On Polynomial Reproduction of Dual FE Bases

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Abstract

We construct local piecewise polynomial dual bases for standard Lagrange finite element spaces which themselves provide maximal polynomial reproduction. This answers a question which came up in connection with the second author's research on mortar finite elements. By means of such dual bases for the Lagrange multiplier, extremely efficient realization of mortar methods on non–matching triangulations can be obtained without losing the optimality of the discretization errors. In contrast to the standard mortar approach, the locality of the basis functions is preserved.

1 Introduction

Consider the Lagrange C^0 finite element space $V = S_q^0(\mathcal{T})$ of arbitrary degree $q \geq 1$ on a one-dimensional partition $\mathcal{T}$, and denote by $\Phi = [\phi_1, \dots, \phi_{\dim V}]$ a suitable nodal basis in V consisting of locally supported functions (details will be given below). We aim at constructing a dual system $\Lambda = [\lambda_1, \dots, \lambda_{\dim V}]$ of locally supported functions belonging to the space $W = S_q^{-1}(\mathcal{T})$ of piecewise polynomial functions of degree q such that the quasi-interpolant projections

$$Qu = \sum_{k=1}^{\dim V} (\lambda_k, u)_{L_2} \phi_k \ , \qquad \tilde{Q}\nu = \sum_{k=1}^{\dim V} (\phi_k, \nu)_{L_2} \lambda_k \ , \tag{1}$$

into the spaces V resp. $\tilde{V} = \operatorname{span} \Lambda \subset W$ satisfy the usual L_p-boundedness conditions and reproduce polynomials of maximal degree. These two conditions are essential for proving optimality of certain mortar finite element methods, see [11] for details.

The construction of dual bases and quasi-interpolants for univariate spline spaces is well-understood (see, e.g., [1, 2, 3, 9]) but tuned mostly to properties of Q. While Q as a projection onto V reproduces all Lagrange finite element functions of degree q and, thus, automatically polynomials of degree $\leq q$, the maximal possible degree of polynomial reproduction of the dual $\tilde{Q}$ (in the specific form stated above) has seemingly not been studied. On the other hand, in most of the biorthogonal wavelet constructions, where polynomial reproduction properties of the dual system are crucial, the space $\tilde{V}$ is usually of a more complicated structure, and cannot be fixed beforehand. See [4], we also note the paper [5] which contains the construction of a wavelet system for sequences of Lagrange C^0 finite element spaces $\{V_j\}$ in any dimension, with an attempt to use similar spaces to define part of the dual system. For the mortar finite element applications, where polynomial reproduction of degree $r = q - 1$ is desirable to guarantee the optimality of the scheme, it has been observed in [10] that for $q = 1$ the classical Clement interpolant reproduces constants. For the case $q = 2$, in [11] a Λ is given such that the corresponding quasi-interpolant $\tilde{Q}$ reproduces constants and linear functions. Although the elementary construction of [11] can be used to generate dual systems for any $q > 2$, it fails to also guarantee polynomial reproduction of sufficiently high degree. Analogous questions are of certain interest in the two-dimensional case [8].

In this note, we prove two results. First, we show that in the univariate case the maximal degree of polynomial reproduction for $\tilde{Q}$ is q for arbitrary $q \geq 1$. The construction is explicit and can easily

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be implemented although we do not claim that it leads to the most efficient implementation. It also works in the case when additional boundary conditions are included in the definition of V. The results depend only on the local mesh ratio, i.e., are suitable for locally refined partitions. Explicit examples are given for $q \le 3$ and uniform partitions. Secondly, we show that our purely algebraic approach is not restricted to the univariate case by constructing a dual basis Λ for quadratic C^0 finite elements ($q = 2$) on two-dimensional triangulations satisfying a mild regularity condition such that $\tilde{Q}$ reproduces linear polynomials ($r = 1$). This extends the much simpler result for $q = 1$, $r = 0$, mentioned in [8] and later in [6].

2 1D: Construction for $q \ge 2$

In this section, rather than formulating the final result and then proving it, we go step by step through the algebraic construction of suitable dual systems in the univariate finite element case. Generalizations to other spline spaces, though possible, will not be pursued here.

We fix $q \ge 2$ (the case $q = 1$ will be handled later). Let $\mathcal{T} = \{\Delta^n : n = 1, \ldots, N\}$ be the partition of a univariate interval $I = [a, b]$ into consecutive intervals Δ^k of length h_k. On the interval $[-1, 1]$, we define the system of polynomials

$$\mathbf{P} = [p_0, p_2, \ldots, p_q, p_1] \,, \qquad (2)$$

where $p_0(t) = (1 - t)/2$, $p_1(t) = (1 + t)/2$. The remaining polynomials p_k of degree $k = 2, \ldots, q$ are supposed to vanish at $t = \pm 1$ and form an orthonormal system on $[-1, 1]$ (with weight $w(t) \equiv 1$). Note that p_k is even (odd) whenever $k \ge 2$ is even (odd). These polynomials are unique up to sign choices (and can be described in terms of Jacobi polynomials). Since we do not use any specifics of this system in the proofs below, we will not bore the reader with details. All function systems are written as row vectors, the matrix notation used below will be consistent with this assumption and with the order of the functions within a system as shown.

Obviously, a basis in W is given by

$$\tilde{\Phi} = [\tilde{\Phi}_{\Delta^1}, \ldots, \tilde{\Phi}_{\Delta^N}] \,,$$

where $\tilde{\Phi}_{\Delta^n} = [\tilde{\phi}_{0,n}, \ldots, \tilde{\phi}_{q,n}]$ is the (unscaled) transformation of the system $\mathbf{P}$ from $[-1, 1]$ to Δ^n. For further reference, let $\tilde{\Phi}'_{\Delta^n} = [\tilde{\phi}_{1,n}, \ldots, \tilde{\phi}_{q-1,n}]$.

As was mentioned above, our construction does not use whether or not the definition of V involves homogeneous boundary conditions. In order to convince the reader and have both Dirichlet and Neumann boundary conditions treated at once, we will from now on work with the following modified definition of V:

$$V = \{u \in S_q^0(\mathcal{T}) \,:\, u(a) = 0\} \,. \qquad (3)$$

We found it convenient to use

$$\Phi = [\tilde{\Phi}'_{\Delta^1}, \tilde{\phi}_{q,1} + \tilde{\phi}_{0,2}, \tilde{\Phi}'_{\Delta^2}, \ldots, \tilde{\phi}_{q,N-1} + \tilde{\phi}_{0,N}, \tilde{\Phi}'_{\Delta^N}, \tilde{\phi}_{q,N}] \,.$$

as basis in V. In this form, it is sometimes called hierarchical finite element basis (not to be confused with the more recent hierarchical multilevel basis). Note that spline literature [9, 2] prefers to work with B-spline bases while nodal basis functions based on local Lagrange interpolation is the standard choice of finite element theory. Using matrix notation, we can write $\Phi = \tilde{\Phi} X^T$. The rectangular matrix X takes the block-matrix form

$$X = \begin{pmatrix} 0 & \mathrm{Id}_{q-1} & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{q-1} & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & 0 & 0 \\ & & \vdots & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \mathrm{Id}_q \end{pmatrix} ,$$

where columns have dimensions 1, $q-1$, 1, 1, $q-1$, 1,..., 1, q and rows have dimensions $q-1$, 1, $q-1$, 1,..., 1, q, respectively. It is straightforward to see that X factors as

$$X = (\Sigma \ \ 0)U \ , \tag{4}$$

where U is an orthogonal matrix of the block-matrix form

$$U = \begin{pmatrix} 0 & \mathrm{Id}_{q-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{q-1} & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & & 0 & 0 \\ & & \vdots & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \mathrm{Id}_q \\ 1 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} , \tag{5}$$

while Σ is a square diagonal matrix

$$\Sigma = \mathrm{diag}(\mathrm{Id}_{q-1}, \sqrt{2}, \mathrm{Id}_{q-1}, \sqrt{2}, \dots, \sqrt{2}, \mathrm{Id}_q) \tag{6}$$

of dimension $\dim V$, and 0 stands for a zero matrices of appropriate sizes.

