Interface preconditioners and multilevel extension operators

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1 Abstract framework

Interface problems arising in nonoverlapping domain decomposition methods can often be viewed as restriction of the original elliptic problem to a discrete trace space. As has been demonstrated in many particular examples, this connection can also be used for the construction of interface preconditioners and extension operators. We present an abstract additive Schwarz framework for this deduction, with applications to multilevel schemes.

We assume familiarity with the theory of stable space splittings and additive Schwarz methods for solving symmetric positive definite (spd) variational problems in a Hilbert space [2, 16, 10, 14]. Let the bilinear forms $a(\cdot,\cdot)$ and $b_j(\cdot,\cdot)$ be bounded and spd on V and V_j, respectively, and $R_j : V_j \to V$ be linear operators. Then we call

$$\{V; a\} \cong \sum_j R_j \{V_j; b_j\} \tag{1}$$

a *stable space splitting*, with stability constants $0 < \mu_1 \le \mu_2 < \infty$, if

$$\mu_1 a(v,v) \le |||v|||^2 := \inf_{v_j \in V_j :\, v = \sum_j R_j v_j} \sum_j b_j(v_j, v_j) \le \mu_2 a(v,v) \quad \forall\, v \in V \ . \tag{2}$$

If $V_j \subset V$ and if R_j represents the natural embeddings then R_j will be omitted. The upper estimate is equivalent to assuming that for any $v \in V$ and any $\epsilon > 0$ we can find a decomposition

$$v = \sum_j R_j v_j \ , \ \ v_j \in V_j \ \ : \quad \sum_j b_j(v_j, v_j) \le (1+\epsilon)\mu_2 a(v,v) \ , \tag{3}$$

while the lower bound in (2) is often reduced to the verification of so-called strengthened Cauchy-Schwarz inequalities [16].

Given a continuous linear operator $T : V \to X$, its range $\hat V \equiv \mathcal{R}(T) \subset X$, can be converted into a Hilbert space if we define a norm on $\hat V$ as follows:

$$\|\hat v\|_{\hat V} = \inf_{v \in V \,:\, \hat v = Tv} \|v\|_V \qquad \forall\, \hat v \in \hat V \ . \tag{4}$$

E.g., if V is a function space on a domain Ω, and T represents a properly defined trace operator to a subset Γ of Ω then this is the common *implicit* definition of the trace norm. The construction of equivalent *explicit* (or *intrinsic*) norms is one of the major problems in the theory of trace spaces. Given a $\hat{v} \in \hat{V}$, the construction of a $v \in V$ with $\hat{v} = Tv$ and $\|v\|_V \le C_{ext}\|\hat{v}\|_{\hat{V}}$ is called *extension problem.* Clearly, one is interested in an as small as possible constant $C_{ext} < \infty$, and an efficient realization of the mapping $E : \hat{v} \to v$.

Suppose that $\hat{a}(\cdot,\cdot)$ is a bounded and spd bilinear form on $\hat{V}$. In particular, this means that there are constants absolute constants $0 < C_1, C_2 < \infty$ such that

$$\hat{a}(Tv,Tv) \le C_1 a(v,v) \qquad \forall\, v \in V \ , \tag{5}$$

and that for any $\hat{v} \in \hat{V}$

$$\exists\, v \in V \ : \quad \hat{v} = Tv \ , \ \ a(v,v) \le C_2 \hat{a}(\hat{v},\hat{v}) \ . \tag{6}$$

For example, if

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{7}$$

represents the block form of a linear system $Ax = f$ with spd matrix A and associated bilinear form $a(x,y) = (Ax,y)$ then the bilinear form $s(x_2,y_2) = (Sx_2,y_2)$ defined by the Schur complement system (resulting after elimination of x_1)

$$Sx_2 = g_2 \ , \qquad S = A_{22} - A_{21}A_{11}^{-1}A_{12} \ , \quad g_2 = f_2 - A_{21}A_{11}^{-1}f_1 \ , \tag{8}$$

satisfies $s(x_2,x_2) = \inf_{x_1} a(x,x)$ by construction. This gives the above assumptions (with constants $C_1 = C_2 = 1$) if we identify V with the Euclidean coordinate space $X = (X_1,X_2)^T$, $\hat{V}$ with X_2, $\hat{a}$ with s, and set $T : x = (x_1,x_2)^T \to x_2$.

