Talk I: Multilevel Frames and Space Splittings with Applications to Iterative Methods

Talk II: Multilevel Discretization Schemes for the Single Layer Potential Equation

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Abstract

This is the support material for two talks given at the 1998 Dutch Conference of Numerical Mathematicians. In Talk I (sections 1-3 below), we review the notions of frames and stable space splittings in a Hilbert space setting. While the frame concept was developed as part of (non-harmonic) Fourier analysis, mainly in connection with signal processing applications, the latter theory of stable subspace splittings has led to a better understanding of iterative solvers (multigrid/multilevel resp. domain decomposition methods) for large-scale discretizations of symmetric elliptic variational problems in Sobolev spaces. In Talk II (see section 4), several aspects (apriori and aposteriori compression, preconditioning, resolution of singularities) of solving the single layer potential equation by multiscale methods are discussed. Although our analysis is restricted to the unit square $[0,1]^2$, some observations generalize, and are worth further investigation.

1 Frames

The notion of a *frame* in a Hilbert space V was introduced in [12]. A first survey with emphasis on frames was [16], see also [5, Chapter 3], [9, Chapter 3]. A more recent and comprehensive source is the collection [29] which we recommend for further reading.

Definition 1 *Let $F \equiv \{f_k\}$ be an at most countable system of elements in V.*
a) *F is a* **frame** *in V if there are two constants $0 < A \le B < \infty$ such that*

$$A\|f\|^2 \le \sum_k |(f, f_k)|^2 \le B\|f\|^2 \qquad \forall\, f \in V\ . \tag{1}$$

The optimal constants A, B in (1) are the lower and upper frame bounds, respectively, their ratio B/A defines the condition of F and will be denoted by $\kappa(F)$.
b) *F is a* **Riesz basis** *in V if F is dense in V and there are constants $0 < \tilde{A} \le \tilde{B} < \infty$ such that*

$$\tilde{A}\|f\|^2 \le \sum_k c_k^2 \le \tilde{B}\|f\|^2 \qquad \forall\, f = \sum_k c_k f_k\ . \tag{2}$$

Assuming **b)**, it can indeed be proved that any $f \in V$ possesses a *unique* V–converging series representation

$$f = \sum_k c_k f_k \ , \quad c = (c_k) \in \ell^2 \ . \tag{3}$$

Any complete orthonormal systems $\{e_j\} \subset V$ (CONS) is obviously both a frame and a Riesz basis. Any Riesz basis is a frame (with $A = 1/\tilde{B}$, $B = 1/\tilde{A}$). This follows from comparing (2) with the following result:

Theorem 2 [12] *A system F satifies (1) (i.e., is a frame) if and only if*

$$B^{-1}\|f\|^2 \le |||f|||^2 \equiv \inf_{c\,:\,f=\sum_k c_k f_k} \|c\|^2_{\ell^2} \le A^{-1}\|f\|^2 \quad \forall\, f \in V \ . \tag{4}$$

For (4) to hold, it is implicitly required that any $f \in V$ possesses at least one V–converging series representation (3). Unless a frame F is a Riesz basis, such a series is *nonunique*. Nevertheless, frames can be used to represent elements from V in essentially the same way as CONS or Riesz bases. To produce a representation formula, we need some standard definitions. Let F be a frame in V. Then the *synthesis operator* R given by

$$R \ : \ c = (c_k) \in \ell^2 \quad \longmapsto \quad Rc = \sum_k c_k f_k \in V$$

is well-defined and bounded. Its adjoint $R^* \ : \ V \to \ell^2$ takes the form

$$f \in V \quad \longmapsto \quad R^* f = ((f, f_k)) \in \ell^2$$

and is called *analysis operator*. The boundedness of R and R^* follows exclusively from the upper estimate in the definition (1). The two-sided inequality (1) can be rephrased as

$$A(f,f) \le \|R^* f\|^2_{\ell^2} = (RR^* f, f) \le B(f,f) \qquad \forall f \in V \ ,$$

which shows that the *frame operator* $\mathcal{P} = RR^* \ : \ V \to V$ is symmetric and has a bounded inverse:

$$\mathcal{P} = \mathcal{P}^* \ , \quad A\,\mathrm{Id} \le \mathcal{P} \le B\,\mathrm{Id}\,, \ \|\mathcal{P}\|_{V\to V} = B \ , \quad \frac{1}{B}\mathrm{Id} \le \mathcal{P}^{-1} \le \frac{1}{A}\mathrm{Id}\,, \ \|\mathcal{P}^{-1}\|_{V\to V} = \frac{1}{A} \ .$$