We now give an algebraic description of all systems Λ dual to Φ and belonging to W. Let

$$G_{\Phi,\Psi} = \int_I \Phi^T \cdot \Psi \, dt$$

denote the (rectangular) Gram matrix for any two function systems on the interval I. Obviously, the dual basis $\tilde\Lambda$ for $\tilde\Phi$ in W is uniquely determined by $G_{\tilde\Phi,\tilde\Lambda} = \mathrm{Id}_{\dim W}$ or, in other words, given by

$$\tilde\Lambda = \tilde\Phi G^{-1}_{\tilde\Phi,\tilde\Phi} \ .$$

Note that $G_{\tilde\Phi,\tilde\Phi}$ and $G^{-1}_{\tilde\Phi,\tilde\Phi}$ are symmetric and block-diagonal, with diagonal blocks given by $h_n G_{\mathbf{P},\mathbf{P}}$ and $h_n^{-1} G^{-1}_{\mathbf{P},\mathbf{P}}$, respectively. Thus, we can assume that $\tilde\Lambda$ is explicitly available. Using the ansatz $\Lambda = \tilde\Lambda Y^T$, we find that duality of Λ with Φ is equivalent to

$$\mathrm{Id}_{\dim V} = G_{\Phi,\Lambda} = \int_I X\tilde\Phi^T \cdot \tilde\Phi G^{-1}_{\tilde\Phi,\tilde\Phi} Y^T \, dx = XY^T \ .$$

Using the factorization of X, this shows that we have proved

Lemma 1 *The system $\Lambda = \tilde\Lambda Y^T = \tilde\Phi G^{-1}_{\tilde\Phi,\tilde\Phi} Y^T$ from W is dual to Φ if and only if*

$$Y = (\Sigma^{-1} \ \ \hat Z)U \ , \tag{7}$$

where Σ and U were explicitly defined above, and $\hat Z$ is an arbitrary matrix of dimension $\dim V \times (\dim W - \dim V)$.

Now we turn to polynomial reproduction of the quasi-interpolant $\tilde Q$ associated with the pair (Φ, Λ). By definition, the degree of polynomial reproduction of $\tilde Q$ is given as the maximal $r \le q$ such that

$$\tilde Q p = \Lambda \cdot \int_I p \Phi^T dt = p$$

for all polynomials p of degree $\leq r$. Since $\tilde{\Phi}$ is a basis in the containing space W and using the expressions for Λ and Φ, this is equivalent to

$$G_{\tilde{\Phi},\Lambda} X\, x = x \;, \qquad x = \int_I p \tilde{\Phi}^T \, dt \;,$$

where $G_{\tilde{\Phi},\Lambda} = G_{\tilde{\Phi},\tilde{\Phi}} G^{-1}_{\tilde{\Phi},\tilde{\Phi}} Y^T = Y^T$. In other words, if we set $y = Ux$ and use the factorizations (4), (7), then y must satisfy

$$\begin{pmatrix} \mathrm{Id}_{\dim V} & 0 \\ \tilde{Z}^T & 0 \end{pmatrix} y = y$$

for all polynomials p of degree $\leq r$. Here $\tilde{Z} = \Sigma \hat{Z}$ is an arbitrary matrix of dimension $\dim V \times (\dim W - \dim V)$. More explicitly, according to (5), this condition can be written in the form

$$\int_I p\Phi \, dx \cdot Z = \int_I p\Psi \, dx \qquad \forall\, p \,:\, \deg p \leq r \;,$$

where

$$Z = \Sigma^{-1} \tilde{Z} \tilde{\Sigma} = \hat{Z} \tilde{\Sigma} \;, \qquad \tilde{\Sigma} = \mathrm{diag}(1, \sqrt{2}, \ldots, \sqrt{2}) \;, \tag{8}$$

represents again an arbitrary matrix (of the same dimensions as $\tilde{Z}$), and

$$\Psi = [\psi_1, \ldots, \psi_N] \equiv [\tilde{\phi}_{0,1}, \tilde{\phi}_{q,1} - \tilde{\phi}_{0,2}, \ldots, \tilde{\phi}_{q,N-1} - \tilde{\phi}_{0,N}]$$

complements Φ to an alternative basis $[\Phi, \Psi]$ in W.

Thus, if we denote by $\mathcal{P}_r$ any basis in the space $\mathbf{P}_r$ of all polynomials of degree $\leq r$, we have

Lemma 2 *There exists a dual system $\Lambda \subset W$ for Φ such that the quasi-interpolant operator $\tilde{Q}$ from (1) reproduces $\mathbf{P}_r$ if and only if the matrix equation*

$$G_{\mathcal{P}_r,\Phi} Z = G_{\mathcal{P}_r,\Psi} \tag{9}$$

has a solution Z. In turn, this is equivalent to $\mathrm{rk}\, G_{\mathcal{P}_r,\Phi} = \dim \mathbf{P}_r \leq \dim V$, i.e., there exists at least one subsystem $\Phi_r \subset \Phi$ of size $\dim \mathbf{P}_r$ such that

$$\det G_{\mathcal{P}_r,\Phi_r} \neq 0 \;. \tag{10}$$

Proof. The first statement has been already established. But solvability of (9) is equivalent to

$$\mathrm{rk}\, G_{\mathcal{P}_r,\Phi} = \mathrm{rk}\,(G_{\mathcal{P}_r,\Phi} \;\; G_{\mathcal{P}_r,\Psi}) = \mathrm{rk}\, G_{\mathcal{P}_r,[\Phi,\Psi]} \;.$$

Since $[\Phi, \Psi]$ is a basis in W and $\mathbf{P}_r \subset W$, the above rank must be $\dim \mathbf{P}_r$ which implies the other statements of Lemma 2.

Remark 1. Having in mind the local support requirement for the dual basis functions as well as good L_p-stability bounds of the quasi-interpolants (1), the following practical strategy is advisable for actually using Lemma 2:

a) For each function ψ_l from the system Ψ, choose a suitable subset Φ_r from Φ such that (10) is satisfied, and solve the non-singular linear system

$$G_{\mathcal{P}_r,\Phi_r} \xi = G_{\mathcal{P}_r,\psi_l} \;. \tag{11}$$

Define the l-th column in Z by associating the components of ξ to the positions in this column by correspondence to the functions in the chosen Φ_r and leaving zeros in positions corresponding to ϕ_k not in Φ_r. Obviously, this Z satisfies (9). Each column of Z has $\leq \dim \mathbf{P}_r$ non-zero entries.

A desirable constraint on the choice of Φ_r is to make sure that the supports of $\phi_j \in \Phi_r$ are as close as possible to the support of the corresponding ψ_l, e.g., are enclosed in a set Δ_{ψ_l} which is obtained by extending $\mathrm{supp}\, \psi_l$ by a fixed number of intervals Δ^k to the left and right. From this and a bound on the local meshsize ratio $\max_{|i-k|=1} h_i/h_k$, the locality of the λ_j and uniform L_p-stability bounds for the quasi-interpolants can be derived. Since this is standard, we will not go into details.

b) After Z has been found, we can determine the dual system in the form

$$\Lambda = \tilde{\Phi} R^T \ , \qquad R = (\Sigma^{-1} \ \ Z\tilde{\Sigma}^{-1})UG^{-1}_{\tilde{\Phi},\tilde{\Phi}} = (\mathrm{Id}_{\dim V} \ \ Z)\tilde{U}G^{-1}_{\tilde{\Phi},\tilde{\Phi}} \ , \tag{12}$$

where $\tilde{U} = \mathrm{diag}(\Sigma^{-1}, \tilde{\Sigma}^{-1})U$ can be found from (6), (8), (5).

Note that all this can be implemented without ever storing globally defined objects. The geometry of the underlying partition $\mathcal{T}$ comes in through Z and, implicitly, through the scaling of the diagonal blocks of $G^{-1}_{\tilde{\Phi},\tilde{\Phi}}$.

Remark 2. The matrix approach which we used to characterize duality and degree of polynomial reproduction is convenient since it directly leads to an implementation. It rests on a few assumptions which can easily be generalized to other situations, e.g., finite element spaces in higher dimensions, see below in section 4. Clearly, in each application one has to find out the proper replacements for U, Σ, $\tilde{\Sigma}$, and to materialize the choice for Φ_r such that (10) can be verified.