We present the assumptions for a generic construction of a stable space splitting for $\{\hat{V};\hat{a}\}$ if a splitting (1) is available for $\{V;a\}$. Suppose that T_j acts on V_j, with range $\hat{V}_j$, and that bounded spd forms $\hat{b}_j$ on $\hat{V}_j$ are introduced (in complete analogy with the introduction of $\hat{V}$ and $\hat{a}$. We assume that all appearing constants are uniform w.r.t. j. In particular,

$$\hat{b}_j(T_jv_j,T_jv_j) \le C_3 b_j(v_j,v_j) \qquad \forall\, v_j \in V_j \ , \tag{9}$$

for some C_3, independently of j. Let $E_j : \hat{V}_j \to V_j$ be right inverses of T_j, $T_jE_j = \mathrm{Id}_{\hat{V}_j}$, such that

$$b_j(E_j\hat{v}_j, E_j\hat{v}_j) \le C_4 \hat{b}_j(\hat{v}_j,\hat{v}_j) \qquad \forall\, v_j \in V_j \ , \tag{10}$$

for some C_4, independently of j. The last condition is on the kernels of the operators T_j and TR_j: For all j,

$$\mathcal{N}(T_j) \subset \mathcal{N}(TR_j) \ . \tag{11}$$

Theorem 1 *Under the assumptions (5), (6), (9), (10), and (11), the stability of (1) implies the stability of the space splitting*

$$\{\hat{V};\hat{a}\} \cong \sum_j \hat{R}_j\{\hat{V}_j;\hat{b}_j\} \qquad (\ \hat{R}_j = TR_jE_j\ )\ , \tag{12}$$

with stability constants $\hat{\mu}_1, \hat{\mu}_2$ *satisfying*

$$0 < (C_1C_4)^{-1}\mu_1 \le \hat{\mu}_1 \le \hat{\mu}_2 \le C_2C_3\mu_2 < \infty\ . \tag{13}$$

Proof. Let $\hat{v} = \sum_j \hat{R}_j\hat{v}_j$ be any representation of $\hat{v}$. Set $v_j = E_j\hat{v}_j$ and $v = \sum_j R_jv_j$. Obviously,

$$Tv = \sum_j TR_jv_j = \sum_j TR_jE_j\hat{v}_j = \sum_j \hat{R}_j\hat{v}_j = \hat{v}\ ,$$

and by (5), (10), and the lower estimate in (2), we have

$$\begin{aligned}\hat{a}(\hat{v},\hat{v}) &= \hat{a}(Tv,Tv) \le C_1a(v,v) \le C_1\mu_1^{-1}\sum_j b_j(v_j,v_j)\\ &= C_1\mu_1^{-1}\sum_j b_j(E_j\hat{v}_j,E_j\hat{v}_j) \le C_1C_4\mu_1^{-1}\sum_j \hat{b}_j(\hat{v}_j,\hat{v}_j)\ .\end{aligned}$$

This gives the lower stability estimate for (12).

In the other direction, take a v with $\hat{v} = Tv$ satisfying (6), and decompose it optimally in the sense of (3) using the upper stability estimate in (2). Set $u_j = E_jT_jv_j$, $\hat{v}_j = T_jv_j$, and observe that $T_j(u_j - v_j) = (T_jE_j - \mathrm{Id}_{\hat{V}_j})T_jv_j = 0$. Thus, by (11) $0 = TR_j(u_j - v_j) = TR_jE_jT_jv_j - TR_jv_j = \hat{R}_j\hat{v}_j - TR_jv_j$. This implies

$$\hat{v} = Tv = \sum_j TR_jv_j = \sum_j \hat{R}_j\hat{v}_j\ ,$$

and, taking into account also (9),

$$\begin{aligned}\sum_j \hat{b}_j(\hat{v}_j,\hat{v}_j) &= \sum_j \hat{b}_j(T_jv_j,T_jv_j) \le C_3\sum_j b_j(v_j,v_j)\\ &\le (1+\epsilon)C_3\mu_2a(v,v) \le (1+\epsilon)C_2C_3\mu_2\hat{a}(\hat{v},\hat{v})\ .\end{aligned}$$

Letting $\epsilon \to 0$ finishes the proof of Theorem 1. Note that we could have replaced (11) by the slightly weaker condition $\mathcal{R}(\mathrm{Id}_{V_j} - E_jT_j) \subset \mathcal{N}(TR_j)$.

