ \tilde{s}_{2e_2}^{1,1} &= (I_1 \phi^1)\left(\frac{1}{4}, 1\right) = \frac{1}{2}p_0\left(\frac{1}{4}, 0\right) = -\frac{1}{32} , \end{aligned}$$

all other $\tilde{s}_\alpha^{1,1}$ can be obtained by symmetry arguments, or vanish. Analogous calculations lead to the values of $\tilde{s}_\alpha^{l,m}$ for other l, m . Figure 4 shows the final result of these elementary computations ($N_0 = 2$).

$s = 1$					
λ	mult	rank	$m(b_0, \lambda)$	$m(\tilde{b}_0, \lambda)$	cond
4.0000000000	1	1	0.0	0.0	5.8
1.0000000000	4	4	0.08868	0.0	176.8
0.5253180854	1	1	0.07819	0.021982	1364.6
0.5181139248	1	1	0.07314	0.020874	111.4
$s = 0$					
1.0000000000	1	1	0.11111	0.0	5.8
0.2500000000	4	4	0.00467	0.013398	176.8

Table 1: Rotated Q1 element (edge averages): Leading eigenvalues of K .

The second case is $\lambda_{\max} = 1$. If this happens and if again eigenspaces associated with eigenvalues of absolute value 1 are not deficient, we have $\rho = 1$ and $r = 1$ in Theorem 8. Thus, as a rule, (9) holds true, and the intergrid transfer operator I_j should be successful in multilevel applications.

To further validate this positive result (note that the eigenspace associated with $\lambda = 1$ has usually a relatively large dimension which might cause some doubts), we have used the observation of Remark 6, and implemented the same procedure with the coefficient vector \tilde{b}_0 of $\tilde{B}_0 = \mathcal{L}_S B_0 - B_0$ instead of b_0 . Obviously, from (40), we immediately get

$$\tilde{b}_0 = \sum_{i=1}^{M_1} (\lambda_i - 1) a_i e_i \equiv \sum_{i=1}^{M_1} \tilde{a}_i e_i . \quad (42)$$

Thus, if the second case applies ($\lambda_{\max} = 1$) then all \tilde{a}_i vanish for which either $|\lambda_i| > 1$ or $\lambda_i = 1$. Thus, normally we expect that all eigenvalues λ_i of absolute value ≤ 1 will be discarded in the above procedure, and that the replacement $\tilde{\lambda}_{\max} = |\lambda_{\tilde{i}_{\max}}|$ for λ_{\max} will be less than 1. If this is the case then Remark 6 independently yields (10) resp. (9) since the relevant eigenvalues λ_i for the Krylov space $V(\mathcal{L}_S, \tilde{B}_0)$ satisfy $|\lambda_i| < 1$.

The results of the numerical calculations for the rotated Q1 element and the I_j defined by averaging are given in Table 1. In the first two columns, we show the distinct leading eigenvalues of K obtained by the MATLAB command *condeig* and their multiplicity (after reordering using *sort*). Next, for each such λ the rank of the associated system $\{e_i : \lambda_i = \lambda\}$ of eigenvectors has been determined by *rank*. In columns 4 and 5, the values

$$m(b, \lambda) = \max_{i: \lambda_i = \lambda} |\tilde{e}_j^* b|$$

are shown in the cases $b = b_0$ and $b = \tilde{b}_0$. In the last column the maximal condition number for each group of eigenvectors is given. Finally, in the lower part of the table, the same calculations are reported on for the case $s = 0$, where T and B_0 are both replaced by a 2×2 unit cell (i.e., the corresponding matrix function coincides with the constant unit matrix), and S was multiplied by $\frac{1}{2}$ to obtain the correct scaling factor. The good index set did not change even though we have started from another B_0 .

On the basis of these calculations, we have for $s = 1$

$$\lambda_{\max} = 1 , \quad \tilde{\lambda}_{\max} = 0.525318 . \quad (43)$$

This means that the iterated intergrid transfer operators \tilde{I}_j for the nonconforming rotated Q1 element satisfy (9) which confirms the results presented in [19, 6]. The situation does not change if the norm of interest includes a L_2 -term.

We have also investigated the standard intergrid transfer operator for another variant of nonconforming rotated Q1 elements where the degrees of freedom are based on interpolation at the

where $E_i(\theta)$ is the Hermitean matrix function associated with the eigenvector e_i . Thus, if b_0 denotes the coefficient vector of B_0 with respect to the standard basis \mathcal{N}_{S, B_0} then (39) is equivalent to

$$b_0 = \sum_{i=1}^{M_1} a_i e_i . \quad (40)$$

Thus, in order to detect the non-zero coefficients a_i in (39) as required by Theorem 8, we can use the biorthogonality relations (38) which give $a_i = \tilde{e}_i^* b_0$. As a practical consequence, an eigenvalue λ of $\mathcal{L}_S|_{V_{S, B_0}}$ does not belong to the spectrum of $\mathcal{L}_{S, B_0} = \mathcal{L}_S|_{V(\mathcal{L}_{S, B_0})}$ if and only if

$$\tilde{e}_i^* b_0 = 0 \quad \forall i : \lambda_i = \lambda . \quad (41)$$

In this form, the statement remains true also without guaranteeing (38) for multiple eigenvalues. Our MATLAB program does therefore the following:

- Computation (and ordering) of the eigenvalues and associated eigenvectors of K and K^T by using the *condeig* and *sort* commands. This leads to the sets of vectors e_i and \tilde{e}_i such that

$$K e_i = \lambda_i e_i , \quad K^T \tilde{e}_i = \lambda_i^* \tilde{e}_i , \quad i = 1, \dots, M_1 ,$$

where λ_i are the eigenvalues of K .

- Computation of b_0 (corresponding to B_0).
- Starting from $i = 1$, for all λ_i with $|\lambda_i| \geq 1$, we check whether λ_i is simple or multiple. Assume for simplicity that the eigenvalues and associated eigenvectors are ordered such that multiple eigenvalues always correspond to an index interval. Let $i_1, i_1 + 1, \dots, i_2$ be the indices associated with λ_i , i.e., $i_1 \leq i \leq i_2$ and

$$\lambda_{i_1-1} \neq \lambda_{i_1} = \dots = \lambda_{i_2} \neq \lambda_{i_2+1} .$$

If the rank of the restricted eigensystem $\{e_{i_1}, \dots, e_{i_2}\}$ is maximal (this test has never failed!) and if

$$\tilde{e}_j^* b_0 = 0 , \quad j = i_1, \dots, i_2 ,$$

then the eigenvalues $\lambda_{i_1}, \dots, \lambda_{i_2}$ are discarded, otherwise they are kept.

The restriction to eigenvalues with $|\lambda_i| \geq 1$ in the last step is typical for our applications, in other applications this part should be appropriately adapted. Set $\lambda_{\max} = |\lambda_{i_{\max}}|$ for the largest absolute value of the eigenvalues remaining after the above procedure.

There are two main cases. First, if it happens that $\lambda_{\max} > 1$ (and that the rank of eigenspace associated with $\lambda_{i_{\max}}$ equals the multiplicity of this eigenvalue) then we have $\rho = \lambda_{\max} > 1$ and $r = 1$ in Theorem 8. In the examples below, this happened only for the Morley element. In this case, the critical eigenvalue was simple, and deficiency problems can be excluded with high probability. In all cases we have also monitored the conditioning of the eigenvalue calculations to improve the reliability of our calculations.

As a consequence, in the first case the iterated intergrid transfer operators under consideration do not satisfy the desired uniform bound (9) of the constants in the norm inequality (8). Moreover, asymptotically we should have $\tilde{c}_{j, J} \asymp \lambda_{\max}^{J-j}$, and consequently, λ_{\max} describes the exact rate of exponential blow-up of these constants. According to the theory in [16], this at the same time indicates exponential growth of the condition numbers of the additive multilevel preconditioners briefly described in the introduction. Thus, intergrid transfer operators with $\lambda_{\max} > 1$ are of little interest for multilevel solvers for large scale applications (however, things may slightly change if the same intergrid transfer operators are used as prolongations and restrictions in a true multigrid algorithm, i.e., in a multiplicative fashion).

$$\chi_{\mathcal{I}_{S, B_0}} : \left(\begin{array}{c} \left[\begin{array}{ccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right] & \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right] \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] & \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right] \end{array} \right)$$

Figure 3. Rotated Q1 element (edge averages): Good index set \mathcal{I}_{S, B_0} .