However, it is fair to mention that in the univariate case considered in this section, we could have departed from the Marsden identity and the known explicit formulas for dual functionals for B-spline bases [1, 2, 3, 9], and tried to directly establish a result similar to Lemma 2.

The remaining part of this section is devoted to the construction of subsystems Φ_r for the case $r = q$ of maximal possible degree of polynomial reproduction. Consider $q \geq 2$, and $N \geq 3$, the case $N \leq 2$ is not so much of practical interest, if necessary, it can be dealt with as a simple linear algebra problem. Linear finite elements are similar (see below). Then, automatically, $\dim V \geq \dim \mathbf{P}_q$.

Consider any three consecutive $\Delta^{n-1}, \Delta^n, \Delta^{n+1}$. Without loss of generality, after a suitable linear coordinate transform we can assume that Δ^n coincides with $[-1,1]$ and that the new intervals left and right to $[-1,1]$ have lengths $h = 2h_{n-1}/h_n$ and $h' = 2h_{n+1}/h_n$, respectively. We choose the system

$$\Phi_q = [\tilde{\phi}_{1,n-1}, \tilde{\phi}_{1,n}, \ldots, \tilde{\phi}_{q-1,n}, \tilde{\phi}_{1,n+1}]; , \tag{13}$$

and denote its transformation to $[-h-1,-1] \cup [-1,1] \cup [1, 1+h']$ again by Φ_q. It is easy to see that the functions with support in $[-1,1]$ are exactly the functions $p_2, \ldots, p_q$ from $\mathbf{P}$ while the remaining two functions on the left and right interval are scaled translates of the quadratic bubble function p_2.

In order to prove (10), we have a free choice of the basis in $\mathbf{P}_q$. So, we can choose

$$\mathcal{P}_q = [\tilde{p}_0, \tilde{p}_1, p_2, \ldots, p_q] \ ,$$

where for $s \in [-1,1]$ we set

$$\tilde{p}_0(s) = 1 - \sum_{k=2}^{q} \int_{-1}^{1} p_k(t)\, dt \cdot p_k(s) \ , \quad \tilde{p}_1(s) = s - \sum_{k=2}^{q} \int_{-1}^{1} t p_k(t)\, dt \cdot p_k(s) \ ,$$

and all polynomials are afterwards extended to all of $\mathbb{R}$. We note the following properties of this system:

- Restricted to $[-1,1]$, the basis $\mathcal{P}_q$ is orthogonal. However, we did not normalize the first two functions.
- $\tilde{p}_0$,$p_2, \ldots$ are even, $\tilde{p}_1$,$p_3, \ldots$ are odd.
- All zeros of $\tilde{p}_0$, $\tilde{p}_1$ are in $(-1,1)$, i.e.,

$$\tilde{p}_0(t) > 0 \ , \quad t \in \mathbb{R}\backslash[-1,1] \ , \quad \tilde{p}_1(t) \begin{cases} < 0 \ , & t < -1 \\ > 0 \ , & t > 1 \end{cases} \ . \tag{14}$$

To prove this for $\tilde{p}_0$, assume that it has zeros outside $[-1,1]$. Since $\tilde{p}_0$ is even and of even degree $2q' \leq q$, its zeros are symmetrically located with respect to the origin. Let $0 < x_1 < \ldots < x_s < 1$

be the zeros of odd multiplicity inside $(0,1)$. By assumption, $s \leq q'-1$ (since at least one pair of zeros is outside $[-1,1]$). Recall also that by construction $\tilde{p}_0(-1)=\tilde{p}_0(1)=1$. Let

$$p(x)=(1-t^2)(t^2-x_1^2)\dots(t^2-x_s^2)\ , \quad t\in[-1,1]\ .$$

This polynomial has the same sign as $\tilde{p}_0$ everywhere in $(-1,1)$, with the possible exception of zeros of even multiplicity of $\tilde{p}_0$. Thus,

$$\int_{-1}^{1} p(t)\tilde{p}_0(t)\,dt>0\ .$$

This contradicts the orthogonality property since from $\deg(p)\leq 2q'\leq q$ and $p(-1)=p(1)=0$ we conclude that

$$p\in \operatorname{span}[p_2,\dots,p_q]\ .$$

The same reasoning goes through for $\tilde{p}_1$, we leave this upon the reader.

We are now in position to show that the condition (10) holds for the above Φ_q and $\mathcal{P}_q$. Equivalently, we show that orthogonality of

$$p=a_0\tilde{p}_0+a_1\tilde{p}_1+\sum_{k=2}^{q}a_kp_k$$

to all functions from Φ_q yields $a_k=0$ for all $k=0,\dots,q$. Since Φ_q contains p_l (restricted to $[-1,1]$), $l=2,\dots,q$, the orthogonality of $\mathcal{P}_q$ with respect to $[-1,1]$ immediately gives $a_l=0$ for all these l. Thus, only a_0 and a_1 can be different from zero. Now we test with the two remaining quadratic bubble functions which we will denote by $\hat{p}$ for $[-1-h,-1]$ and $\hat{p}'$ for $[1,1+h]$. But due to the sign properties (14), the 2×2 determinant

$$\det G_{[\tilde{p}_0,\tilde{p}_1],[\hat{p},\hat{p}']}=\begin{vmatrix} >0 & >0\\ <0 & >0\end{vmatrix}>0\ ,$$

is positive (recall that the quadratic bubbles are positive in the interiors of their respective support intervals). This shows $a_0=a_1=0$, and concludes the verification of (10). As a by-product, we see that the inverse $G^{-1}_{\mathcal{P}_q,\Phi_q}$ continuously depends on h,h', which yields continuous dependence of the entries of the solution matrices Z on these parameters after suitable choices for $n=n(l)$ in dependence on $\psi_l\in\Psi$ have been made (see Remark 1 and the examples in the next section). This proves the main result of this section.

Other ad hoc choices of suitable Φ_q might be tried. An example, which can be treated in essentially the same way is

$$\Phi_q=[\tilde{\phi}_{q,n-1}+\tilde{\phi}_{0,n},\tilde{\phi}_{1,n},\dots,\tilde{\phi}_{q-1,n},\tilde{\phi}_{q,n}+\tilde{\phi}_{0,n+1}]\ . \tag{15}$$

It also covers the case $q=1$, which will be considered in the next section. We do not claim that (13) or (15) lead to minimally supported λ_k. Dual systems with λ_k of small support have been constructed in [11] for $q\leq 2$ and $r=q-1$. We were not successful with proving a general result where the supports of $\phi_j\in\Phi_q$ are confined to only 2 intervals. However, we are confident that such a choice is possible (see the examples for $q=2,3$ in the next section).

As was mentioned in the introduction, for the mortar finite element applications polynomial reproduction of degree $q-1$ would suffice, for which simpler constructions might be possible. Our above proof implies that for

$$\Phi_{q-1}=[\tilde{\phi}_{q,n-1}+\tilde{\phi}_{0,n},\tilde{\phi}_{1,n},\dots,\tilde{\phi}_{q-2,n},\tilde{\phi}_{q,n}+\tilde{\phi}_{0,n+1}]\ , \tag{16}$$

the relation (10) can be shown in the same way (clearly, with $\mathcal{P}_q$ replaced by a suitable $\mathcal{P}_{q-1}$). For $q\leq 3$, some simpler choices will be examined in the next section.

Remark 3. Replacing in the mortar finite element approach the standard Lagrange multiplier space on each interface by the span of our biorthogonal basis functions yields optimal a priori estimates for the discretization errors. Here, we have to work with Dirichlet boundary conditions for V on both endpoints of I. In [11, Conditions (Sa)–(Sd)], sufficient conditions are given to obtain a discretization error of order h^q and h^{q+1} in the H^1– and L^2–norm, respectively. Almost all conditions, as locality, polynomial reproduction of degree $q-1$, and $c\|\phi_k\|_{L_2}^{-1} \le \|\lambda_k\|_{L_2} \le C\|\phi_k\|_{L_2}^{-1}$ are satisfied by construction. To establish the optimality, it is sufficient to show that there exist k_1 and k_2 such that

$$\int_I \lambda_{k_1} \tilde{\phi}_{0,1}\, dx \neq 0, \quad \int_I \lambda_{k_2} \tilde{\phi}_{q,N}\, dx \neq 0, \tag{17}$$

and that $k_i \neq lq$, $1 \le l \le N-1$, $i = 1,2$, see [11, Conditions (Sc) and (Sd)]. We consider the case $i = 1$, the same reasoning goes through for $i = 2$. Using the representation $\Lambda = \tilde{\Lambda} U^T (\Sigma^{-1} \;\; Z\tilde{\Sigma}^{-1})^T$, we find that (17) is equivalent to $z_1 \neq 0$, where z_1 denotes the first column of Z. Following the construction in Remark 1a) and choosing Φ_q (or Φ_{q-1}) according to (13) for $n = 2$, we find that the components $(z_1)_{lq}$ vanish for all $1 \le l \le N-1$ while $\xi = G^{-1}_{\mathcal{P}_r,\Phi_r} G_{\mathcal{P}_r,\psi_1} \neq 0$ implies $z_1 \neq 0$. Thus the optimality of the mortar method is guaranteed.