Corollary 2 *Under the assumptions of Theorem 1, given an arbitrary* $\hat{v} \in \hat{V}$*, let* $\hat{v}_j \in \hat{V}_j$ *be such that the analog of (3) holds:*

$$\hat{v} = \sum_j \hat{R}_j\hat{v}_j\ :\quad \sum_j \hat{b}_j(\hat{v}_j,\hat{v}_j) \le C_5\hat{a}(\hat{v},\hat{v})\ . \tag{14}$$

Then a suitable extension $u = E\hat{v} \in V$ *of* $\hat{v}$ *satisfying* $\hat{v} = Tu$ *is given by*

$$u = \sum_j R_j E_j \hat{v}_j \ : \quad a(u,u) \le C_4 C_5 \hat{a}(\hat{v},\hat{v}) \le C_1 C_4 C_5 \inf_{v \in V \,:\, \hat{v} = Tv} a(v,v) \ . \tag{15}$$

If $\hat{v}_j = \hat{Q}_j \hat{v}$ *for some linear operators* $\hat{Q}_j \ : \ \hat{V} \to \hat{V}_j$ *then the extension operator* $E = \sum_j R_j E_j Q_j \ : \ \hat{V} \to V$ *is linear and bounded.*

A particular instance, where this abstract framework for constructing additive Schwarz preconditioners and extension operators can be applied, are *multilevel splittings* based on a hierarchy of spaces

$$V_0 \xrightarrow{I_1} V_1 \xrightarrow{I_2} \dots \xrightarrow{I_J} V_J \xrightarrow{I_{J+1}} \dots \ , \tag{16}$$

where $I_j \ : \ V_{j-1} \to V_j$ are given prolongation operators acting between subsequent spaces of the hierarchy (again, nestedness $V_{j-1} \subset V_j$ is not assumed). Setting $V = V_J$ and assuming that $a = a_J$ is a bounded and spd bilinear form on V_J, the associated multilevel splitting is given by

$$\{V_J; a_J\} \cong \sum_{j=0}^{J} R_j^J \{V_j; b_j\} \ , \qquad R_j^k = I_k \dots I_{j+1} \ , \ \ 0 \le j < k \ . \tag{17}$$

Suppose that (17) is stable, uniformly w.r.t. J. Assume that the sequence T_j is given, and that $T = T_J$. Furthermore, let (5), (6) hold with $a, \hat{a}$ replaced by $a_J, \hat{a}_J$ and with constants independent of J. We also keep requirements (9), (10), while (11) is replaced by the existence of another set of linear operators

$$\hat{I}_j \ : \ \hat{V}_{j-1} \to \hat{V}_j \ : \quad T_j I_j = \hat{I}_j T_{j-1} \ , \quad j \ge 1 \ . \tag{18}$$

This new requirement implies (11) since

$$T_J R_j^J = (T_J I_J) I_{J-1} \dots I_{j+1} = \hat{I}_J (T_{J-1} I_{J-1}) \dots I_{j+1} = \dots = \hat{I}_J \dots \hat{I}_{j+1} T_j \ .$$

Thus, Theorem 1 and Corollary 2 are applicable, leading to splittings and extension operators for $\{\hat{V}_J; \hat{a}_J\}$ and J-independent constants in the estimates. Moreover, by the same reasoning we obtain

$$\hat{R}_j^J = T_J R_j^J E_j = \hat{I}_J \dots \hat{I}_{j+1} T_j E_j = \hat{I}_J \dots \hat{I}_{j+1} \ , \quad 0 \le j < J < \infty \ ,$$

i.e., the recursive definition of the R_j^J carries over to the new embedding operators $\hat{R}_j^J$. This is important for the efficient implementation of additive and multiplicative Schwarz methods for solving variational problems associated with the ladder

$$\hat{V}_0 \xrightarrow{\hat{I}_1} \hat{V}_1 \xrightarrow{\hat{I}_2} \dots \xrightarrow{\hat{I}_J} \hat{V}_J \xrightarrow{\hat{I}_{J+1}} \dots \tag{19}$$

and the bilinear forms $\hat{a}_J$.