Here, A, B are the frame bounds of F. As a consequence, the spectral condition number of $\mathcal{P}$ coincides with the frame condition:

$$\kappa(\mathcal{P}) = \|\mathcal{P}\|_{V\to V}\|\mathcal{P}^{-1}\|_{V\to V} = \kappa(F) \ .$$

Obviously,

$$f = \sum_k (\mathcal{P}^{-1} f, f_k) f_k = \sum_k (f, \mathcal{P}^{-1} f_k) f_k \qquad \forall\, f \in V \ . \tag{5}$$

The system $\tilde{F} = \{\tilde{f}_k = \mathcal{P}^{-1} f_k\}$ is called *dual frame.* It is easy to see that $\tilde{F}$ is indeed a frame, with frame operator $\mathcal{P}^{-1}$. Finally, note that there is another interesting operator $\tilde{\mathcal{P}} = R^* R \ : \ \ell^2 \to \ell^2$ which is also symmetric (in ℓ^2) but not necessarily invertible. Its matrix representation $\tilde{\mathcal{P}} = (\tilde{\mathcal{P}}_{k,j} = (f_j, f_k))$ suggests the name *Gramian of F* for $\tilde{\mathcal{P}}$. $\tilde{\mathcal{P}}$ can also be used for characterizing properties of a frame.

(5) is the desired canonical decomposition-reconstruction formula. It even gives the *best* representation (3) with respect to F such that the infimum in (4) is achieved. Its practical use requires, in one way or the other, to compute $\mathcal{P}^{-1}$ on certain elements of V, or equivalently, to solve the operator equation

$$\mathcal{P} g = h$$

for given $h \in V$. It was already proposed in [12] (see also [30, Section 8.2]) that *Richardson iteration*

$$g^{(n+1)} = g^{(n)} + \omega(h - \mathcal{P} g^{(n)})\,, \quad n \ge 0\ , g^{(0)} \in V \ , \quad \omega = 2/(A+B) \ , \tag{6}$$

could be used. The convergence rate of the iteration (6) is given by

$$\rho_R = \rho(\mathrm{Id} - \omega\mathcal{P}) = 1 - \frac{2}{1+\kappa(F)} \ ,$$

it exclusively depends on the frame condition. Since $\mathcal{P}$ is symmetric, Richardson iteration can be replaced by the *conjugate gradient method* which would result in an even better convergence rate and avoid knowledge of good bounds for A, B. Other iterative methods might be tried as well.

There is another tricky point. In many applications, the theoretical investigations are for infinite frames (in infinite-dimensional V) while the algorithms work with sections $F_n = \{f_1, \ldots, f_n\}$ of the frame. An example from [4] shows that one should be cautious. Let

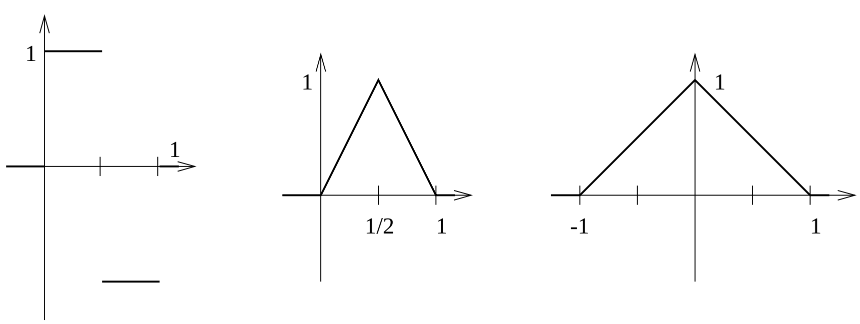
$$F = \{f_1 = e_1, f_2 = e_1 + e_2/2, \ldots, f_k = e_{k-1} + e_k/k, \ldots\} \ ,$$

where $\{e_k\}$ is a CONS in V. F is a frame. However, if one considers its sections F_n as frames in the subspaces span $F_n \subset V$, then the corresponding lower frame bounds A_n deteriorate as $n \to \infty$. It can be shown that $\kappa(F_n) \geq (n!)^2$. Thus, working with the sections F_n of a frame F needs special care. In contrast, if F is a Riesz basis then the inequalities in (2) are automatically preserved for any subsystem, with the same (or better) constants, which yields $\kappa(F_n) \leq \kappa(F)$.