3 1D: Examples for $q \le 3$

The aim of this section is to illustrate the above theoretical result, and to provide explicit formulas for $q \le 3$, at least for the case of uniform partitions. We decided not to use (13) but rather go with (15) and other choices with smaller support. I.e., we will assume from now on that $I = [0, N]$, $\Delta^n = [n-1, n]$, $n = 1, \ldots, N$, and $q \le 3$. For our convenience, we will fix the basis for $\tilde{\Phi}_{\Delta^1}$ on $\Delta^1 = [0,1]$ by defining

$$\tilde{\Phi}_{\Delta^1} = \begin{cases} [1-t, t]\,, & q = 1\,, \\ [1-t, 6t(1-t), t]\,, & q = 2\,, \\ [1-t, 6t(1-t), 10t(1-t)(2t-1), t]\,, & q = 3\,. \end{cases}$$

The bases $\tilde{\Phi}_{\Delta^n}$ are obtained by translation. This will lead to the following n-independent formulas for the diagonal blocks $G^{-1}_{\tilde{\Phi}_{\Delta^n},\tilde{\Phi}_{\Delta^n}}$ of $G^{-1}_{\tilde{\Phi},\tilde{\Phi}}$:

$$G^{-1}_{\tilde{\Phi}_{\Delta^n},\tilde{\Phi}_{\Delta^n}} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 9 & -5 & 3 \\ -5 & 5 & -5 \\ 3 & -5 & 9 \end{pmatrix}, \quad \begin{pmatrix} 16 & -5 & 7 & -4 \\ -5 & 5 & 0 & -5 \\ 7 & 0 & 7 & -7 \\ -4 & -5 & -7 & 16 \end{pmatrix}, \tag{18}$$

for $q = 1, 2, 3$, respectively. The explicit formulas for $\Sigma, \tilde{\Sigma}, U$ only differ in the sizes of the identity matrices for different q, and can be recovered from (6), (8), (5). Thus, we have all ingredients ready for using (12), with the exception of the matrix Z. The construction of Z is described in Remark 1, a), and depends on q and the desired degree of polynomial reproduction $r \le q$. Although we provide calculations and explicit results only for uniform partitions, the construction can be used for non-uniform partitions without essential changes.

Case $q = 1$: We provide the details for the case $r = 1$ of maximal degree of polynomial reproduction. Let $N \ge 3$, and choose the set Φ_1 according to (15), where $n = n(l)$ will depend on ψ_l as follows: for $l \le 2$ we take $n(l) = 2$, and for $l > 2$ we set $n(l) = l - 1$. To explore obvious symmetries, we also set

$$\mathcal{P}_1 = [p_0(t) = 1,\, p_1(x) = t - (n(l) - \frac{1}{2})]\,.$$

Thus,

$$G_{\mathcal{P}_1,\Phi_1} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad G^{-1}_{\mathcal{P}_1,\Phi_1} = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{pmatrix}$$

do not depend on l.

Now we have to distinguish between $l = 1$ and $l \geq 2$. Since $\psi_1(t) = 1 - t$ on Δ^1 and 0 elsewhere, we compute

$$\xi = G^{-1}_{\mathcal{P}_1,\Phi_1} \begin{pmatrix} (p_0, \psi_1)_{L_2} \\ \\ (p_1, \psi_1)_{L_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \\ -\frac{7}{12} \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ \\ -\frac{1}{3} \end{pmatrix}$$

for $l = 1$. Analogously, for $l \geq 2$ we have

$$\xi = G^{-1}_{\mathcal{P}_1,\Phi_1} \begin{pmatrix} 0 \\ \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \\ -\frac{1}{3} \end{pmatrix} .$$

Thus, Z takes the form

$$Z = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{3} & 0 & & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \cdots & \\ 0 & 0 & 0 & -\frac{1}{3} & & 0 \\ & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & & 0 \end{pmatrix} .$$

Now, all ingredients of the formula (12) have been precomputed, and we present the result of the multiplication:

$$\Lambda = \tilde{\Phi} R^T \, , \qquad R = \begin{pmatrix} 2 & 1 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & & & & & & & \\ -1 & 0 & 0 & 1 & \frac{7}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & & & & & \\ & & & & -\frac{2}{3} & \frac{4}{3} & \frac{7}{3} & -\frac{2}{3} & & & & & \\ & & & & & & -\frac{2}{3} & \frac{4}{3} & \ddots & -\frac{2}{3} & \frac{1}{3} & & \\ & & & & & & & & & \frac{7}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ & & & & & & & & & -\frac{2}{3} & \frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ & & & & & & & & & & & -2 & 4 \end{pmatrix} .$$

This gives the expressions of all dual basis functions λ_k with respect to the basis $\tilde{\Phi}$. The functions are depicted in Figure 1 for $N = 5$. The dashed line shows the graph of the corresponding ϕ_k. The first two pictures correspond to basis functions near the left endpoint (Dirichlet b.c.), the third to a generic interior basis function ($3 \leq k \leq N-2$), and the last two to basis functions near the right endpoint (Neumann b.c.).

Case $q = 2$: We will only show the calculations for $r = 2$ (for $r = 1$, see [11]). Suppose $N \geq 2$. For $l = 1$, set $n(l) = 1$, and for $l > 1$, set $n(l) = l - 1$. For given n, fix

$$\mathcal{P}_2 = [p_0(t) = 1,\, p_1(t) = t - n(l), p_2(t) = (t - n(l))^2] \, ,$$

and, in contrast to Section 2, let Φ_2 consist of the 3 basis functions from Φ with support in $\Delta^n \cup \Delta^{n+1}$