Corollary 3 *Under the assumptions made, the multilevel splittings associated with (19),*

$$\{\hat{V}_J;\hat{a}_J\} \cong \sum_{j=0}^{J} \hat{R}_j^J\{\hat{V}_j;\hat{b}_j\}\ , \qquad \hat{R}_j^k = \hat{I}_k \dots \hat{I}_{j+1}\ ,\ \ 0 \le j < k\ , \tag{20}$$

are stable, with uniformly bounded stability constants.

2 An application

Applications of Theorem 1 for preconditioning Schur complement problems in substructuring methods for finite element discretizations of second order elliptic boundary value problems are well-known (see [11], [17, section 2.3 and 7.1], [7] for some special cases and further references). Thess [15] has recently worked on applications to fourth order plate and shell problems. The Stokes problem has been treated in [6].

To illustrate how the results of the above abstract framework apply, we will derive a discrete harmonic multilevel extension operator of optimal complexity for linear finite element boundary data on locally refined meshes (for similar results on preconditioning, see [8]). The main assumption on the admissible local refinement process is, roughly speaking, as follows: The refinement on the $(d-1)$-dimensional boundary Γ of Ω is inherited from an analogous refinement process for the d-dimensional domain Ω. For simplicity, let Ω be a bounded polyhedral domain in $\mathbf{R}^d$, $d \ge 2$, equipped with a nested sequence of regular, quasi-uniform simplicial partitions $\mathcal{T}_j$ of element diameter $\approx 2^{-j}$, $j \ge 0$. Let $\tilde{V}_j$ be the space of linear finite elements w.r.t. $\mathcal{T}_j$, the standard basis function in $\tilde{V}_j$ associated with a vertex $P_{j,i}$ of $\mathcal{T}_j$ is denoted by $N_{j,i}$. *Nested refinement* is modelled by selecting a finite increasing sequence

$$\emptyset = \Omega'_0 \subset \Omega'_1 \subset \dots \Omega'_J \subset \Omega'_{J+1} = \Omega\ , \tag{21}$$

where each Ω'_j, $1 \le j \le J$, is or empty or the closure of the union of some simplices from $\mathcal{T}_{j-1}$. The sequence $\Omega_j = \mathrm{clos}(\Omega\backslash\Omega'_j)$ is decreasing and, roughly speaking, represents the simplices of $\mathcal{T}_{j-1}$ which are refined at level j. The spaces V_j are spanned by all basis functions $N_{j,i}$ for which $P_{j,i}$ belongs to the interior of Ω_j (relative to Ω). As the finite element space V corresponding to this refinement process we take $V = \sum_{j=0}^{J} V_j$. It is easy to see that an algebraic basis $\mathcal{N}$ for V is given by all $N_{j,i} \in V_j\backslash V_{j+1}$, $j \le J$. The construction is schematically illustrated for square partitions and bilinear finite element spaces $(d = 2)$ in Figure 1. The refinement regions Ω_j are indicated by a dashed line. Since there is a one-to-one relationship between the basis functions $N_{j,i}$ and nodal points (vertices) in the corresponding partition, we have chosen to indicate the bases in V_j and the basis functions in $\mathcal{N}$ by marking the nodal points, in dependence on the level number j. Although V is not a traditional finite element space associated with an triangulation, its approximation power on each domain Ω_j

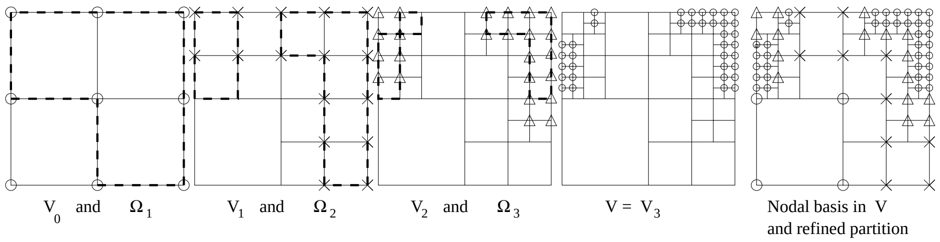

Figure 1: Nested refinement: Regions Ω_j and nodal points of bases in V_j and V $(J=3)$

is almost as good as that of $\tilde{V}_j$ (and less good on Ω'_j). Uniform refinement corresponds to the case $\Omega'_j=\emptyset$, $0\le j\le J$. Infinite-dimensional situations are covered by setting $J=\infty$.