In the computations below, we have used $V_{S, B_0} = V_{\mathcal{I}_{S, B_0}}$ equipped with the standard basis

$$\mathcal{N}_{S, B_0} = \{X(k, n, \beta) : (k, n, \beta) \in \mathcal{J}_{S, B_0}\},$$

where the cell representations of the basis functions are given by

$$X(k, n, \beta)_\alpha^{l, m} = \begin{cases} 1 & , \quad l = k, m = n, \alpha = \beta \\ 1 & , \quad m = k, l = n, \alpha = -\beta \\ 0 & , \quad \text{otherwise} \end{cases},$$

and the index set \mathcal{J}_{S, B_0} is given by all triples (k, n, β) satisfying $1 \leq k \leq n \leq M$, $\beta \in (I_{S, B_0})^{k, n}$ and, if $k = n$, additionally $\beta_1 + \beta_2 > 0$ or $-\beta_1 = \beta_2 \geq 0$. I.e., to each position in Figure 3 marked by $\mathbf{1}$, there corresponds a basis element given by the characteristic function of the self-conjugate index set generated by this position. With this, we can derive a dimension formula for V_{S, B_0} :

$$M_1 \equiv \dim V_{S, B_0} = \frac{1}{2}(\#\mathcal{I}_{S, B_0} + M^*), \quad \#\mathcal{I}_{S, B_0} = \sum_{l, m} \#(I_{S, B_0})^{l, m},$$

where M^* ($0 \leq M^* \leq M$) is the number of index sets in the diagonal for which $0 \in (I_{S, B_0})^{m, m}$ (with the exception of exotic cases, we usually have $M^* = M$). From Figure 3, we have $M_1 = 42$ for the example of this subsection.

In this basis \mathcal{N}_{S, B_0} a matrix representation K of the operator $\mathcal{L}_S|_{V_{S, B_0}}$ can be found by evaluating $\mathcal{L}_S X(k, n, \beta)$ for each $(k, n, \beta) \in \mathcal{J}_{S, B_0}$. K is a real $M_1 \times M_1$ matrix.

Assume for the following discussion that K is diagonalizable, and that the MATLAB routine *eig* has provided the eigenvalues $\{\lambda_1, \dots, \lambda_{M_1}\}$, and a linearly independent set of associated eigenvectors $\{e_1, \dots, e_{M_1}\}$ within a certain sufficiently small tolerance (note that *condeig* and *rank* can be used to gain some numerical support for this hypothesis). Unfortunately, we do not know whether the above K satisfies this assumption automatically, in the computations reported on below the eigenspaces of interest never did show a ‘numerical’ sign of deficiency. We will assume that the natural ordering of the eigenvalues is already by decreasing absolute value. Since K is real, the possibly complex eigenvalues occur in pairs. Since $K^* = K^T$ is also diagonalizable, we have essentially the same properties for the eigenvalues $\tilde{\lambda}_i = \lambda_i^*$ and eigenvectors \tilde{e}_i of K^* . Without loss of generality, we may assume that the eigenvector systems $\{e_i\}$ and $\{\tilde{e}_i\}$ are biorthogonal, i.e.,

$$\tilde{e}_j^* e_i = \delta_{ij}, \quad i, j = 1, \dots, M_1. \quad (38)$$

Since $B_0(\theta) \in V_{S, B_0}$, we have

$$B_0(\theta) = \sum_{i=1}^{M_1} a_i E_i(\theta), \quad (39)$$

reduce to the 3×3 arrays shown above. Below, we will automatically take care about reducing array sizes, without explicitly mentioning it.

We have used the standard MATLAB commands *conj*, *fliplr*, *flipud*, *conv2* which by default act on two-dimensional arrays (matrices). Replacing the cell entries by d -dimensional arrays potentially covers the case of other dimensions d . Note that in our applications all coefficients remain real-valued, thus, the above use of *conj* is superfluous.

Because it is central to the paper, let us also outline the implementation of the transfer operator \mathcal{L}_S . According to its definition in terms of symbol functions, given a matrix function $X(\theta)$ by its cell representation X , one computes the cell product $Y = ht(S)XS$ as described before. In order to get the cell representation \tilde{X} of $\tilde{X}(\theta) = (\mathcal{L}_S X)(\theta)$, according to (27) it remains to pick suitable subarrays. E.g., if N_0 and N ($N_0 \leq N \leq 2N_0$) characterize as above the sizes of the index boxes suitable for S , $ht(S)$, and X , then (after applying *conv2* twice) the arrays $Y^{l,m}$ are associated with index boxes $[-2N_0 - N, N + 2N_0]^2$, and the pseudo-MATLAB assignment $\tilde{X}\{l, m\} = Y\{l, m\}(-2N : 2 : 2N, -2N : 2 : 2N)$ would be appropriate.

Before we can set up the matrix eigenvalue problem corresponding to the operator \mathcal{L}_{S, B_0} as described in Section 2, we need to define a suitable V_{S, B_0} . The following notation will be useful. Let us consider $M \times M$ cells \mathcal{I} (so-called index sets), the entries $I^{l,m}$ of which are finite subsets of \mathbb{Z}^d . By $\chi_{\mathcal{I}}$ we denote the characteristic function for the index set \mathcal{I} , i.e., the entries of its cell representation are given by

$$(\chi_{\mathcal{I}})_{\alpha}^{l,m} = \begin{cases} 1 & , \quad \alpha \in I^{l,m} \\ 0 & , \quad \alpha \notin I^{l,m} \end{cases} .$$

Since we are working exclusively with Hermitean matrix functions, we will assume that our index sets are self-conjugate, i.e.,

$$\alpha \in I^{l,m} \iff -\alpha \in I^{m,l} \quad \forall \alpha \in \mathbb{Z}^2 ,$$

or, in other words, that $ht(\chi_{\mathcal{I}}) = \chi_{\mathcal{I}}$. By $V_{\mathcal{I}}$ we denote the linear space of all Hermitean matrix functions with Fourier spectrum (componentwise) contained in the self-conjugate index set \mathcal{I} . A convenient way to describe this property is to define the spectrum of a matrix function $X(\theta)$ as the (not necessarily self-conjugate) index set $\mathcal{I}(X)$ given componentwise by

$$I(X)^{l,m} = \{\alpha : x_{\alpha}^{l,m} \neq 0\} \quad (X^{l,m}(\theta) = \sum_{\alpha \in \mathbb{Z}^2} x_{\alpha}^{l,m} e^{-i\alpha\theta}) .$$

Then

$$V_{\mathcal{I}} = \{X(\theta) : X^*(\theta) = X(\theta), I(X)^{l,m} \subset I^{l,m}, l, m = 1, \dots, M\} .$$

An index set \mathcal{I} is called a *good index set* for \mathcal{L}_S if $V_{\mathcal{I}}$ is invariant under the transfer operator \mathcal{L}_S . The existence of good index sets is clear, e.g., the index set \mathcal{I}_N consisting of index boxes $[-N, N]^2$ with $N = \max(2N_0, N_1)$ will do (see Section 2).

Here is a simple procedure for constructing a possibly much smaller good index set I with the additional property that $B_0(\theta) \in V_{\mathcal{I}}$. Set $\hat{S} = \chi_{\mathcal{I}(S)}$, i.e., replace in the coefficient arrays of S all non-zero coefficients by 1, and $X^{(0)} = \chi_{\mathcal{I}(B_0)}$. The sizes of the arrays in the cell representations of S and X are characterized by integers N_0 and N , respectively. Then define recursively the sequence $\{X^{(n)}\}$ by performing the following steps. From the representation of $X^{(n)}$, compute first the representation $Y^{(n)}$ of $\mathcal{L}_{\hat{S}} X^{(n)}$ as discussed above. $Y^{(n)}$ consists of arrays of non-negative integer coefficients. After this, set $X^{(n+1)} = \chi_{\mathcal{I}(Y^{(n)} + X^{(n)})}$ which is again a cell representation with 0 and 1 in the arrays. Obviously, we have

Lemma 9 *The sequence $\{X^{(n+1)}\}$ is increasing, i.e., $X^{(n+1)} \geq X^{(n)}$, and bounded by $\chi_{\mathcal{I}_N}$. Therefore, there exists a maximal element characterized by a finite n^* which can be determined as the smallest n such that $X^{(n+1)} = X^{(n)}$. Then $\mathcal{I}_{S, B_0} \equiv I(X^{(n^*)})$ is a good index set for \mathcal{L}_S .*

The proof is left as an exercise upon the reader.