Frames have been considered mainly in connection with image and signal processing applications. The most prominent investigations are connected with **irregular sampling** ([12],[30, Chapter 8]), **Gabor frames** ([30, Chapter 3 and 7],[14],[13]), and **multilevel systems** originating from some kind of multiresolution analysis of subspaces $\{V_j\}$. E.g., given a nonzero function $\psi \in L_2(\mathbb{R})$, we can define *wavelet-like systems* by using integer shifts and dyadic dilation:

$$F_\psi = \{\psi_{j,i}(t) = 2^{j/2}\psi(2^j t - i) \ , \quad j, i \in \mathbb{Z} \ \}.$$

More information can be found in [9, 5, 29, 30]. The classical counterparts of this construction are the Haar and the Faber-Schauder system. These are obtained if the functions ψ depicted in Figure 1 a) and b) are used, respectively. Both choices lead to linearly independent systems. In the Haar case, the resulting wavelet system F_ψ is even a CONS in $L_2(\mathbb{R})$. Moreover, after suitable scaling it is a Riesz basis for the Sobolev spaces $H^s(\mathbb{R})$ with $-1/2 < s < 1/2$. The Faber-Schauder system associated with the ψ in Figure 1 b) is not a Riesz basis in $L_2(\mathbb{R})$ but in $H^s(\mathbb{R})$, $1/2 < s < 3/2$. The system F_ψ resulting from the hat function in Figure 1 c) leads to a system which contains redundancy, and yields frames (not Riesz bases) in $H^s(\mathbb{R})$ if $0 < s < 3/2$. All these systems have generalizations to higher dimensions and to the finite element setting on bounded domains. See [22, 6, 21, 8] for further results.



a) Haar wavelet b) Faber-Schauder 'wavelet' (hierarchical basis) c) BPX 'wavelet' (multilevel nodal basis frame)

Figure 1: Three basic 'wavelets' ψ

2 Stable space splittings

We come to the concept of stable space splittings which originated from work on the theoretical foundation of fictitious domain, domain decomposition methods, and multigrid methods [11, 32] for variational problems. It generalizes the frame concept in two directions: The individual f_k are replaced by Hilbert spaces V_j, and the assumption $f_k \in V$ is relaxed (instead of requiring $V_j \subset V$ we only assume the existence of suitable mappings $R_j : V_j \to V$, the scalar product on V_j need not be inherited from V). This allows for a broader range of applications to be covered (outer approximation schemes, block-iterative schemes, etc.). However, many basic ideas remain the same (this will be expressed by the notation used below).

Again, let V be the basic Hilbert space, with $(\cdot,\cdot)$ resp. $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_{V'\times V}$ as basic scalar product resp. duality pairing. Consider a *symmetric V-elliptic variational problem*

$$u \in V \;:\quad a(u,v) = \langle f, v\rangle \qquad \forall\, v \in V \tag{7}$$

to be solved. (7) is equivalent to the operator equation $Au = f$, where $f \in V'$ and $A : V \to V'$ is defined by $\langle Au, v\rangle = a(u,v)$. Since symmetry and V-ellipticity require *symmetry, continuity*, and *coercivity* of the bilinear form $a(\cdot,\cdot)$, $\{V; a(\cdot,\cdot)\}$ (i.e., the space V equipped with the scalar product $a(\cdot,\cdot)$) is an isomorphic copy of $\{V;(\cdot,\cdot)\}$.

Let V_j, $j = 1,2,\ldots$, be an at most countable family of Hilbert spaces, with $(\cdot,\cdot)_j, \langle\cdot,\cdot\rangle_j$ introduced similarly. To each V_j we assign its own symmetric V_j-elliptic bilinear form $b_j(\cdot,\cdot) : V_j \times V_j \to \mathbb{R}$ which in particular means that $\{V_j; b_j(\cdot,\cdot)\}$ are Hilbert spaces. The V_j and $b_j(\cdot,\cdot)$ will be used to create auxiliary variational problems, and to compose from their solution operators an approximate inverse to A. The latter is then used as a preconditioner in an iterative method for solving (7), see section 3 for the details. It is not assumed that the V_j are subspaces of V (but it is implicit that they correspond to certain portions of V, see below).