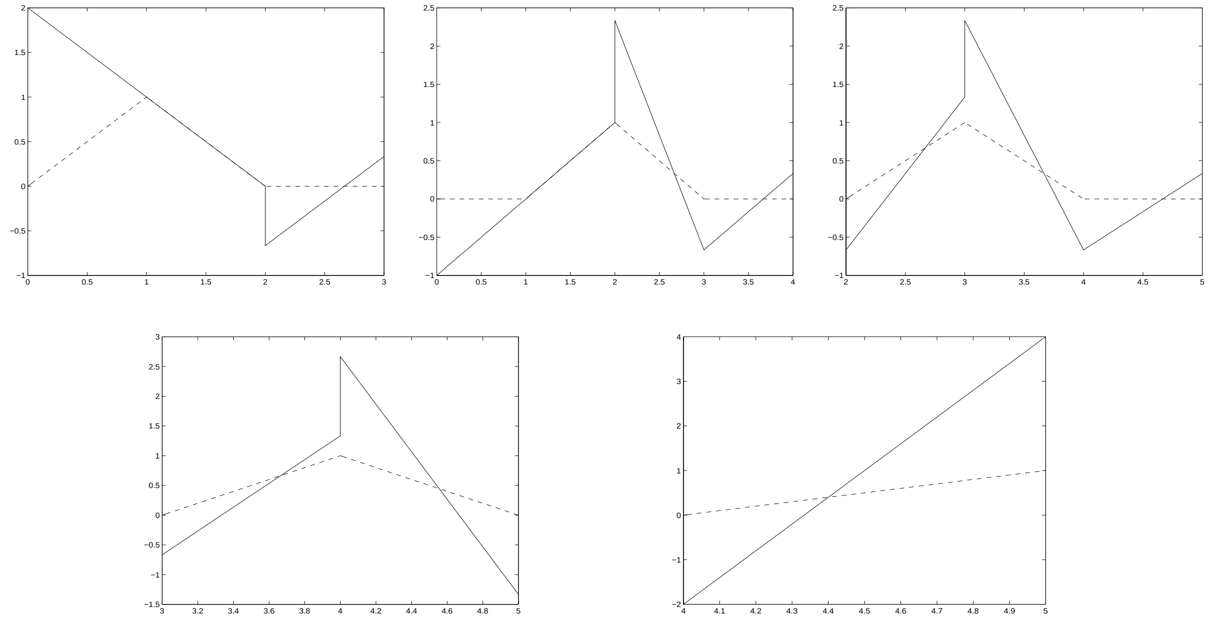

Figure 1: Dual basis for $q = r = 1$

(one hat function and two quadratic bubbles). Thus, again independently of l,

$$G_{\mathcal{P}_2,\Phi_2} = \begin{pmatrix} 1 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{10} & \frac{1}{6} & \frac{3}{10} \end{pmatrix} , \qquad G^{-1}_{\mathcal{P}_2,\Phi_2} = \begin{pmatrix} -\frac{5}{8} & -1 & \frac{15}{4} \\ \frac{9}{4} & 0 & -\frac{15}{2} \\ -\frac{5}{8} & 1 & \frac{15}{4} \end{pmatrix} .$$

For $l = 1$ we have

$$\xi = G^{-1}_{\mathcal{P}_2,\Phi_2} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{3} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{23}{24} \\ -\frac{3}{4} \\ \frac{7}{24} \end{pmatrix} ,$$

while for $l > 1$

$$\xi = G^{-1}_{\mathcal{P}_2,\Phi_2} \begin{pmatrix} 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ -\frac{1}{3} \end{pmatrix} .$$

This allows to define Z and to compute the final representation of the dual basis $\Lambda = \tilde{\Phi} R^T$ from

$$R = \begin{pmatrix}
\frac{33}{8} & -\frac{5}{8} & -\frac{5}{8} & -\frac{3}{2} & \frac{5}{6} & -\frac{1}{2} & & & & & & & \\
-\frac{21}{4} & \frac{5}{4} & \frac{9}{4} & \frac{9}{2} & -\frac{5}{2} & \frac{3}{2} & & & & & & & \\
\frac{17}{8} & -\frac{5}{8} & -\frac{5}{8} & -3 & \frac{10}{3} & -3 & & -\frac{3}{2} & \frac{5}{6} & -\frac{1}{2} & & & \\
 & & & \frac{3}{2} & -\frac{5}{2} & \frac{9}{2} & \ddots & \frac{9}{2} & -\frac{5}{2} & \frac{3}{2} & & & \\
 & & & -\frac{1}{2} & \frac{5}{6} & -\frac{3}{2} & & -3 & \frac{10}{3} & -3 & -\frac{3}{2} & \frac{5}{6} & -\frac{1}{2} \\
 & & & & & & & \frac{3}{2} & -\frac{5}{2} & \frac{9}{2} & \frac{9}{2} & -\frac{5}{2} & \frac{3}{2} \\
 & & & & & & & -\frac{1}{2} & \frac{5}{6} & -\frac{3}{2} & -\frac{7}{2} & \frac{25}{6} & -\frac{9}{2} \\
 & & & & & & & & & & 3 & -5 & 9
\end{pmatrix}.$$

The dual functions are depicted in Figure 2. The graphs in the first row correspond to basis functions near the left endpoint (Dirichlet b.c.), in the second row to interior basis functions, and the last row to basis functions near the right endpoint (Neumann b.c.). Note that the support of λ_k is contained in ≤ 3 neighboring Δ^n close to the support of the corresponding ϕ_k.

Case $q = 3$: We will detail the construction for $r = 2$ which is sufficient for the mortar finite element applications. We keep the same definition of $n(l)$, $\mathcal{P}_2$, and Φ_2 as before. Thus, Z does not changed either (since Ψ is independent of $q \geq 2$). The changes are only in the block sizes of U and Σ, and in $G^{-1}_{\tilde{\Phi},\tilde{\Phi}}$, see (5), (6), (18). Multiplication according to (12) yields the formula $\Lambda = \tilde{\Lambda} R^T$ for the dual basis, where R is given by

$$\begin{pmatrix}
\frac{29}{3} & -\frac{5}{8} & \frac{133}{24} & -\frac{37}{6} & -\frac{8}{3} & \frac{5}{6} & -\frac{7}{6} & \frac{2}{3} & & & & & & & & & \\
7 & 0 & 7 & -7 & & & & & & & & & & & & & \\
-14 & \frac{5}{4} & -\frac{35}{4} & 11 & 8 & -\frac{5}{2} & \frac{7}{2} & -2 & & & & & & & & & \\
\frac{16}{3} & -\frac{5}{8} & \frac{77}{24} & -\frac{23}{6} & -3 & \frac{10}{3} & 0 & -3 & & -\frac{8}{3} & \frac{5}{6} & -\frac{7}{6} & \frac{2}{3} & & & & \\
 & & & & 7 & 0 & 7 & -7 & & & & & & & & & \\
 & & & & -2 & -\frac{5}{2} & -\frac{7}{2} & 8 & \ddots & 8 & -\frac{5}{2} & \frac{7}{2} & -2 & & & & \\
 & & & & \frac{2}{3} & \frac{5}{6} & \frac{7}{6} & -\frac{8}{3} & & -3 & \frac{10}{3} & 0 & -3 & -\frac{8}{3} & \frac{5}{6} & -\frac{7}{6} & \frac{2}{3} \\
 & & & & & & & & & 7 & 0 & 7 & -7 & & & & \\
 & & & & & & & & & -2 & -\frac{5}{2} & -\frac{7}{2} & 8 & 8 & -\frac{5}{2} & \frac{7}{2} & -2 \\
 & & & & & & & & & \frac{2}{3} & \frac{5}{6} & \frac{7}{6} & -\frac{8}{3} & -\frac{7}{3} & \frac{25}{6} & \frac{7}{6} & -\frac{17}{3} \\
 & & & & & & & & & & & & & 7 & 0 & 7 & -7 \\
 & & & & & & & & & & & & & -4 & -5 & -7 & 16
\end{pmatrix}.$$

The dual functions for the interior shape functions (hat function, quadratic and cubic bubbles) are shown in Figure 3. Let us finally note that there is a more compact construction (compared to (13) or