In the case of the rotated Q1 element, with S and B_0 as shown above, this procedure leads to the choice of $N = 2$ and to \mathcal{I}_{S, B_0} as depicted in Figure 3.

$$T : \left(\begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \right) \quad B_0 : \left(\begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \right)$$

Figure 2. Rotated Q1 element (edge averages): T and B_0 for $s = 1$.

Note that the introduction of a suitable equivalent norm does not change, up to constants, the asymptotical behavior of the norms of iterated intergrid transfer operators. However, it may change the size of the computational Krylov space (and, thus, the size of the eigenvalue problem to be considered). We will therefore choose equivalent norms, where the resulting representations of T resp. B_0 are as compact as possible.

3.1 Direct methods

We come to the details of computing the needed spectral properties of \mathcal{L}_S by direct methods for eigenvalue problems. The discussion is illustrated for the rotated Q1 case. We assume that the entries of the coefficient arrays in the representations for S , T have been precomputed. All further computations can conveniently be carried out using this type of data structure. To give an example, we discuss how to compute the representation of B_0 . Recall that $B_0 = T^*T$, or in terms of matrix functions $B_0(\theta) = T^*(\theta)T(\theta)$. To carry out these multiplications (and other operations needed in the sequel) in an efficient way, MATLAB's version V provides the necessary tools (readers not familiar with MATLAB should better skip the information on implementational details). We will identify T (and, simultaneously, the associated matrix function $T(\theta)$) with a cell of coefficient arrays which we call the *cell representation* of T . In the above example, this results in a 3×2 cell whose entries $T\{l, m\} \equiv T^{l,m}$ are given by the 3×3 arrays shown above. Analogously, cell representations of other operators resp. matrix functions are defined. We immediately see that T^* corresponds to the Hermitean transpose $ht(T)$ of T which is defined as a cell of size 2×3 , with the entries

$$ht(T)^{l,m} = \text{flipud}(\text{fliplr}(\text{conj}(T^{m,l}))) .$$

The product of two matrix functions $A(\theta), B(\theta)$ (represented by cells A and B of coefficient arrays of compatible dimensions as introduced above) is naturally given by a cell AB with the entries

$$AB^{l,m} = \sum_k \text{conv2}(A^{l,k}, B^{k,m}) .$$

Linear combinations $aA + bB$ resp. order relations $A \leq B$ of cell representations are defined componentwise by

$$(aA + bB)^{l,m} = aA^{l,m} + bB^{l,m}$$

resp. by

$$A_\alpha^{l,m} \leq B_\alpha^{l,m} \quad \forall \alpha \in \mathbb{Z}^2 ,$$

$l, m = 1, \dots, M$. Applying the product operation to $A = ht(T)$ and $B = T$, we get the cell representation of B_0 . Formally, in the above example the result would be a 2×2 cell of 5×5 coefficient arrays (corresponding to an index box $[-2, 2]^2$). Since the support of all these arrays (the index set of the non-zero entries) is contained in a smaller index box $[-1, 1]^2$, we could immediately

$av(v_1, e) = av(v_0, e)$. Otherwise, there are two adjacent squares E, E' such that $e \subset \partial E \cap \partial E'$, and we set

$$av(v_1, e) = \frac{1}{2}(av(v_0|_E, e) + av(v_0|_{E'}, e)).$$

Again, the general I_j is obtained by dilation.

In [6, Lemma 2.3], it was shown that I_j is bounded, both with respect to the L_2 -norm and the discrete H^1 -seminorm, uniformly in j . Also, Lemma 2.4 of the same paper essentially established the uniform boundedness of the iterated transfer operators \tilde{I}_j , i.e., proved the validity of (9). We will recover this result following the approach of section 2. The entries of the associated subdivision operator S which coincide with the values of $\tilde{s}_\gamma^{l,m}$ multiplied by the scaling factor 2^{s-1} can be found from equations (2.9-10) of [6]. We show them in the following Figure 1 for $s = 1$ (in all examples, these values are given for the s associated with the order of the natural energy space, and depicted in form of two-dimensional cells of two-dimensional coefficient arrays (the so-called masks) corresponding to the index box $[-N_0, N_0]^2$).

$$S : \frac{1}{8} \left(\begin{array}{c} \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 8 & 8 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \left[\begin{array}{ccccc} -1 & 1 & 1 & -1 & 0 \\ 1 & 5 & 5 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & -1 & 1 & -1 \end{array} \right] \quad \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 1 & 0 \\ 0 & 1 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array} \right)$$

Figure 1. Rotated Q1 element (edge averages): S for $s = 1$.

It is easy to check that for any

$$v_0 = \sum_{m=1}^2 \sum_{\alpha \in \mathbf{Z}^2} c_\alpha^m \phi_{0,\alpha}^m \in V_0,$$

we have by local estimates for the H^1 -seminorm for functions from Q_1^{rot}

$$\|v_0\|_{0,H^1}^2 \asymp |v_0|_{0,1}^2 \equiv \sum_{\alpha \in \mathbf{Z}^2} ((c_{\alpha+\epsilon^2}^1 - c_\alpha^1)^2 + (c_{\alpha+\epsilon^1}^2 - c_\alpha^2)^2 + (c_{\alpha+\epsilon^2}^1 + c_\alpha^1 - c_{\alpha+\epsilon^1}^2 - c_\alpha^2)^2) \quad (36)$$

while

$$\|v_0\|_{L_2}^2 \asymp |v_0|_{0,0}^2 \equiv \sum_{\alpha \in \mathbf{Z}^2} ((c_\alpha^1)^2 + (c_\alpha^2)^2) \quad (37)$$

according to the Riesz basis property (14) of \mathcal{N}_0 . With these discrete norms of our choice, we get explicit representations for the Toeplitz operators T and B_0 . For $s = 1$, the value $N_1 = 1$ is suitable, and the result is depicted in Figure 2.

While in Section 3 the discussion is illustrated by considering the simple nonconforming rotated Q1 element, the application of the methods to other popular low-order finite element constructions is documented in Section 4.

3 Algorithmical details: The rotated Q1 element

We come to the application of the above theory to some nonconforming elements for second-order elliptic problems (Section 3 and Subsection 4.1, 4.2) and fourth-order problems (Subsection 4.3, 4.4). All examples are in two space dimensions ($d = 2$).

In the present section, we deal with the rectangular rotated Q1 element, where we have $M = 2$. In Subsection 3.1, the solution of the outlined eigenvalue problem by direct methods is discussed while in Subsection 3.2 the iterative method is described in more detail. For all subsequent examples collected in Section 4, where the number M is larger, we will be relatively brief. The reader may easily include other examples of interest by computing the matrix representations S, T of the intergrid transfer operator and the norms under consideration.

The rotated Q1 element has been introduced in [20] for general quadrilateral partitions, for a detailed investigation of multilevel and multigrid algorithms in the case of partitions into squares, see [6]. Let \mathcal{T}_0 be the square partition of \mathbb{R}^2 of side-length 1, with the vertices at the integer points in \mathbb{Z}^2 . The edges of \mathcal{T}_0 will be denoted by $e_\alpha^1 (= e_0^1 + \alpha), e_\alpha^2 (= e_0^2 + \alpha), \alpha \in \mathbb{Z}^d$, where e_0^1, e_0^2 denote the edges represented by the unit vectors $e^1 = (1, 0), e^2 = (0, 1) \in \mathbb{Z}^2$ placed at the origin. Analogously, the square elements of the partition are denoted by E_α , where E_0 is the unit square with the edges $e_0^1, e_0^2, e_{\alpha+e^2}^1, e_{\alpha+e^1}^2$.