Denote the Hilbert sum of this family by $\tilde V$, i.e., for

$$\tilde u = (u_j),\ \tilde v = (v_j), \quad u_j, v_j \in V_j \ \ \forall\, j$$

set

$$\tilde a(\tilde u, \tilde v) = \sum_j b_j(u_j, v_j)$$

which makes sense as a scalar product on

$$\tilde V = \{\tilde u \,:\, \tilde a(\tilde u,\tilde u) < \infty\}\,.$$

Finally, consider bounded linear mappings $R_j \;:\; V_j \to V$. Formally, they can be considered as the components of an operator $R : \tilde V \to V$ given by $R\tilde u = \sum_j R_j u_j$.

Definition 3 *([22]) The system $\{\{V_j; b_j\}, R_j\}$ gives rise to a* **stable splitting** *of $\{V;a\}$ which will be expressed by the short-hand notation*

$$\{V; a\} \cong \sum_j R_j \{V_j; b_j\}\,, \tag{8}$$

if there are two constants $0 < \tilde A \le \tilde B < \infty$ such that

$$\tilde A a(u,u) \le |||u|||^2 \equiv \inf_{\tilde u \in \tilde V:\, u = R\tilde u} \tilde a(\tilde u, \tilde u) \le \tilde B a(u,u) \quad \forall\, u \in V\,. \tag{9}$$

The optimal constants $\tilde A, \tilde B$ in (9) will be called lower and upper stability constants, and their ratio $\kappa = \tilde B / \tilde A$ condition of the splitting (8).

It should be noted that (9) implicitly requires that R makes sense (convergence of the sum if infinitely many V_j are involved) and yields a bounded operator from $\tilde{V}$ *onto* V, i.e., $\mathrm{ran}(R) = V$. The similarity of this definition with (4) in Theorem 2 is obvious. The adjoint $R^* : V \to \tilde{V}$ is defined as

$$R^* \;:\; u \in V \quad \longmapsto R^* u = (R_1^* u, R_2^* u, \ldots) \in \tilde{V} \;,$$

where the components $R_j^* : V \to V_j$ are determined by solving the auxiliary variational problems:

$$b_j(R_j^* u, v_j) = a(u, R_j v_j) \qquad \forall\, v_j \in V_j \;. \tag{10}$$

Introduce the bounded linear operators

$$\mathcal{P} = RR^* \;:\; u \in V \longmapsto \mathcal{P}u = \sum_j T_j u \in V \quad (T_j = R_j R_j^* \;:\; V \to V) \tag{11}$$

and

$$\tilde{\mathcal{P}} = R^* R \;:\; \tilde{u} \to \tilde{\mathcal{P}}\tilde{u} \in \tilde{V} \tag{12}$$

where $\tilde{\mathcal{P}}$ can be considered as operator matrix with entries $\tilde{P}_{jk} = R_j^* R_k$. Following some tradition [11, 32], $\mathcal{P}$ is called *Schwarz operator* associated with the stable splitting (8) while the operator matrix associated with $\tilde{\mathcal{P}}$ will be called *extended Schwarz operator* (it is nothing but the generalization of the Gramian for frames discussed in Section 1, and the abstract analog of the matrix of the semi-definite system [17]).

Theorem 4 *The Schwarz operator (11) associated with a stable splitting (8) is symmetric positive definite and has a bounded inverse. Moreover,*

$$|||u|||^2 = a(\mathcal{P}^{-1}u, u) \qquad \forall\, u \in V \;,$$

and

$$\frac{1}{\tilde{B}}\mathrm{Id} \le \mathcal{P} \le \frac{1}{\tilde{A}}\mathrm{Id} \,, \quad \tilde{A}\mathrm{Id} \le \mathcal{P}^{-1} \le \tilde{B}\mathrm{Id} \,, \qquad \kappa(\mathcal{P}) = \kappa \;.$$