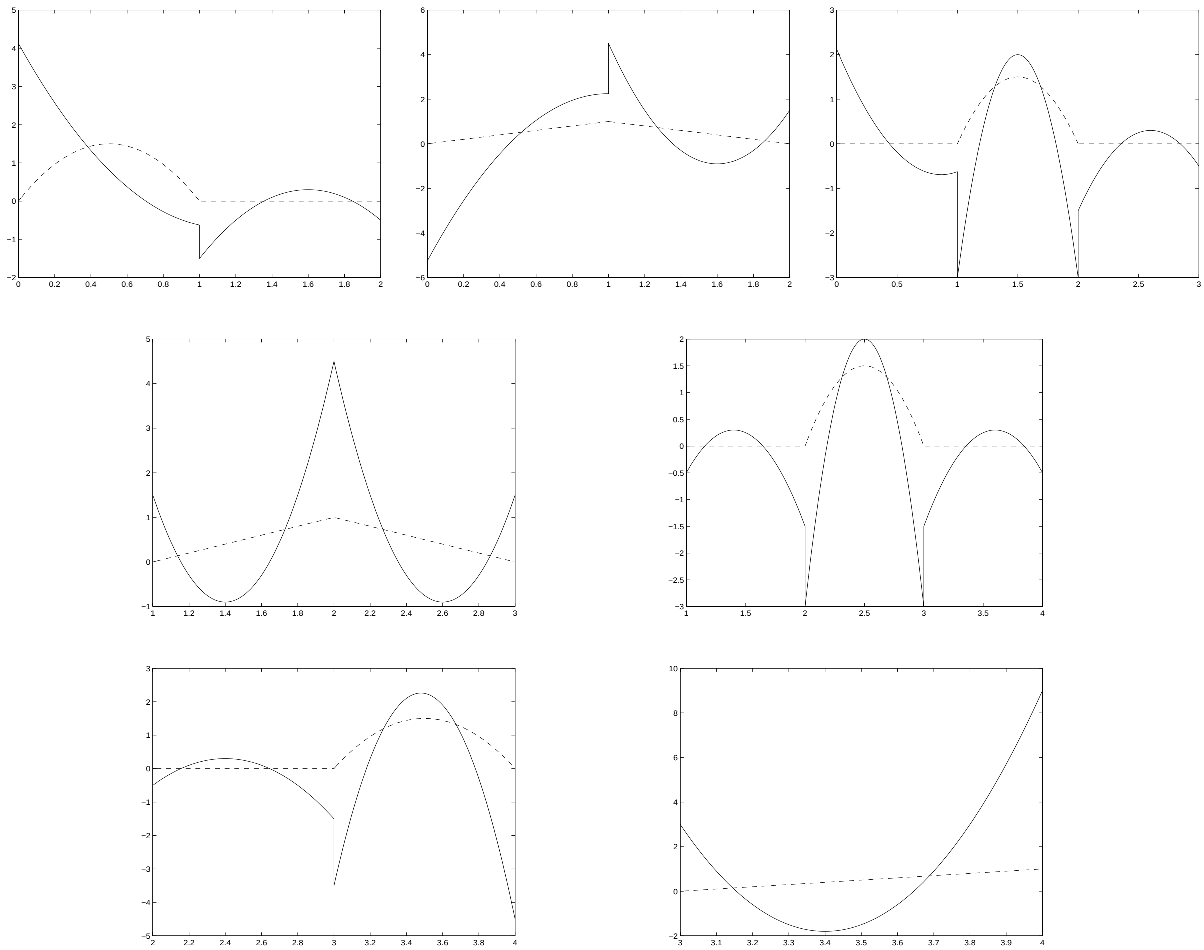

Figure 2: Dual basis for $q = r = 2$

(15)) also for $r = 3$: One can enlarge the above Φ_2 to Φ_3 by adding any of the two cubic bubbles with support in $\Delta^n \cup \Delta^{n+1}$, and enlarge $\mathcal{P}_2$ to $\mathcal{P}_3$ by including the cubic monomial $p_3(t) = (t-n)^3$. To see this, just compute $G_{\mathcal{P}_3,\Phi_3}$ and check its non-singularity. The dual functions for internal basis functions are shown in Figure 4. Note also that the more symmetric choice of a Φ_3 consisting of the quadratic and cubic bubbles with support in $\Delta^n \cup \Delta^{n+1}$ (altogether 4 functions) does not satisfy (10).

Remark 4. Numerical evidence suggests that the choices presented for Φ_3 and for $r = 2, 3$ generalize to all $q > 3$, and can serve as an replacement for (13) resp. (15) in this general case. Note also that the support of any of the λ_k can be confined to ≤ 3 consecutive intervals, for any $q \ge 2$ and any type of boundary condition. Clearly, if $\mathcal{T}$ is non-uniform, the functions will depend on the local mesh-ratio, and be generally discontinuous at all interior knots.

4 2D: Quadratic Lagrange elements

We will show on an example how the above approach generalizes to higher dimensions. We consider spaces V of quadratic Lagrange C^0-elements (i.e., $q = 2$) on a triangulation $\mathcal{T}$ of a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, and show the existence of a dual basis such that $\tilde{Q}$ reproduces linear polynomials locally (i.e., $r = 1$) under a certain regularity condition on $\mathcal{T}$. We will outline the construction for the case of zero Dirichlet boundary conditions. The changes for arbitrary Dirichlet-Neumann boundary

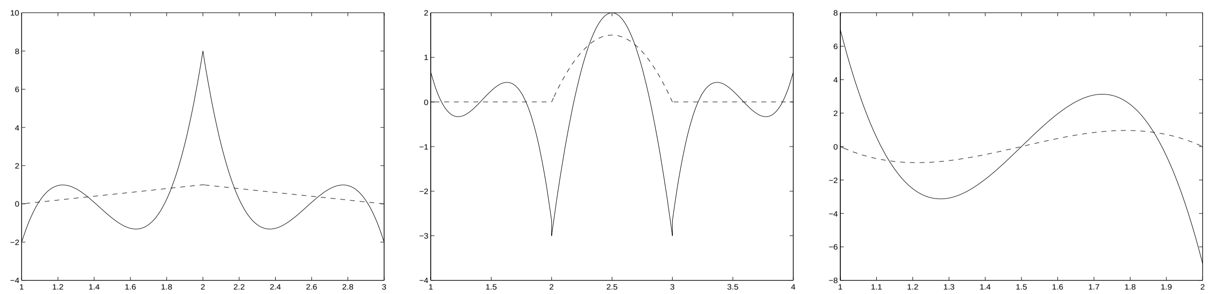

Figure 3: Interior dual basis functions for $q = 3$, $r = 2$

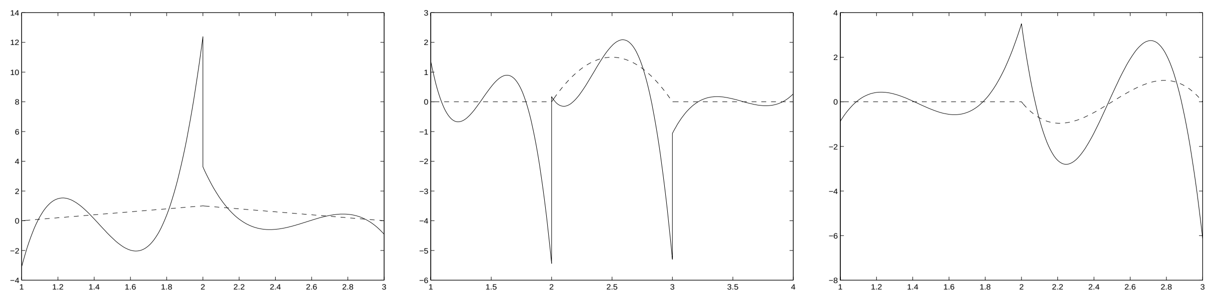

Figure 4: Interior dual basis functions for $q = 3$, $r = 3$

conditions are obvious and left to the reader. To establish notation, let $\mathcal{V}$ and $\mathcal{E}$ denote the set of interior vertices and interior edges of $\mathcal{T}$, and let M, K be the number of elements in $\mathcal{V}$ and $\mathcal{E}$, respectively.

The basis $\tilde{\Phi}$ in $W = S_2^{-1}(\mathcal{T})$ (the space of discontinuous piecewise quadratics) is conveniently given by the collection of all elemental nodal shape functions $\tilde{\phi}_{\Delta,P}$ (piecewise linear barycentric coordinate function for vertex P of triangle Δ) and $\tilde{\phi}_{\Delta,e}$ (quadratic tent function associated with triangle Δ and its edge e). Each such function is supported on a single triangle, and there are 6 of them for each Δ. For our convenience, we order the functions in $\tilde{\Phi}$ according to the following rules:

- For each vertex $P \in \mathcal{V}$, we set $\tilde{\Phi}_P = [\tilde{\phi}_{\Delta_1,P}, \ldots, \tilde{\phi}_{\Delta_{N_P},P}]$, where Δ_k, $k = 1, \ldots, N_P$, are the triangles attached to P. The size of $\tilde{\Phi}_P$ varies with P.
- For each interior edge $e \in \mathcal{E}$, set $\tilde{\Phi}_e = [\tilde{\phi}_{\Delta_1,e}, \tilde{\phi}_{\Delta_2,e}]$, where Δ_1, Δ_2 are the two triangles attached to e.
- The system $\tilde{\Phi}_{\partial\Omega}$ of all remaining $\tilde{\phi}_{\Delta,P}, \tilde{\phi}_{\Delta,e}$ contains the shape functions associated with the boundary (if part of the boundary corresponds to Neumann boundary conditions, more $\tilde{\Phi}_P$, $\tilde{\Phi}_e$ need to be defined). The size of this subsystem will be denoted by L.