The space V_0 under consideration is given by three requirements. First, the restrictions of $v_0 \in V_0$ to the squares E_α belong to the 4-dimensional space spanned by the polynomials $\{1, x_1, x_2, x_1^2 - x_2^2\}$, the so-called rotated Q1 finite element space which we will denote by Q_1^{rot} . Note that $\Pi_1 \subset Q_1^{\text{rot}} \subset \Pi_2$, where Π_k stands for the space of multivariate polynomials of total degree $\leq k$. Secondly, the edge averages of the restrictions of v_0 to any two adjacent squares coincide along the common edge. We will denote averages of a function f over a set G by

$$av(f, G) = |G|^{-1} \int_G f(x) dx ,$$

the measure will be clear from the context. E.g., the above continuity requirement can be written as

$$av(v_0|_{E_\alpha}, e_\alpha^1) = av(v_0|_{E_{\alpha-e^2}}, e_\alpha^1) , \quad av(v_0|_{E_\alpha}, e_\alpha^2) = av(v_0|_{E_{\alpha-e^1}}, e_\alpha^2) \quad \forall \alpha \in \mathbb{Z}^2 .$$

The third requirement is $V_0 \subset L_2(\mathbb{R}^2)$. Obviously, the \mathbb{Z}^2 -shifts of the two functions ϕ^1, ϕ^2 defined by

$$\phi^m|_{E_\alpha} \in Q_1^{\text{rot}} \quad \forall \alpha \in \mathbb{Z}^2 , \quad av(\phi^m, e_\alpha^k) = \begin{cases} 1 & , \quad m = k, \alpha = 0 \\ 0 & , \quad \text{otherwise} \end{cases} ,$$

form the required Riesz basis \mathcal{N}_0 . Thus, in this subsection we have $M = 2$. The hierarchies $\mathcal{T}_j, V_j, \mathcal{N}_j$ are constructed by dyadic dilation as described in section 2.

We are interested in studying the discrete $H^1(\mathbb{R}^2)$ norm of intergrid transfer operators which we can split into the L_2 -norm ($s = 0$) and the discrete H^1 -seminorm

$$|f|_{0, H^1}^2 = \sum_{\alpha \in \mathbb{Z}^2} \int_{E_\alpha} |\nabla f|^2 dx \quad (s = 1) .$$

The intergrid transfer operator of primary interest is defined by averaging nodal values with respect to the next grid (which is the most common approach in the practice of multilevel/multigrid solvers for nonconforming finite element discretizations). More precisely, to define $v_1 = I_1 v_0$ we prescribe $av(v_1, e)$ with respect to the edges e of \mathcal{T}_1 as follows. If e belongs to the interior of some E_α then

Theorem 8 *Let the sequences $\{V_j\}$, $\{I_j\}$, and the seminorms $|\cdot|_{j,s}$ be introduced in the dilation-shift-invariant fashion as described above. Let S resp. T denote the associated subdivision resp. Toeplitz operators. Consider the matrix transfer operator \mathcal{L}_S given by (12). Then the invariant Krylov space $V(\mathcal{L}_S, B_0)$ generated from $B_0(\theta) = T^*(\theta)T(\theta)$ is finite-dimensional. Let ρ be the spectral radius of the restricted operator \mathcal{L}_{S, B_0} , and r the dimension of the largest Jordan block associated with eigenvalues λ of maximal absolute value ($|\lambda| = \rho$). Then*

$$|\tilde{I}_j^J v_j|_{J,s} \asymp (J-j+1)^{r-1} \rho^{J-j} 2^{2sj} \|v_j\|_{L_2(\mathbb{R}^d)}^2 \quad (35)$$

holds for arbitrary $v_j \in V_j$, and $0 \leq j \leq J$. The uniform estimate (10) for $c_j(S, T)$ holds if and only if either $\rho < 1$ or $\rho = 1$ and $r = 1$.

In the remainder of this section we briefly discuss how to obtain ρ and r in practice. The straightforward approach would be first to solve the complete eigenvalue problem for \mathcal{L}_S restricted to a suitable invariant subspace V_{S, B_0} containing B_0 (i.e., its Jordan decomposition), and to find the eigenmatrix decomposition of $B_0(\theta)$. From this decomposition, according to the argument used in the proof of Lemma 7, we can then detect the numbers of interest. This would circumvent the difficult task of explicitly characterizing the Krylov space $V(\mathcal{L}_S, B_0)$. Unfortunately, the computation of Jordan forms is not a numerically stable procedure. In the examples below, we show how to find a relatively small V_{S, B_0} which will be described by a matrix version of a ‘good index set’

$$\mathcal{I}_{S, B_0} = (I_{S, B_0}^{l,m} : l, m = 1, \dots, M) .$$

I.e., V_{S, B_0} will be defined as the set of all Hermitean matrix functions $X(\theta)$ where the Fourier spectrum of the entries $X^{l,m}(\theta)$ belongs to the corresponding index set $I_{S, B_0}^{l,m} \subset \mathbb{Z}^d$. After choosing a standard basis in V_{S, B_0} , we found the matrix representation K of the matrix transfer operator and solved the eigenvalue problem for both K and K^* . We then checked for the largest eigenvalue of K such that the corresponding eigenmatrix enters the eigenmatrix decomposition of B_0 (here, the eigenvectors of K^* are needed). For multiple eigenvalues, we have also computed the rank of the numerically obtained system of eigenvectors. It turned out that non-trivial Jordan blocks for the multiple eigenvalues of interest did never occur. I.e., we always had $r = 1$ but there is no proof for this observation.

We have used standard MATLAB routines such as *condeig* and *rank* to check these facts numerically. We just note that the new version MATLAB V is well-suited (due to the new array and cell structures) for quickly setting up a program that solves the outlined problems and helps to gain further inside into them. Since the dimension of suitable spaces V_S containing $V(\mathcal{L}_S, B_0)$ was in the range from about 30 to several hundreds, and only a few eigenspaces with large eigenvalues are of interest, all this is still a reasonable task, and serious numerical instabilities have not been observed so far. More details will be given in Subsection 3.1.

Alternatively, according to (34) one might consider iterative procedures of Krylov space type (such as power or simultaneous iteration, or Arnoldi type methods) for the operator \mathcal{L}_S with starting value B_0 . Here, the main difficulty is that due to rounding these iterations very soon leave the subspace $V(\mathcal{L}_S, B_0)$ and yield the spectral radius of \mathcal{L}_S with respect to another invariant subspace containing eigenmatrices with larger absolute value (which are not present in the basis decomposition of B_0). It turns out that in the applications (in full analogy with the similar problems arising in investigations on the regularity of multiwavelets [8, 15, 13, 12, 21]), these spurious eigenmatrices are connected with the invariance of certain spaces of multivariate polynomials under the action of the intergrid transfer operators I_j . Thus, it seems to be possible to deal with the ‘dangerous’ round-off perturbations that are associated with these specific eigenmatrices. For $M = 1$, this technique has been successfully used in [14] for a similar application. There, rather large eigenvalue problems of dimension up to 10^5 occurred for which direct eigenvalue solver would have failed. See Subsection 3.2 for more information on the spectrum of \mathcal{L}_S , and for details on the iterative method.

then to $[-2N_0 - N_1, 2N_0 + N_1]^d$. Thus, according to (27), the index box for $B_1(\theta) = (\mathcal{L}_S B_0)(\theta)$ is safely contained in $[-N_0 - N_1/2, N_0 + N_1/2]^d$. Iterating this argument one sees that the union of all index boxes of all B_j is, at least, contained in $[-N, N]^d \cap \mathbb{Z}^d$, where $N = \max(2N_0, N_1)$. Thus, the above Krylov space is finite-dimensional, and its dimension is bounded by $M^2(2N+1)^2$ (which can be reduced by roughly a factor of 2 if the Hermitean structure of all B_j is taken into account). A more careful estimation of the Fourier spectra can further reduce this dimension, and is related to the notion of ‘good index sets’, compare [9]. We will return to this point in connection with the examples discussed in Section 3.

Since $\|\lambda_{\max}(B(\theta))\|_{L_\infty}$ is obviously a norm on the space of all Hermitean matrix functions with polynomial entries, and in particular on the Krylov space $V(\mathcal{L}_S, B_0)$, we can estimate the quantities $c_j(S, T)$ in (29) by the spectral radius $\rho \equiv \rho(\mathcal{L}_S, B_0)$ of the operator \mathcal{L}_S restricted to its finite-dimensional invariant subspace $V(\mathcal{L}_S, B_0)$ which we will denote for short by \mathcal{L}_{S, B_0} . Indeed, we can immediately prove the following

Lemma 7 *For any norm $\|\cdot\|$ on $V(\mathcal{L}_S, B_0)$ we have*

$$c_0(j+1)^{r-1}\rho^j \leq \|B_j\| \leq c_1(j+1)^{r-1}\rho^j, \quad j \geq 0, \quad (33)$$

with two positive constants c_0, c_1 , where r is the size of the largest Jordan block corresponding to an eigenvalue λ of \mathcal{L}_{S, B_0} satisfying $|\lambda| = \rho$.