With $\phi = R\tilde{\phi} \in V$, $\tilde{\phi} \in \tilde{V}$ defined from f in an appropriate way, $u \in V$ solves the variational problem (7) if it solves the operator equation

$$\mathcal{P}u = \phi \;, \tag{13}$$

or, equivalently, $u = R\tilde{u}$ for any solution $\tilde{u} \in \tilde{V}$ of the operator equation

$$\tilde{\mathcal{P}}\tilde{u} = \tilde{\phi} \;. \tag{14}$$

The computational aspect of these reformulations of (7) will be discussed in section 3. Standard examples of stable space splittings are discussed in [22, 26, 27]. A particularly important example, with deep connections to approximation, function space and interpolation theory, are **multilevel splittings** associated with a hierarchy of spaces

$$V_0 \to V_1 \to \ldots \to V_{j-1} \to V_j \to \ldots \to V \;, \tag{15}$$

where the relation $V_{j-1} \to V_j$ is described by an *embedding operator* $I_j : V_{j-1} \to V_j$ (these I_j are also called *prolongations* or *intergrid transfer operators*). If the spaces are *nested*, i.e., if $V_{j-1} \subset V_j$, then the natural embeddings are often the prefered choice. Define $R_j^J = I_J \ldots I_{j-1} : V_j \to V_J$, $0 \le j \le J$, and $R_j = \lim_{J\to\infty} R_j^J : V \to V$ (the existence of the latter operators needs verification). The multilevel splittings of interest are

$$\{V_J; a_J\} \cong \sum_{j=0}^{J} R_j^J \{V_j; b_j\} \;, \qquad \{V; a\} \cong \sum_{j=0}^{\infty} R_j \{V_j; b_j\} \;. \tag{16}$$

In applications to differential and integral equations on a domain Ω, where (7) is related to energy minimization in Sobolev norms, the b_j are given by scaled L_2–scalar products. Then the verification of the stability of the splittings in (16) can be reduced to the study of *Jackson-Bernstein inequalities* and *approximation spaces* associated with (15) [2, 22, 6]. Other techniques (e.g., using information on strengthened Cauchy-Schwarz inequalities

$$a(R_j u_j, R_l u_l) \leq \gamma_{jl} b_j(u_j, u_j)^{1/2} b_l(u_l, u_l)^{1/2} \qquad \forall\, u_j \in V_j\,,\, u_l \in V_l\,) \tag{17}$$

can be found in [32, 34, 1].

We can introduce some general operations on stable space splittings which allow us to modify a given one, in order to adapt it to a specific application or to optimize the implementation with respect to a given hardware platform. In [18, 22], we described *refinement* (replace some of the components $\{V_j; b_j\}$ of a splitting by stable splittings of their own), *clustering* (the inverse operation), and *selection* (replace some V_j by subspaces $\hat{V}_j \subset V_j$ or drop some components; this operation corresponds to selecting subsystems of a frame, and may lead to a detoriation of the condition number of the splitting). Furthermore, *tensor-product techniques* can be explored [19] to obtain splittings for higher-dimensional applications. Another variation is to consider *mappings of stable splittings* to produce stable splittings for the range of a certain operator $T : V \to \hat{V} = \mathrm{ran}(T)$. This applies, e.g., to problems associated with trace spaces [25].

3 Iterative solvers

We come to some consequences of the notion of stable space splittings for the construction of iterative solution methods for solving the variational problem (7). Theorem 4 provides the tools. Assume that the splitting (8) is stable, and recall that

$$a(u,v) = \langle Au, v\rangle\ , \quad b_j(u_j, v_j) = \langle B_j u_j, v_j\rangle_j\ ,$$

defines invertible operators $A : V \to V'$, $B_j : V_j \to V_j'$. Introduce the dual operators $R_j' : V' \to V_j'$ by

$$\langle R_j' f, v_j\rangle_j = \langle f, R_j v_j\rangle \qquad \forall\, f \in V',\ v_j \in V_j\ .$$

It follows that $R_j^* = B_j^{-1} R_j' A$. Note that $\tilde{\phi}_j = B_j^{-1} R_j' f$ is the right choice in (13-14). Thus, the additive Schwarz operator has the representation $\mathcal{P} = CA$, where

$$C = \sum_j R_j B_j^{-1} R_j' \equiv \sum_j \hat{T}_j : V' \to V \tag{18}$$

satisfies $C' = C$. Obviously, C can be considered as a (symmetric) *preconditioner* or *approximate inverse* for A, and the switch from (7) to the equivalent formulation (13) as a *preconditioning method*, the quality of which critically depends on the condition κ of the splitting.

Following this setup, several iterative methods for solving (7) based on auxiliary subproblems associated with the given stable splitting can be introduced and analyzed (see [32, 34, 1, 22, 18]). To be practical, consider a *finite stable splitting*

$$\{V; a\} \cong \sum_{j=0}^{J} R_j \{V_j; b_j\} \qquad (\,\dim V_j = N_j < \infty\,,\ \dim V = N < \infty\,)\ . \tag{19}$$

The following basic algorithms associated with (19) have been formulated in [32]:

(AS) Additive Schwarz method. Starting with an initial guess $u^{(0)} \in V$, repeat

$$u^{(n+1)} = u^{(n)} + \omega \sum_{j=1}^{J} \hat{T}_j (f - Au^{(n)})\ ,$$

until a stopping criteria is satisfied.