Thus, after fixing some ordering in $\mathcal{V}$ and $\mathcal{E}$, we set

$$\tilde{\Phi} = [\tilde{\Phi}_{\mathcal{V}}, \tilde{\Phi}_{\mathcal{E}}, \tilde{\Phi}_{\partial\Omega}] , \quad \tilde{\Phi}_{\mathcal{V}} = [\tilde{\Phi}_{P_1}, \ldots, \tilde{\Phi}_{P_M}] , \ \ \tilde{\Phi}_{\mathcal{E}} = [\tilde{\Phi}_{e_1}, \ldots, \tilde{\Phi}_{e_K}] .$$

Let O_N be a fixed orthogonal matrix of dimension N, the first row of which coincides with the vector $a_N := (N^{-1/2}, \ldots, N^{-1/2})$. The remaining $(N-1) \times N$ matrix will be denoted by O'_N. Obviously, the definitions

$$[\phi_P, \psi_{P,1}, \ldots, \psi_{P,N_P-1}] = \sqrt{N_P}\tilde{\Phi}_P O_{N_P}^T , \qquad [\phi_e, \psi_e] = \sqrt{2}\tilde{\Phi}_e O_2^T ,$$

create the standard basis

$$\Phi = [\Phi_{\mathcal{V}}, \Phi_{\mathcal{E}}] , \quad \Phi_{\mathcal{V}} = [\phi_{P_1}, \ldots, \phi_{P_M}] , \ \ \Phi_{\mathcal{E}} = [\phi_{e_1}, \ldots, \phi_{e_K}] ,$$

in $V = S^0_{2,0}(\mathcal{T})$, and its complement $\Psi = [\Psi_{\mathcal{V}}, \Psi_{\mathcal{E}}, \Psi_{\partial\Omega}]$, where

$$\Psi_{\mathcal{V}} = [\psi_{P_1,1}, \ldots, \psi_{P_1,N_1-1}, \ldots, \psi_{P_M,1}, \ldots, \psi_{P_M,N_M-1}] \ , \ \ \Psi_{\mathcal{E}} = [\psi_{e_1}, \ldots, \psi_{e_K}] \ , \ \ \Psi_{\partial\Omega} = \tilde{\Phi}_{\partial\Omega} \ ,$$

in W. For convenience, we have set $N_k = N_{P_k}$.

Now, it is easy to see that the whole construction of Section 2 goes through step by step. I.e., we arrive at the explicit formula (12), where now U has the block structure

$$U = \begin{pmatrix} a_{N_1} & & & & & & \\ & \ddots & & & & & \\ & & a_{N_M} & & & & \\ & & & a_2 & & & \\ & & & & \ddots & & \\ & & & & & a_2 & \\ O'_{N_1} & & & & & & \\ & \ddots & & & & & \\ & & O'_{N_M} & & & & \\ & & & O'_2 & & & \\ & & & & \ddots & & \\ & & & & & O'_2 & \\ & & & & & & \mathrm{Id}_L \end{pmatrix} \ ,$$

and

$$\Sigma = \mathrm{diag}(\sqrt{N_1}, \ldots, \sqrt{N_M}, \sqrt{2}\mathrm{Id}_K) \ , \qquad \tilde{\Sigma} = \mathrm{diag}(\sqrt{N_1}\mathrm{Id}_{N_1-1}, \ldots, \sqrt{N_M}\mathrm{Id}_{N_M-1}, \sqrt{2}\mathrm{Id}_K, \mathrm{Id}_L) \ .$$

Recall that the rows of U are ordered following the ordering of the system $[\Phi, \Psi]$ while the columns of U are ordered according to $\tilde{\Phi}$. The diagonal matrices Σ and $\tilde{\Sigma}$ are related to Φ and Ψ, respectively. Finally, note that after reordering the basis functions in $\tilde{\Phi}$ with respect to the supporting triangles $\Delta \in \mathcal{T}$, the matrices $G_{\tilde{\Phi},\tilde{\Phi}}$ and $G^{-1}_{\tilde{\Phi},\tilde{\Phi}}$ become block-diagonal, with the diagonal blocks being multiples of fixed matrices, respectively.

For a fixed degree r of polynomial reproduction, the matrix Z can be found from Lemma 2 by choosing suitable subsystems $\Phi_r \subset \Phi$ and bases $\mathcal{P}_r$ in the space $\mathbf{P}_r$ of bivariate polynomials of degree $\leq r$ such that $G_{\mathcal{P}_r,\Phi_r}$ is square and invertible. The Φ_r is chosen in dependence of the corresponding ψ_l which enters the right-hand side of the small linear system (11) to be solved for determining the columns of Z. Clearly, to make the resulting system Λ and the quasi-interpolants (1) interesting for practical applications, the support of the functions in Φ_r should be close to the support of the corresponding functions in Ψ, and the inverse of $G_{\mathcal{P}_r,\Phi_r}$ should depend smoothly on the metric and topological characteristics of the local mesh.

We will look into a particular but generic choice for Φ_1, i.e., for the case $r = 1$ of linear polynomial reproduction, where $\dim \mathbf{P}_1 = 3$. Namely, given any interior triangle Δ, we set

$$\Phi_1 = [\phi_{e_1}, \phi_{e_2}, \phi_{e_3}] \ ,$$

where $e_1, e_2, e_3 \in \mathcal{E}$ denote the 3 interior edges of Δ. Using the affine invariance of both Φ_1 and $\mathbf{P}_1$ we can without loss of generality assume that Δ is equilateral, with area $A = 1$. All the other notation can be found in Figure 5. The area of the triangle Δ_k, attached to Δ along e_k, is denoted by A_k. Let us fix the basis in $\mathbf{P}_1$ by setting $\mathcal{P}_1 = [p_1, p_2, p_3]$ where p_k denotes the extension of the barycentric coordinate function associated with the vertex P_k of the center triangle Δ to all of $\mathbb{R}^2$ which is defined by requiring $p_k \in P_1$ and $p_k(P_l) = \delta_{kl}, \ k, l = 1, 2, 3$.

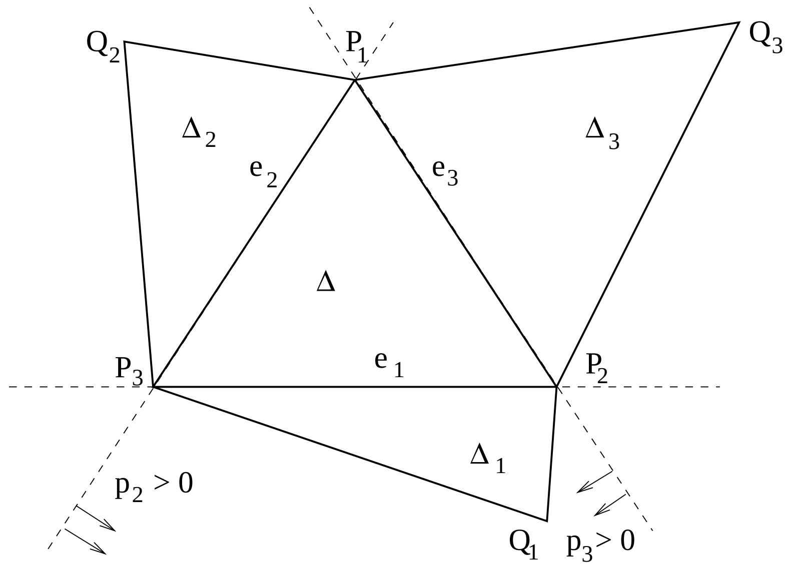

Figure 5: Notation for Lemma 3

Lemma 3 *In addition to the assumptions made, let the triangles in Figure 5 satisfy the following condition: For each $k = 1, 2, 3$, the diagonal $P_k Q_k$ belongs to the closure of the corresponding quadrilateral $\Delta \cup \Delta_k$. Then the determinant of $G_{\mathcal{P}_1,\Phi_1}$ is positive and depends continuously on the location of the Q_k. If the additional geometric assumption is dropped, the matrix $G_{\mathcal{P}_1,\Phi_1}$ may become singular.*