Note that despite the nice structure of the matrix functions, the operator \mathcal{L}_S is neither symmetric nor normal which means that Jordan blocks could possibly appear. For the proof of (33), let us denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues associated with different Jordan blocks of \mathcal{L}_{S, B_0} , and by $\{X_{k,1}(\theta), \dots, X_{k,r_k}\}$ the eigenmatrices and associated eigenmatrices for each λ_k , i.e., we have

$$\begin{aligned} \mathcal{L}_S X_{k,1} &= \lambda_k X_{k,1} + X_{k,2}, \\ &\dots \\ \mathcal{L}_S X_{k,r_k-1} &= \lambda_k X_{k,r_k-1} + X_{k,r_k}, \\ \mathcal{L}_S X_{k,r_k} &= \lambda_k X_{k,r_k}. \end{aligned}$$

for all $k = 1, \dots, n$. The union of all these eigenmatrices is a basis in the Krylov space $V(\mathcal{L}_S, B_0)$. If we consider the decomposition of an arbitrary $B(\theta) \in V(\mathcal{L}_S, B_0)$ with respect to this basis

$$B = \sum_{k=1}^n \sum_{l=1}^{r_k} a_{k,l} X_{k,l},$$

then

$$\|B\|_1 \equiv \sum_{k=1}^n \sum_{l=1}^{r_k} |a_{k,l}| \asymp \|B\|$$

is an equivalent norm. Since $V(\mathcal{L}_S, B_0)$ is generated from B_0 , we necessarily have $a_{k,1} \neq 0$, $k = 1, \dots, n$, for the coefficients of B_0 . Thus,

$$B_j = \mathcal{L}_S^j B_0 = \sum_{k=1}^n \sum_{l=1}^{r_k} \left(\sum_{i=0}^{\min(r_k-1, j)} \binom{j}{i} \lambda_k^{j-i} a_{k,l-i} \right) X_{k,l}$$

(for convenience, we have set $a_{k,0} = a_{k,-1} = \dots = 0$). This gives asymptotically

$$\|B_j\| \asymp \|B_j\|_1 \asymp \sum_{k=1}^n j^{r_k-1} |\lambda_k|^j. \quad (34)$$

To see (34) observe that the dominating coefficient term in the above basis representation of B_j is the one with $l = r_k, i = r_k - 1, k = 1, \dots, n$. Since (34) implies (33), Lemma 7 is established.

For the convenience of the reader, we subsume the results obtained so far (compare Lemma 1, 3, and 7) in the following theorem which can be considered as the main theoretical result.

Though this is actually the inequality that is needed in the theory of [19], one might ask for the true analog of (8) where the expression $2^{2sj} \|v_j\|_{L_2(\mathbb{R}^d)}^2$ in the right-hand side of (30) is replaced by $|v_j|_{j,s}^2$. In other words, we are asking about the optimal constant $c = \tilde{c}_j(S, T)$ in the inequality

$$\|TS^j c\|_{(\ell^2(\mathbf{Z}^d))^K}^2 \leq c \|Tc\|_{(\ell^2(\mathbf{Z}^d))^K}^2 \quad (31)$$

for arbitrary $c \in (\ell^2(\mathbf{Z}^d))^M$. According to the above considerations, this is equivalent to searching for the best constant in

$$(B_j(\theta)c(\theta), c(\theta))_{L_2(\mathbf{T}^d)} \leq c(B_0(\theta)c(\theta), c(\theta))_{L_2(\mathbf{T}^d)}$$

for all $c(\theta) \in (L_2(\mathbf{T}^d))^M$, and leads to an generalized eigenvalue problem $B_j(\theta)x = \lambda B_0(\theta)x$ to be solved for the maximal eigenvalue for each $\theta \in \mathbf{T}^d$. With this replacement, the analog of (29) holds for $\tilde{c}_j(S, T)$.

Remark 5 An alternative approach to the above estimates would be to find out whether there exists another subdivision operator $U : (\ell^2(\mathbf{Z}^d))^K \rightarrow (\ell^2(\mathbf{Z}^d))^K$ such that $TS = UT$ holds identically on $(\ell^2(\mathbf{Z}^d))^M$. Then we have the identity $TS^j c = U^j Tc$, and the relevant matrix functions to be studied would have the expressions $\mathcal{L}_T \mathcal{L}_U^{j-1} \text{Id}$ (for the statement of Lemma 3) resp. $\mathcal{L}_U^{j-1} \text{Id}$ for proving bounds such as in (31) (in this latter case, one has to restrict the attention to the range of T which makes the task not easier). In both cases, Id denotes the identity matrix operator, with diagonal entries $\equiv 1$, and non-diagonal entries $\equiv 0$. We did not pursue these alternatives further.

Remark 6 Here is a more technical remark which will be useful below. We can write the norm expression $\|TS^j c\|_{(\ell^2(\mathbf{Z}^d))^K}^2$ as a telescopic sum

$$\begin{aligned} \|TS^j c\|_{(\ell^2(\mathbf{Z}^d))^K}^2 &= \|Tc\|_{(\ell^2(\mathbf{Z}^d))^K}^2 + \sum_{k=1}^j (\|TS^k c\|_{(\ell^2(\mathbf{Z}^d))^K}^2 - \|TS^{k-1} c\|_{(\ell^2(\mathbf{Z}^d))^K}^2) \\ &= \|Tc\|_{(\ell^2(\mathbf{Z}^d))^K}^2 + \sum_{k=1}^j ((S^*)^{k-1} (S^* T^* TS - T^* T) S^{k-1} c, c)_{(\ell^2(\mathbf{Z}^d))^M} \\ &= \|Tc\|_{(\ell^2(\mathbf{Z}^d))^K}^2 + \sum_{k=1}^j ((\mathcal{L}_S^{k-1} \tilde{B}_0)(\theta) c(\theta), c(\theta))_{L_2(\mathbf{T}^d)}, \end{aligned}$$

where $\tilde{B}_0 = \mathcal{L}_S B_0 - B_0$. The essence of this modification will be demonstrated below, just to give it a name: the Krylov space for the operator \mathcal{L}_S generated by \tilde{B}_0 is obviously contained in that generated from B_0 , and smaller by a possible subspace associated with ‘eigenmatrices’ for the operator \mathcal{L}_S corresponding to the eigenvalue 1.

Still, our formulas are not very useful: computing the L_∞ -bounds of the matrix functions $B_j(\theta)$ seems not to be a simple task. The important observation is that (under the above assumptions on S, T) the Krylov space

$$V(\mathcal{L}_S, B_0) = \text{span}\{\mathcal{L}_S^k B_0 : k \geq 0\} \quad (32)$$

is a finite-dimensional subspace of the space of all Hermitean matrix functions with $L_2(\mathbf{T}^d)$ -entries. This fact is well-known in the literature, and follows directly from the finite support of the Fourier coefficient sequences of all entries in $S(\theta)$, $B_0(\theta)$, in conjunction with the definition (12) of \mathcal{L}_S and (27). Indeed, let $[-N_0, N_0]^d \cap \mathbf{Z}^d$ be the smallest index box such that it contains the indices of all non-vanishing Fourier coefficients of all entries of $S(\theta)$. Automatically, the same box fits $S^*(\theta)$, too. Let $[-N_1, N_1]^d \cap \mathbf{Z}^d$ be the index box suitable for $B_0(\theta) = T^*(\theta)T(\theta)$. The indices of non-vanishing Fourier coefficients of the entries of the matrix product $S^*(\theta)B_0(\theta)S^*(\theta)$ belong

Throughout the paper, by $a^*(\theta)$ resp. $A^*(\theta)$ we denote the Hermitean transpose of a vector function resp. of a (rectangular) matrix function. Since for any $a(\theta) \in L^2(\mathbb{T}^d)$ we have

$$2^{-d} \sum_{e \in \{0,1\}^d} a\left(\frac{\theta}{2} + \pi e\right) = \sum_{\alpha \in \mathbb{Z}^d} a_{2\alpha} e^{-i\alpha\theta}, \quad (27)$$

the matrix $\tilde{A}(\theta)$ in the last expression is again 2π -periodic, and can easily be calculated from the Fourier representation of the elements of $A(\theta)$.