This construction has been discussed in [10, section 4.2.3] and (without proofs and in a slightly restricted form) in [12]. Since the exposition in [10] contains some index errors which were brought to our attention by S. V. Nepmnyaschikh, we decided to repeat some details. Along the lines of [10, section 4.2.3] it follows that

$$\{V;(\cdot,\cdot)_{H^1(\Omega)}\}\cong\sum_{j=0}^{J}\{V_j,2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\}\cong\sum_{j=0}^{J}\sum_{i\,:\,N_{j,i}\in V_j}\{V_{j,i},2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\}\ , \tag{22}$$

where $V_{j,i}$ are the one-dimensional spaces spanned by individual $N_{j,i}$, and with bounds for the stability constants that are independent on J and $\{\Omega'_j\}$. The second, refined splitting (involving the $V_{j,i}$) follows from the first one by simply using the L_2-stability of the finite element bases (this fact will be used below without further mentioning). The stability of the splitting (22) expresses nothing but the optimality of BPX multilevel preconditioning in the nested refinement case, and can be derived from the corresponding result in the uniform refinement case

$$\{\tilde{V}_J;(\cdot,\cdot)_{H^1(\Omega)}\}\cong\sum_{j=0}^{J}\{\tilde{V}_j,2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\}\ ,\quad J\ge 0\ , \tag{23}$$

(see [10, Theorem 19]) by quasi-interpolant techniques. The latter are also useful for the efficient implementation of good multilevel decompositions of finite element functions. To derive (22) from (23) we construct a sequence of linear, uniformly L_2-bounded projections $P_j\,:\,L_2(\Omega)\to\tilde{V}_j$ such that $Q_j=P_j-P_{j-1}$ maps V into V_j, $0\le j\le J$ (set $P_{-1}=0$). We use specific quasi-interpolants: Set

$$P_jv(x)=\sum_i(v,\lambda_{j,i})_{L_2(\Omega)}N_{j,i}(x)\ , \tag{24}$$

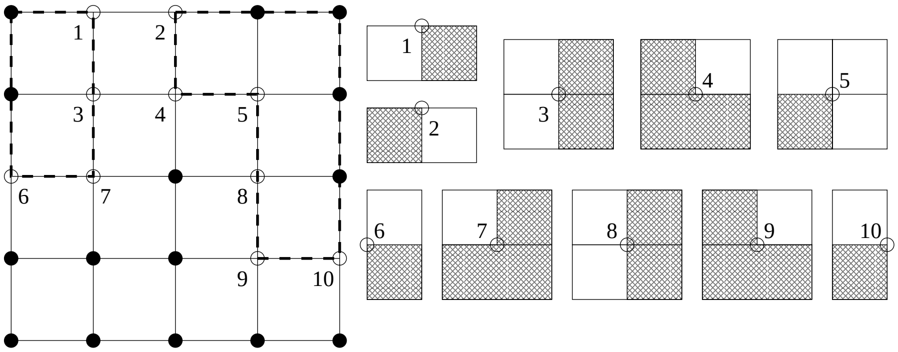

Figure 2: Definition of $\Lambda_{j,i}$ with $j = 1$ for the refinement process of Figure 1

where the functions $\lambda_{j,i}$ are piecewise linear (but not continuous!) on the simplices of $\mathcal{T}_j$ and have support $\Lambda_{j,i}$ consisting of a few simplices attached to $P_{j,i}$. More precisely, we define

$$\Lambda_{j,i} = \begin{cases} \operatorname{supp} N_{j,i} & \text{if } \operatorname{supp} N_{j,i} \subset \Omega_{j+1} \\ \operatorname{supp} N_{j,i} \cap \Omega'_{j+1} & \text{otherwise} \end{cases} . \tag{25}$$

Since, by definition, the sets Ω'_{j+1}, Ω_{j+1} are the closure of certain unions of simplices from $\mathcal{T}_j$, any $\Lambda_{j,i}$ contains at least one simplex of $\mathcal{T}_j$ completely. To ensure the projection property w.r.t. V_j, we define

$$\lambda_{j,i}(x) = (N_{j,i}, 1)^{-1}_{L_2(\Lambda_{j,i})} \psi_{j,i}(x) , \quad x \in \Lambda_{j,i} , \tag{26}$$

where the piecewise linear function $\psi_{j,i}$ has value $d + 1$ at $P_{j,i}$, and -1 at all other vertices of simplices attached to $P_{j,i}$. This definition is illustrated (again for $d = 2$ and the square refinement of Figure 1) in Figure 2. The dashed regions schematically show the $\Lambda_{j,i}$ for the enumerated nodal points while for all other vertices $\Lambda_{j,i} = \operatorname{supp} N_{j,i}$.