(MS) Multiplicative Schwarz method. Starting with an initial guess $u^{(0)} \in V$, repeat

$$\begin{aligned} v^{(J+1)} &= u^{(n)} , \\ v^{(j)} &= v^{(j+1)} + \omega \hat{T}_j (f - A v^{(j+1)}) , \quad j = J, \ldots, 0 , \\ u^{(n+1)} &= v^{(0)} , \end{aligned}$$

until a stopping criteria is satisfied.

Variations (pcg-iterations, symmetric multiplicative methods) are possible. Note that the ordering of the subproblems has impact only on the multiplicative method **(MS)**. The *relaxation parameter* $\omega > 0$ can be used to properly *scale* the subproblems, and to enhance the convergence behavior. A special case of **(AS)** is the iteration (6) mentioned in connection with frame decompositions.

An elegant way to analyze the above iterations is to rewrite them in terms of classical iterative methods applied to the operator matrix $\tilde{\mathcal{P}}$ (which is now of size $J+1$) as proposed in [17, 18]. Richardson iteration and the SOR-method applied to the "matrix" problem (14) in $\tilde{V}$ transform into the iterations **(AS)** and **(MS)** in V, respectively, if the mapping R is applied. This leads to the following convergence result:

Theorem 5 *Let (19) be a finite stable space splitting, with stability constants $\tilde{A}, \tilde{B}$, and condition κ.*
a) *The additive method* **(AS)** *converges for $0 < \omega < 2\tilde{A}$. The optimal convergence rate is achieved for $\omega^* = 2\tilde{A}\tilde{B}/(\tilde{A} + \tilde{B})$:*

$$\rho^*_{AS} = \inf_{0<\omega<2/\tilde{A}} \rho_{AS,\omega} = 1 - \frac{2}{1+\kappa} . \tag{20}$$

b) *For the multiplicative algorithm* **(MS)**, *convergence is guaranteed if $0 < \omega < 2/\gamma$, where $\gamma = \max_j \gamma_{jj} \le 1/\tilde{A}$ (the γ_{jj} are defined in (17)). The optimal convergence rate can be estimated by*

$$(\rho^*_{MS})^2 = \inf_{0<\omega<2/\gamma} (\rho^*_{MS,\omega})^2 \le 1 - \frac{1}{\log_2(4(J+1)) \cdot \kappa} . \tag{21}$$

In this generality, these estimates are the best possible ones (see [23]). They show the importance of having well-conditioned splittings. Improved results for the multiplicative method can be found in, e.g., [32, 34, 1]. If the splitting is of multilevel type (16) then the iterations **(AS)** and **(MS)** can be interpreted as a V-cycle preconditioner and a V-cycle multigrid algorithm, respectively. For a comprehensive treatment of multigrid theory in the framework of subspace correction algorithms, see [1]. Additive multilevel preconditioning, especially based on multilevel frames and Riesz bases in Sobolev spaces, is emphasized in [22, 6, 7, 27]. Domain decomposition algorithms are treated in [3, 31, 33]. The concept of space splittings has also been applied to discretizations for elliptic systems (Stokes and Maxwell equations). Generalizations to nonsymmetric, indefinite, and unstructured problems have been attempted, with mixed success.

4 Multilevel schemes for the single layer potential equations

The *single layer potential equation*

$$Tf \equiv \frac{1}{4\pi} \int_\Gamma \frac{f(y)}{|x-y|} \, dy = g(x) , \tag{22}$$

is the prototype of an operator equation associated with a symmetric elliptic pseudodifferential operator $T : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ of order -1. One concrete application are capacity calculations of electrically charged bodies in $\mathbb{R}^3$ where Γ is the surface of the body, $g(x) = 1$ on Γ, and $f(x)$ represents the charge density. The capacity itself is defined by

$$C = C(\Gamma) = \frac{1}{4\pi} \int_\Gamma f(y) \, dy \tag{23}$$

Similar problems arise for *interface problems* for second order elliptic boundary value problems when Neumann data have to be determined from Dirichlet data.