Let us apply this observation to the following expression:

$$\begin{aligned} \|TS^j c\|_{(\ell^2(\mathbb{Z}^d))^\kappa}^2 &= \|T(\theta)S(\theta) \dots S(2^{j-1}\theta)c(2^j\theta)\|_{L_2(\mathbb{T}^d)}^2 \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} c^*(2^j\theta) \prod_{k=j-1}^1 S^*(2^k\theta)T^*(\theta)T(\theta) \prod_{k=1}^{j-1} S(2^k\theta)c(2^j\theta) d\theta \\ &= \underbrace{(B_0(\theta) \prod_{k=1}^{j-1} S(2^k\theta)c(2^j\theta), \prod_{k=1}^{j-1} S(2^k\theta)c(2^j\theta))}_{\equiv \Delta_j} \end{aligned}$$

where $B_0(\theta) = T^*(\theta)T(\theta)$. Denoting

$$a(\theta) = b(\theta) = \prod_{k=2}^{j-1} S(2^k\theta)c(2^j\theta), \quad A(\theta) = S^*(\theta)B_0(\theta)S(\theta),$$

we can apply the above identity and arrive at

$$\Delta_j = (B_1(\theta) \prod_{k=1}^{j-2} S(\theta)c(2^{j-2}\theta), \prod_{k=1}^{j-2} S(2^k\theta)c(2^{j-1}\theta))_{L_2(\mathbb{T}^d)} \equiv \Delta_{j-1}.$$

For the matrix $B_1(\theta)$ we observe that

$$B_1(\theta) = \tilde{A}(\theta) = (\mathcal{L}_S B_0)(\theta),$$

where the transfer operator was introduced in (12). Now repeating this argument with Δ_{j-1} and so on, we arrive at

Lemma 3 *For the operators S and T as introduced above, we have*

$$\|TS^j c\|_{(\ell^2(\mathbb{Z}^d))^\kappa}^2 = (B_j(\theta)c(\theta), c(\theta))_{L_2(\mathbb{T}^d)} \quad \forall c \in (\ell^2(\mathbb{Z}^d))^M, \quad j \geq 0, \quad (28)$$

where $B_j(\theta) = (\mathcal{L}_S^j B_0)(\theta)$ and $B_0(\theta) = T^*(\theta)T(\theta)$ are Hermitean matrix functions on \mathbb{T}^d . The matrix transfer operator \mathcal{L}_S is defined in (12). Consequently,

$$c_j(S, T) = \|TS^j\|_{(\ell^2(\mathbb{Z}^d))^M \rightarrow (\ell^2(\mathbb{Z}^d))^\kappa}^2 = \|\lambda_{\max}(B_j(\theta))\|_{L_\infty(\mathbb{T}^d)}, \quad (29)$$

compare with (10), (11).

Remark 4 *When combined with the Riesz basis property of \mathcal{N}_j expressed by (18), and with (21), Lemma 1 and 3 yield the estimate*

$$|\tilde{I}_j^J v_j|_{J,s}^2 \leq cc_{J-j}(S, T)2^{2sj} \|v_j\|_{L_2(\mathbb{R}^d)}^2. \quad (30)$$

the elements it is necessary and sufficient to ‘test’ inequalities such as (8) resp. (9) on the nodal basis functions $\phi_{j,i}$. Since the operators I_j of practical interest act locally (in the sense that the support of $\tilde{I}_j^J \phi_{j,i}$ is contained in a finite neighborhood of the support of $\phi_{j,i}$, i.e., in a finite union of cells from \mathcal{T}_j), negative results for the \mathbb{R}^d -case carry over immediately. Thus, though of limited value for answering the general problem, the methods presented in this paper help to discard ‘bad’ intergrid transfer operators and to look for better ones. In a purely mathematical context, they are connected to other areas of current research, e.g., to the study of refinable vector functions, and might have some other applications.

We now turn to Fourier techniques to further transform the problem. For a given sequence $a = (a_\alpha : \alpha \in \mathbb{Z}^d) \in \ell^2(\mathbb{Z}^d)$, we denote by

$$a(\theta) = \sum_{\alpha \in \mathbb{Z}^d} a_\alpha e^{-i\alpha\theta}$$

the associated periodic function in $L^2(\mathbb{T}^d)$. Analogously, for $c \in (\ell^2(\mathbb{Z}^d))^M$, we introduce the vector function $c(\theta) = (c^1(\theta), \dots, c^M(\theta))^T$, and with the operators T and S we associate matrix functions

$$T(\theta) = (t^{k,m}(\theta) : k = 1, \dots, K, m = 1, \dots, M)$$

of size $K \times M$ and

$$S(\theta) = (s^{l,m}(\theta) : l, m = 1, \dots, M)$$

of size $M \times M$, respectively. From now on, the usual scalar products in the spaces $\ell^2(\mathbb{Z}^d)$ resp. $L^2(\mathbb{T}^d)$ of complex-valued sequences resp. functions will be considered. The scalar product in $L_2(\mathbb{T}^d)$ includes the scaling factor $(2\pi)^{-d}$ such that the above mapping gives an exact isometry between the two Hilbert spaces:

$$(a, b)_{\ell^2(\mathbb{Z}^d)} = (a(\theta), b(\theta))_{L_2(\mathbb{T}^d)} .$$

It is well-known (the interested reader is encouraged to carry out the elementary calculations) that

$$(Tc)(\theta) = T(\theta)c(\theta) , \quad (Sc)(\theta) = S(\theta)c(2\theta) . \quad (26)$$

Thus, the natural counterparts of the operators S, T acting on $(\ell^2(\mathbb{Z}^d))^M$ are simple matrix multiplication operators defined on $(L_2(\mathbb{T}^d))^M$. Note that due to the finite support property of their defining sequences, all entries of the operator functions are Laurent polynomials (multivariate trigonometric polynomials in complex form). Therefore, the operators T, S can formally be extended beyond $(\ell^2(\mathbb{Z}^d))^M$ resp. $(L_2(\mathbb{T}^d))^M$ to generalized vector functions. This fact will be utilized later.

Let us first check the following identity involving a matrix function $A(\theta)$ and two vector functions $a(\theta), b(\theta)$ of size M :

$$\begin{aligned} (A(\theta)a(2\theta), b(2\theta))_{L_2(\mathbb{T}^d)} &= (2\pi)^{-d} \int_{\mathbb{T}^d} b^*(2\theta)A(\theta)a(2\theta) d\theta \\ &= (2\pi)^{-d} \sum_{\epsilon \in \{0,1\}^d} \int_{\frac{1}{2}\mathbb{T}^d} b^*(2\theta)A(\theta + \pi\epsilon)a(2\theta) d\theta \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} b^*(\theta) \underbrace{\left(2^{-d} \sum_{\epsilon \in \{0,1\}^d} A\left(\frac{\theta}{2} + \pi\epsilon\right) \right)}_{\tilde{A}(\theta)} a(\theta) d\theta \\ &= (\tilde{A}(\theta)a(\theta), b(\theta))_{L_2(\mathbb{T}^d)} . \end{aligned}$$

norms can be covered by taking linear combinations of $|\cdot|_{j,s}$ -seminorms with different s . Note that the factor in (21) is such that for $s = 0$ the choice $|\cdot|_{0,j} = \|\cdot\|_{L_2(\mathbb{R}^d)}$ is included.

What remains is to introduce the intergrid transfer operators I_j . Again, we will describe $I_1 : V_0 \rightarrow V_1$ in detail, the general case will be defined by dilation:

$$I_j v_{j-1} = I_1 v_0(2^{j-1}\cdot), \quad v_0 = v_{j-1}(2^{-(j-1)}\cdot) \quad (22)$$

for all $v_{j-1} \in V_{j-1}$ and $j > 1$. For I_1 , we naturally assume invariance with respect to \mathbb{Z}^d -shifts, locality, and linearity. I.e., let the action of the operator I_1 on the basis functions $\phi^m = \phi_{0,0}^m$ be given by

$$I_1 \phi^m = \sum_{l=1}^M \sum_{\gamma \in \mathbb{Z}^d} \tilde{s}_\gamma^{l,m} \phi_{1,\gamma}^l, \quad m = 1, \dots, M, \quad (23)$$

where all sequences $\tilde{s}^{l,m}$ are finitely supported (this corresponds to the locality requirement). Then, using linearity and shift-invariance, we obtain

$$\begin{aligned} I_1 v_0 &= \sum_{m=1}^M \sum_{\alpha} c_\alpha^m I_1 \phi_{0,\alpha}^m \\ &= \sum_{m=1}^M \sum_{\alpha} \sum_{l=1}^M \sum_{\gamma} c_\alpha^m \tilde{s}_\gamma^{l,m} \phi_{1,\gamma+2\alpha}^l \\ &= \sum_{l=1}^M \sum_{\beta} \left(\sum_{m=1}^M \sum_{\alpha} \tilde{s}_{\beta-2\alpha}^{l,m} c_\alpha^m \right) \phi_{1,\beta}^l \end{aligned}$$

(recall for the understanding that a shift by $\alpha \in \mathbb{Z}^d$ in the argument of $\phi_{1,\gamma}^l$ corresponds to an index shift by 2α). This gives the formula for the coefficients of $I_1 v_0$ in the basis \mathcal{N}_1 as a specific linear transformation of the coefficients of v_0 in the basis \mathcal{N}_0 . For convenience, we include a factor $2^{s-d/2}$ in the definition of the resulting matrix operator S in $(\ell^2(\mathbb{Z}^d))^M$, that is, we set $s_\alpha^{l,m} = 2^{s-d/2} \tilde{s}_\alpha^{l,m}$, and define

$$(Sc)_\beta^l = \sum_{m=1}^M \sum_{\alpha \in \mathbb{Z}^d} s_{\beta-2\alpha}^{l,m} c_\alpha^m, \quad \beta \in \mathbb{Z}^d, \quad l = 1, \dots, M. \quad (24)$$

In the literature, such operators are studied under the name subdivision operators, we refer, e.g., to [4].