The reader can easily verify by using barycentric coordinates that

$$(N_{j,l}, \lambda_{j,i})_{L_2(\Omega)} = \delta_{i,l} , \qquad \forall\, i, l ,$$

which ensures the projection property of P_j w.r.t. $\tilde{V}_j$. Moreover, we have

$$|(v, \lambda_{j,i})_{L_2(\Omega)}|^2 \le C 2^{jd} \|v\|^2_{L_2(\Lambda_{j,i})} ,$$

which implies the uniform L_2-boundedness of the P_j:

$$\|P_j v\|^2_{L_2(\Omega)} \le C 2^{-jd} \sum_i |(v, \lambda_{j,i})_{L_2(\Omega)}|^2 \le C \sum_i \|v\|^2_{L_2(\Lambda_{j,i})} \le C(d+1) \|v\|^2_{L_2(\Omega)} .$$

Finally, if $v \in V$ then v is a linear finite element function w.r.t. $\mathcal{T}_{j-1}$ outside Ω_j. Thus, by our choice of the $\Lambda_{j,i}$, both P_j and P_{j-1} reproduce v on Ω'_j exactly.

This gives $Q_j v = 0$ on this set and since $Q_j v \in \tilde{V}_j$ we arrive at $Q_j v \in V_j$ for any $v \in V$.

Now, from the stability of the splitting (23) we have

$$\mu_1 \|\tilde{u}_J\|^2_{H^1(\Omega)} \leq \inf_{\tilde{v}_j \in \tilde{V}_j \,:\, \tilde{u}_J = \sum_{j=0}^J \tilde{v}_j} \sum_{j=0}^{J} 2^{2j} \|\tilde{v}_j\|_{L_2(\Omega)} \leq \mu_2 \|\tilde{u}_J\|^2_{H^1(\Omega)} \ . \tag{27}$$

Since $V_j \subset \tilde{V}_j$ and $V \subset \tilde{V}_J$, the lower estimate in (27) immediately gives the lower stability estimate for (22):

$$\mu_1 \|v\|^2_{H^1(\Omega)} \leq \inf_{v_j \in V_j \,:\, v = \sum_{j=0}^J v_j} \sum_{j=0}^{J} 2^{2j} \|v_j\|_{L_2(\Omega)} \ .$$

for all $v \in V$, with the same constant μ_1.

Let us derive the opposite inequality (with a possibly different constant μ_2). According to the upper estimate in (27), for any given $v \in V \subset \tilde{V}_J$ we can choose a representation $v = \sum_{j=0}^J \tilde{v}_j$ with $\tilde{v}_j \in \tilde{V}_j$ such that

$$\sum_{j=0}^{\infty} 2^{2j} \|\tilde{v}_j\|^2_{L_2(\Omega)} \leq 2\mu_2 \|v\|^2_{H^1(\Omega)} \ .$$

Now, since the P_j are projections onto $\tilde{V}_j$ and $V \subset \tilde{V}_J$ we have

$$v = \sum_{l=0}^{J} Q_l v \quad \left(= \sum_{l=0}^{J} \sum_{j=0} Q_l \tilde{v}_j \right) \ .$$

Moreover, we have $Q_l \tilde{v}_j = 0$ if $j < l$, and therefore

$$\|Q_l v\|^2_{L_2(\Omega)} \leq (\sum_{j=l}^{J} \|Q_l \tilde{v}_j\|_{L_2(\Omega)})^2 \leq C_\epsilon \sum_{j=l}^{J} 2^{\epsilon(j-l)} \|\tilde{v}_j\|^2_{L_2(\Omega)} \ ,$$

where $\epsilon > 0$ (here, the uniform L_2 boundedness of the P_l has been used). Fixing any $\epsilon \in (0, 2)$, we finally get

$$\begin{aligned} \inf_{v_j \in V_j \,:\, v = \sum_{j=0}^J v_j} \sum_{j=0}^{J} 2^{2j} \|v_j\|_{L_2(\Omega)} &\leq \sum_{l=0}^{J} 2^{2l} \|Q_l v\|^2_{L_2(\Omega)} \\ &\leq C \sum_{j=0}^{J} 2^{\epsilon j} \|\tilde{v}_j\|^2_{L_2(\Omega)} \sum_{l=0}^{j} 2^{(2-\epsilon)l} \\ &\leq \sum_{j=0}^{J} 2^{2j} \|\tilde{v}_j\|^2_{L_2(\Omega)} \leq 2C\mu_2 \|v\|^2_{H^1(\Omega)} \ . \end{aligned}$$