The numerical analysis of an integral equation such as (22) faces a number of difficulties:

- The solution theory "lives" in $H^{-1/2}(\Omega)$, the natural energy space for (22). Sobolev norms of negative order are not so well-investigated in connection with standard discretization schemes.

- For non-smooth bodies, e.g., if Γ contains corners and edges, the solutions exhibit strong singularities.

- The stiffness matrices A_N associated with any typical discretization space V_N (e.g., boundary element or spectral approximations) are dense matrices. Moreover, almost independently of the used discretization spaces and methods, the reliable numerical computation of the entries of the stiffness matrices represents a serious bottleneck. The singularity of the kernel $k(x,y) = 1/|x-y|$ along the 'diagonal' $x = y$ and the parametrization of the surface Γ represent additional challenges.

- The condition of A_N grows with dim V_N. Although this growth is moderate compared with second order elliptic equations, preconditioning needs to be considered.

These challenges have attracted many researchers. In particular, *hp-methods* and *wavelet methods* (combined with *matrix compression*) are under investigation. See [6, 20] for some references.

In Talk 2, we first report theoretical and numerical results [24] on *preconditioning low-order boundary element discretizations* for (22) using semi-orthogonal wavelet splittings. This material is closely related to Talk 1, and further illustrates the machinery of stable splittings in the Sobolev context. In particular, we derive the exact asymptotical properties of multilevel preconditioners based on Haar-type L_2-orthogonal bases in the case of piecewise constant elements.

The main part of this talk is devoted to our recent joint work with Griebel and Schiekofer [20, 28] on using *sparse grid spaces* to partly overcome the above-mentioned difficulties. This second part is restricted to the case of a *square screen* $\Gamma = [0,1]^2$ (and generalizes to Γ composed of a few tensor-product faces). In this case, sparse grid spaces can be constructed by looking at special sections of *tensor-product systems* obtained from univariate spline systems. E.g., if $\{H_{j,i}\}$ denotes the one-dimensional Haar system on $[0,1]$ (the *level index* j is determined by the requirement that $H_{j,i}$ is constant on dyadic intervals of length 2^{-j}) then

$$\hat{V}_J = \mathrm{span}\{H_{\mathbf{j},\mathbf{i}} = H_{j_1,i_1} \otimes H_{j_2,i_2} \,:\, j_1 + j_2 \le J\}, \quad J \ge 0 ,$$

is the definition of standard sparse grid spaces for piecewise constant functions on $[0,1]^2$. If V_J denotes the standard *full grid spaces* (piecewise constant functions on a square grid of sidelength 2^{-J}) then $\hat{V}_J \subset V_J$ and, most importantly, dim $\hat{V}_J \approx J2^J << 2^{2J} =$ dim V_J. As is well-known for H^s-approximation with $s \ge 0$, under additional regularity assumptions (existence of certain higher order mixed derivatives) the use of $\hat{V}_J$ instead of V_J leads to good approximation rates with a small number of degrees of freedom. We make the approximation power of $\hat{V}_J$ precise for $s < 0$, and observe some reduced efficiency of the sparse grid approach for this case.

Our numerical experiments for Galerkin discretizations of (22) in the case $s = -1/2$, however, looked much more promising than predicted by the error analysis. We traced this phenomenon back to the favorable properties of tensor-product systems (as considered in the construction of sparse grid spaces) for the resolution of *edge singularities*. It is shown in [28] to which extent optimally constructed *adaptive sparse grid spaces* may be superior over traditional adaptive wavelet spaces in the case of (22). In particular, we can show that we can choose $\le N$ Haar functions $H_{\mathbf{j},\mathbf{i}}$ such that using them as ansatz functions in a Galerkin scheme for the capacity problem ($g(x) = 1$, $\Gamma = [0,1]^2$ in (22)) leads to capacity approximations C_N with an error rate of

$$|C - C_N| = \mathrm{O}(N^{-5/2}) , \qquad N \to \infty .$$

Traditional adaptive wavelet schemes cannot reach rates better than $O(N^{-1})$ for the same problem. The practical results for capacity computations are very impressive if moderate accuracy (relative error $\approx 10^{-3}$) suffices (asymptotically, for very high resolution, hp-methods will be superior). This topic is related to investigations on nonlinear best N-term approximation [10], and to the problem of how to properly incorporate *anisotropic refinement*. It should be mentioned that most of our observations are not yet fully practical, and need further evaluation.

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