The following lemma is now straightforward (just put the definitions together, the additional factor included into the definition of S takes care of the factor in (21)).

Lemma 1 *Under the above assumptions, the following identity holds for the iterated integrid transfer operator $\tilde{I}_j^J = I_J \cdots I_{j+1}$ and all $v_j \in V_j$, $0 \leq j \leq J$:*

$$\|\tilde{I}_j^J v_j\|_{J,s} = \|TS^{J-j}c\|_{(\ell^2(\mathbb{Z}^d))^\kappa}, \quad (25)$$

where c is the coefficient vector of v_j defined by (17).

Remark 2 *Examples of operators T and S will be given in the next section, for $d = 2$, and with emphasis of nonconforming finite elements. Lemma 1 is the key to using properties of subdivision and Toeplitz operators for estimating the quantities mentioned in the introduction. As a rule, estimates for the shift-invariant \mathbb{R}^d -case can be carried over to nice domains by special extension procedures (see, e.g., [6] for the nonconforming rotated Q1 element on uniform square partitions of a square). In rare cases, the whole hierarchy of underlying triangulations can be mapped to part of a shift-invariant partition (and imbedded into \mathbb{R}^d) such that results can be carried over to a more general setting (see [16] for the nonconforming P1-element). As was outlined in [19], for most of*

be the set of their \mathbb{Z}^d -shifts. The closed span of \mathcal{N}_0 in $L_2(\mathbb{R}^d)$ is denoted by V_0 . A basic assumption is that \mathcal{N}_0 is a Riesz basis for V_0 , i.e., each $v_0 \in V_0$ possesses a unique representation

$$v_0 = \sum_{m=1}^M \sum_{\alpha \in \mathbb{Z}^d} c_\alpha^m \phi_{0,\alpha}^m \quad (13)$$

with a vector of coefficient sequences

$$c = (c^m : m = 1, \dots, M) \in (\ell^2(\mathbb{Z}^d))^M, \quad c^m = (c_\alpha^m : \alpha \in \mathbb{Z}^d) \in \ell^2(\mathbb{Z}^d),$$

such that

$$\|c\|_{(\ell^2(\mathbb{Z}^d))^M} \asymp \|v_0\|_{L_2(\mathbb{R}^d)} \quad \forall v_0 \in V_0. \quad (14)$$

We will use $A \asymp B$ if a two-sided inequality holds between A and B with positive constants that are independent of all quantities in the expressions for A and B . Criteria for (14) to hold are well-known, see, e.g., [11]. In all examples below where the ϕ^m represent finite element nodal basis functions with respect to a uniform, shift-invariant partition \mathcal{T}_0 of \mathbb{R}^d , the Riesz basis property of \mathcal{N}_0 can be checked directly, using the available explicit formulae for c_α^m in terms of simple interpolation functionals applied to v_0 (point values or averages of v_0 and its derivatives).

Given V_0 as described, we generate V_j , $j \geq 1$, and \mathcal{N}_j by dyadic dilation:

$$V_j = \{v_j = v_0(2^j \cdot) : v_0 \in V_0\} \quad (15)$$

and

$$\mathcal{N}_j = \{\phi_{j,\alpha}^m \equiv \phi^m(2^j \cdot - \alpha) : m = 1, \dots, M, \alpha \in \mathbb{Z}^d\}. \quad (16)$$

Obviously, \mathcal{N}_j is a Riesz basis in V_j , where (13), (14) take now the form

$$v_j = \sum_{m=1}^M \sum_{\alpha \in \mathbb{Z}^d} c_\alpha^m \phi_{j,\alpha}^m \quad (17)$$

and

$$2^{-j d/2} \|c\|_{(\ell^2(\mathbb{Z}^d))^M} \asymp \|v_j\|_{L_2(\mathbb{R}^d)} \quad \forall v_j \in V_j. \quad (18)$$

Since we do not assume that the functions ϕ^m are refinable (i.e., belong to V_1), the nestedness (3) is not assumed, either.

We will introduce an abstract semi-norm $|\cdot|_{j,s}$ of order $s = 0, 1, \dots$, satisfying the following properties: On V_0 , it is given via the formula

$$|v_0|_{0,s}^2 = \|Tc\|_{(\ell^2(\mathbb{Z}^d))^K}^2, \quad (19)$$

where c is defined by (13), and the Toeplitz operator $T : (\ell^2(\mathbb{Z}^d))^M \rightarrow (\ell^2(\mathbb{Z}^d))^K$ is given by finitely supported sequences $t^{k,m} = (t_\alpha^{k,m} : \alpha \in \mathbb{Z}^d)$ such that

$$(Tc)_\alpha^k = \sum_{m=1}^M \sum_{\beta \in \mathbb{Z}^d} t_{\alpha-\beta}^{k,m} c_\beta^m, \quad \alpha \in \mathbb{Z}^d, \quad k = 1, \dots, K. \quad (20)$$

Finally, on V_j we define $|\cdot|_{j,s}$ by dilation: By definition of V_j , for each $v_j \in V_j$ there is a unique $v_0 \in V_0$ such that $v_j = v_0(2^j \cdot)$, and we set

$$|v_j|_{j,s}^2 = 2^{(2s-d)j} |v_0|_{0,s}^2. \quad (21)$$

From (19-21) we can easily obtain a discrete representation of $|\cdot|_{j,s}$ as in (19) for all $j \geq 0$. These definitions are modeled after homogeneous energy norms of integer order s on \mathbb{R}^d , inhomogeneous

of the ‘multilevel’ assertions (8) resp. (9) than in the case of the ‘two-level’ inequality (6). See [16, 6] for partial results for two particular low-order nonconforming elements.

In this paper we propose to study an idealized version of (8) resp. (9). We will assume that the spaces V_j form a dyadic hierarchy of shift-invariant subspaces of $L_2(\mathbb{R}^d)$ generated by a finite number of locally supported basis functions $\{\phi^1, \dots, \phi^M\}$ in an L_2 -stable way (in contrast to true multiresolution analyses as typical for multiwavelet constructions, we do not assume nestedness (3)). In the finite element context, this corresponds to the domain $\Omega = \mathbb{R}^d$, equipped with a uniform partition structure. Thus, boundary effects and more general partitions are neglected. We also assume that the intergrid transfer operators I_j act in a shift- and dilation-invariant fashion. Furthermore, the true energy norm $\|u_j\|_{E,j} = \sqrt{a_j(u_j, u_j)}$ is assumed to be equivalent to some homogeneous, shift-invariant discrete Sobolev semi-norm $|u_j|_{j,s}$ of integer order $s = 0, 1, \dots$. How ‘realistic’ these simplifications are, is discussed in Remark 2 of Section 2.