Proof. The proof is based on elementary calculations. We start by stating the formula

$$\int_\Delta \tilde{\phi}_{e_1,\Delta} \cdot \sum_{k=1}^{3} \alpha_k p_k \, dx = \frac{A}{15}(\alpha_1 + 2(\alpha_2 + \alpha_3)) \,,$$

which holds (due to affine invariance of all functions involved) for all triangles. This allows us to compute all scalar products necessary for $G_{\Phi_1,\mathcal{P}_1}$. E.g.,

$$\int_\Omega \tilde{\phi}_{e_1,\Delta} \cdot p_1 \, dx = \int_\Delta \tilde{\phi}_{e_1,\Delta} \cdot p_1 \, dx + \int_{\Delta_1} \tilde{\phi}_{e_1,\Delta} \cdot p_1 \, dx = \frac{A}{15} + \frac{A_1}{15} p_1(Q_1) = \frac{1 - A_1^2}{15}$$

since $p_1(Q_1) = -A_1/A = -A_1$. Since $p_1 + p_2 + p_3 \equiv 1$, we have $p_2(Q_1) + p_3(Q_1) = 1 - p_1(Q_1) = 1 + A_1$ which leads to the ansatz

$$p_2(Q_1) = \frac{1 + A_1 - \epsilon_1}{2} \,, \quad p_3(Q_1) = \frac{1 + A_1 + \epsilon_1}{2} \,,$$

where our geometric assumption implies that $\min(p_2(Q_1), p_3(Q_1)) \geq 0$ or, equivalently, $|\epsilon_1| \leq 1 + A_1$. With this at hand, we compute

$$\int \tilde{\phi}_{e_1,\Delta} \cdot p_2 \, dx = \frac{2A}{15} + \frac{A_1}{15}(2 + p_2(Q_1)) = \frac{4 + 5A_1 + A_1^2 - A_1 \epsilon_1}{30} \,,$$

and, analogously,

$$\int \tilde{\phi}_{e_1,\Delta} \cdot p_3 \, dx = \frac{4 + 5A_1 + A_1^2 + A_1 \epsilon_1}{30} \,.$$

Applying the same analysis to the other rows of $G_{\Phi_1,\mathcal{P}_1}$ and observing that the rows almost completely divide by $(1 + A_k)$, we get the following explicit formula

$$\frac{30^3}{(1 + A_1)(1 + A_2)(1 + A_3)} \det G_{\Phi_1,\mathcal{P}_1} = D \equiv \begin{vmatrix} 2 - 2A_1 & 4 + A_1 - \epsilon_1' & 4 + A_1 + \epsilon_1' \\ 4 + A_2 + \epsilon_2' & 2 - 2A_2 & 4 + A_2 - \epsilon_2' \\ 4 + A_3 - \epsilon_3' & 4 + A_3 + \epsilon_3' & 2 - 2A_3 \end{vmatrix} \,,$$

where $|\epsilon'_k| = A_k|\epsilon_k|/(1+A_k) \le A_k$, $k=1,2,3$ follows from our assumption.

A straightforward calculation reveals that

$$D = 10(3s_2(\mathbf{A}) + 4s_1(\mathbf{A}) + 4 + f(\mathbf{A},\epsilon')) ,$$

where $s_1(\mathbf{x}) = x_1 + x_2 + x_3$, $s_2(\mathbf{x}) = x_1x_2 + x_2x_3 + x_3x_1$ for any $\mathbf{x} \in \mathbb{R}^3$, and

$$f(\mathbf{A},\epsilon') = s_2(\epsilon') + \epsilon'_1(A_2 - A_3) + \epsilon'_2(A_3 - A_1) + \epsilon'_3(A_1 - A_2) .$$

The global minimum of f with respect to the cube $\epsilon'_k \in [-A_k, A_k]$, $k=1,2,3$, is attained on the boundary of this cube, and can be determined easily:

$$f(\mathbf{A},\epsilon') \ge s_2(\mathbf{A}) - 4\max(A_1A_2, A_2A_3, A_3A_1) \ge -3s_2(\mathbf{A})$$

holds for all ϵ'_k of interest. Substitution gives

$$D \ge 40(s_1(\mathbf{A}) + 1) > 40 . \tag{19}$$

since $A_k > 0$, $k=1,2,3$. This shows the assertions of Lemma 3 under the geometric assumptions made. The continuous dependence of the determinant and thus the inverse of $G_{\mathcal{P}_1,\Phi_1}$ on the local topology is obvious.

It remains to provide a counterexample that shows that the above choice for Φ_1 may fail to guarantee the invertibility of $G_{\mathcal{P}_1,\Phi_1}$. Figure 6 contains the counterexample. We claim that if Q_1 is moved to the left, the determinant of $G_{\mathcal{P}_1,\Phi_1}$ will vanish at some point. Indeed, the specification of the example is such that $A = A_1 = A_2 = A_3 = 1$, both Δ and Δ_2 are equilateral (thus, $\epsilon'_2 = 0$), and $\epsilon'_3 = 1$ since Q_3 belongs to the extension of e_1. Thus, according to the above formula, $D = \alpha\epsilon'_1 + \beta$ is a linear function with respect to ϵ'_1, with slope $\alpha = 10(\epsilon'_2 + \epsilon'_3 + A_2 - A_3) = 10$ and $\beta = 250$ (since for $\epsilon'_1 = 0$ the geometric assumption is satisfied and therefore (19) is valid). Thus, moving Q_1 sufficiently far to the left or, equivalently, decreasing ϵ'_1, we finally hit a zero value for D. This proves our claim. A similar, even simpler example is given by $1 = A = A_1 = A_2 > A_3 > 0$, $\epsilon'_2 = \epsilon'_3 = 0$, and Q_1 moving to the left (the slope α in the above formula for D is then $\alpha = 10(A_2 - A_3) > 0$).

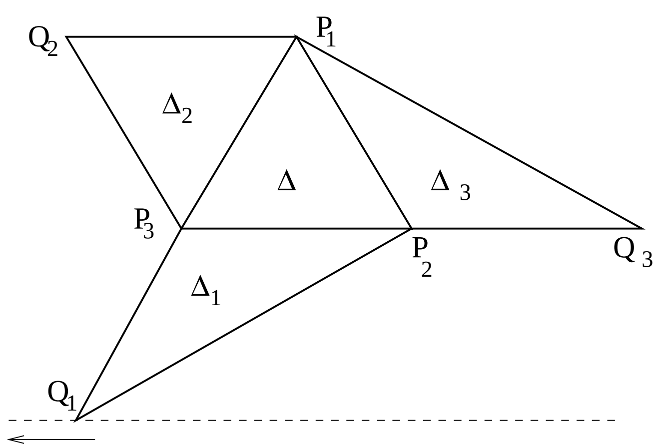

Figure 6: Counterexample with $\det G_{\mathcal{P}_1,\Phi_1} = 0$

Remark 5. The above results indicate that for two-dimensional triangulations the local geometry plays a more subtle role. A triangulation-dependent choice of Φ_1 might be necessary. This needs further research. For the moment being, we can only state that linear polynomials are reproduced by $\tilde{Q}$ if the triangulation satisfies the additional condition of Lemma 3 for any pair of triangles (or almost everywhere). If a coarse triangulation satisfies this condition for all pairs of triangles touching along an edge, and is afterwards refined by uniform dyadic refinement or by bisection, then it obviously preserves this property. Green refinement may lead to "bad pairs" of triangles, diagonal flipping for mesh-improvement is another critical operation. Note that bisection can be used to reduce the number of "bad pairs" of triangles.

Remark 6. In contrast to biorthogonal wavelet systems, the spaces $\tilde{V} = \mathrm{span}\,\Lambda$ spanned by dual systems obtained from our construction are not refinable. I.e., if $\mathcal{T}'$ is a proper refinement of $\mathcal{T}$ we cannot expect to have $\tilde{V} \subset \tilde{V}'$. However, since we still have $W \subset W'$, i.e., refinability for the larger spaces of piecewise polynomials, efficient multilevel computations involving the quasi-interpolants (1) can easily be set up. This was suggested in [7] in connection with extension operators and interface problem preconditioners.

Another possible modification is to start the construction of dual bases with a finite element space W corresponding to a refined partition $\mathcal{T}'$ rather than with the space of non-smooth piecewise polynomials on the same $\mathcal{T}$. This could make the resulting Λ suitable for applications, where higher smoothness of the functions in the dual system is required. However, for use as Lagrange multiplier subspaces of $H^{-1/2}$ in the mortar finite element method this is not essential.

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