When switching to the coefficient representations of functions from V_j with respect to the natural bases (generated by the dyadic shifts and dilations of ϕ^m , $m = 1, \dots, M$), all the above operations can be interpreted as linear operators acting on vectors of $\ell^2(\mathbb{Z}^d)$ sequences. Due to the assumed invariances with respect to dyadic dilations and shifts, these operators are similar to Toeplitz resp. subdivision operators occurring in wavelet analysis. E.g., the inequality (9) can essentially be replaced by

$$c_j(S, T) \equiv \|(S^*)^j T^* T S^j\|_{(\ell^2(\mathbb{Z}^d))^M} \leq c < \infty, \quad (10)$$

where S is a subdivision operator describing the action of I_j , and T is a Toeplitz operator associated with the discrete Sobolev semi-norm. As will be detailed in Section 2, such operators can be represented via standard Fourier techniques by periodic $M \times M$ matrix functions on \mathbb{T}^d . Provided that both S and T act locally (as is the case in all relevant applications), these matrices consist of Laurent polynomials. We will prove that

$$c_j(S, T) = \|\lambda_{\max}((\mathcal{L}_S^j B_0)(\theta))\|_{L_\infty(\mathbb{T}^d)}, \quad (11)$$

where $B_0(\theta) = T^*(\theta)T(\theta)$ and

$$(\mathcal{L}_S B)(\theta) = 2^{-d} \sum_{e \in \{0,1\}^d} S^*(\frac{\theta}{2} + \pi e) B(\frac{\theta}{2} + \pi e) S(\frac{\theta}{2} + \pi e), \quad \theta \in \mathbb{T}^d, \quad (12)$$

is an operator acting on Hermitean periodic matrix functions. Exactly the same operator appears in recent studies on the regularity of multiwavelets [8, 15, 12, 13, 21].

After introducing in detail to the terminology and proving the above mentioned results, we give in Section 2 the exact asymptotics for norms of iterated intergrid transfer operators. In particular, necessary and sufficient conditions for (9) resp. (10) to hold are stated. Our criteria are based on solving a particular finite-dimensional eigenvalue problem associated with \mathcal{L}_S . For the latter, both direct and iterative solution strategies have been tried. Algorithmical details are explained and justified in Section 3 where the discussion is illustrated by the example of the so-called rotated Q1 element. Though this element is simple, it is a good test case since it has a number of features which are typical for the present application.

Section 4 is devoted to the study of intergrid transfer operators for a number of low-order nonconforming finite elements used in the discretization of standard second- and fourth-order elliptic boundary value problems. We apply the theory developed in Section 2 and 3, and confirm the numerical results previously obtained in [19].

2 Theory

We first introduce the hierarchy of subspaces V_j of $L_2(\mathbb{R}^d)$. Let $\phi^1, \dots, \phi^M \in L_2(\mathbb{R}^d)$ be compactly supported functions, and let

$$\mathcal{N}_0 = \{\phi_{0,\alpha}^m \equiv \phi^m(\cdot - \alpha) : \alpha \in \mathbb{Z}^d, m = 1, \dots, M\}$$

this naturally results in a subspace splitting

$$V_J = \sum_{j=0}^J V_j = \sum_{j=0}^J \sum_i V_{j,i}$$

as required by the abstract theory of additive Schwarz preconditioners, see [22, 23, 17]. The preconditioner C_J associated with such a subspace splitting can be defined by the recursion

$$C_0 = S_0, \quad C_j = I_j C_{j-1} I_j^T + S_j, \quad j = 1, \dots, J, \quad (4)$$

where the S_j represent simple spd approximations to A_j^{-1} (usually diagonal matrices such as $(\text{diag}(A_j))^{-1}$ will do), and the rectangular matrices I_j describe the natural embedding $V_{j-1} \subset V_j$. The recursion (4) exhibits the typical structure of a multigrid V-cycle (for the connection of multilevel preconditioning and multigrid theory, consult [1, 10] or the previously cited sources).

If the nestedness condition (3) is violated which is often the case for complicated conforming and all nonconforming finite element constructions, one can still proceed in the same way as before but now special attention has to be paid to the design of the intergrid transfer operator

$$I_j : V_{j-1} \rightarrow V_j, \quad j \geq 1 \quad (5)$$

(we will use the same notation for operators acting on V_j spaces and their matrix representations in the standard bases \mathcal{N}_j). Typically, assumptions on the size of the constants c_j^* in inequalities for the energy norm of I_j such as

$$a_j(I_j u_{j-1}, I_j u_{j-1}) \leq c_j^* a_{j-1}(u_{j-1}, u_{j-1}) \quad \forall u_{j-1} \in V_{j-1}, \quad (6)$$

are of importance (compare [2], [3], [1, Section 4], and the references cited therein). On the basis of (6) a general theory of optimal W-cycle multigrid convergence resp. variable V-cycle multigrid preconditioners has been developed. However, in almost all situations of practical interest, the observed size of the constants c_j^* is too large to yield realistic results for V-cycle multigrid convergence resp. multilevel preconditioning as discussed in the present paper.

Recently, it was observed [16, 19, 6, 5] that (6) would be better replaced by analogous inequalities for the iterated intergrid operators

$$\tilde{I}_j^J = I_J \cdots I_{j+1} : V_j \rightarrow V_J, \quad 0 \leq j \leq J, \quad (7)$$

such as

$$a_J(\tilde{I}_j^J u_j, \tilde{I}_j^J u_j) \leq \tilde{c}_{j,J} a_j(u_j, u_j) \quad \forall u_j \in V_j. \quad (8)$$

On the one hand, the additive Schwarz theory indicates that these are the operators which need to be controlled since

$$V_J = \sum_{j=0}^J \tilde{I}_j^J V_j = \sum_{j=0}^J \sum_i \tilde{I}_j^J V_{j,i}$$

are the subspace splittings associated with the preconditioner (4). On the other hand, numerical evidence [19, 5] shows that for many standard examples of intergrid transfer operators I_j for nonconforming elements, the constants in (8) remain uniformly bounded:

$$\tilde{c}_{j,J} \leq \tilde{c} < \infty \quad \forall j, J : 0 \leq j \leq J. \quad (9)$$

This indicates that the trivial estimate

$$\tilde{c}_{j,J} \leq c_{j+1}^* \cdots c_J^*,$$

which comes from a straightforward repeated application of (6) and suggests an exponential growth of the bound, is too pessimistic. Unfortunately, to give rigorous proofs is much harder in the case

On norm bounds for iterated intergrid transfer operators

P. Oswald
SCAI, GMD
Schloß Birlinghoven
D-53754 Sankt Augustin, Germany
E-mail:peter.oswald@gmd.de

June 2, 1997

Abstract

In the theory of multilevel methods for nonconforming finite element methods, exact bounds for iterated intergrid transfer operators in the discrete energy norm are desirable. In the case of uniform refinement, the problem can be modeled in a multiresolution analysis of non-nested subspaces of $L_2(\mathbb{R}^d)$ which is obtained by dyadic dilation from a finitely generated shift-invariant finite element space with respect to the coarsest partition. The desired operator norms can then be expressed by spectral properties of certain transfer operators acting on matrix functions, similarly to recent approaches to estimating the regularity of multiwavelets. It is shown that the desired norm estimates can be reduced to solving specific finite-dimensional eigenvalue problems. For the latter, both direct and iterative methods have been developed. The theory is tested on some of the popular low order nonconforming finite elements.

1 Introduction

We will deal with a hierarchy of variational problems

$$\text{Find } u_j \in V_j : a_j(u_j, v_j) = \Phi_j(v_j) \quad \forall v_j \in V_j, \quad (1)$$

where $a_j(\cdot, \cdot)$ is a symmetric positive definite (spd) bilinear form, and Φ_j a linear functional on V_j , $j \geq 0$. Throughout the paper, the problems (1) are to be considered as the ‘restrictions’ to V_j of the variational formulation of a certain symmetric elliptic boundary value problem of a partial differential equation in a domain $\Omega \subset \mathbb{R}^d$. The spaces V_j denote finite element spaces of a certain type associated with a sequence of uniform partitions \mathcal{T}_j of Ω generated by a standard dyadic refinement process. Under these circumstances, the linear systems

$$A_j x_j = f_j \quad (2)$$

obtained from discretizing (1) with respect to the standard nodal bases $\mathcal{N}_j = \{\phi_{j,i}\}$ of the finite element spaces V_j , $j \geq 0$, are spd and sparse. Since, as a rule, their condition numbers grow exponentially with j , the efficient iterative solution of (2) requires additional efforts, at least, for larger j .

We consider symmetric multilevel preconditioners for preconditioning (2) for some $j = J$ (by J we will denote the index of the temporarily fixed ‘computational subspace’). These are based on an additive splitting of V_J obtained from the spaces V_j with $j \leq J$ resp. from all one-dimensional spaces $V_{j,i}$ spanned by the individual basis functions $\phi_{j,i}$ with $j \leq J$. In a nested situation, where

$$V_0 \subset V_1 \subset \dots \subset V_j \subset \dots, \quad (3)$$