This concludes the proof of (22).

Let us now consider the extension problem (a typical application might consist in determining a proper partial solution satisfying inhomogeneous Dirichlet boundary data for iteratively solving a second order elliptic problem, or the construction of a discrete harmonic extension operator for use in domain decomposition or mortar problems). Let Γ be the boundary of Ω, and set $\hat{V} = V|_\Gamma$ (note that everything extends to polyhedral manifolds Γ consisting of $(d-1)$-dimensional faces of simplices in $\mathcal{T}_0$). Obviously, $\hat{V} = \sum_{j=0}^{J} \hat{V}_j$, where $\hat{V}_j \equiv V_j|_\Gamma \subset \hat{\tilde{V}}_j \equiv \tilde{V}_j|_\Gamma$. Note that the sequence $\{\hat{V}_j\}$ could have been derived by a nested refinement procedure on Γ similar to the one described above (consider the triangulations $\hat{\mathcal{T}}_j$ induced from $\mathcal{T}_j$ on Γ, and use $\hat{\Omega}'_j = \Omega'_j \cap \Gamma$ for the increasing sequence (21)). A basis in $\hat{V}_j$ is given by all non-vanishing

$$\hat{N}_{j,i} = N_{j,i}|_\Gamma \ , \quad N_{j,i} \in V_j \ .$$

Accordingly, a basis $\hat{\mathcal{N}}$ in $\hat{V}$ is given by all $\hat{N}_{j,i} \in \hat{V}_j \backslash \hat{V}_{j+1}$, $j \le J$.

Let $T : V \to \hat{V}$, $T_{j,i} : V_{j,i} \to \hat{V}_{j,i}$ coincide with the restriction of the trace operator $|_\Gamma$ to the corresponding spaces (if $P_{j,i} \notin \Gamma$ then $\hat{V}_{j,i}$ is the null space and $T_{j,i}$ the null operator). Since the operators $R_{j,i}$ represent the natural embeddings for $V_{j,i} \subset V$, condition (11) is automatically fulfilled. Define extension operators $E_{j,i} : \hat{V}_{j,i} \to V_{j,i}$ by simply setting $E_{j,i}(c\hat{N}_{j,i}) = cN_{j,i}$. Combining the $E_{j,i}$ with j fixed leads to the usual level-j extension-by-zero operator $E_j : \hat{V}_j \to V_j$. Obviously, the assumptions (9), (10) are satisfied if we set

$$b_{j,i}(u,v) = 2^{2j}(u,v)_{L_2(\Omega)} \ , \qquad \hat{b}_{j,i}(\hat{u},\hat{v}) = 2^{j}(\hat{u},\hat{v})_{L_2(\Gamma)} \ .$$

From the first part of Theorem 1 it follows that

$$\{\hat{V}; (\cdot,\cdot)_{\hat{V}}\} \cong \sum_{j=0}^{J} \{\hat{V}_j, 2^j(\cdot,\cdot)_{L_2(\Gamma)}\} \cong \sum_{j=0}^{J} \sum_{i\,:\,\hat{N}_{j,i}\in\hat{V}_j} \{\hat{V}_{j,i}, 2^j(\cdot,\cdot)_{L_2(\Gamma)}\} \tag{28}$$

is also a stable splitting.

The next steps are as follows. First, it can be shown that

$$\|\hat{v}\|_{\hat{V}} \approx \|\hat{v}\|_{H^{1/2}(\Gamma)} \equiv \inf_{u\in H^1(\Omega)\,:\,v=Tu} \|u\|_{H^1(\Omega)} \qquad \forall\, \hat{v} \in \hat{V} \ . \tag{29}$$

This relates the $\|\cdot\|_{\hat{V}}$ norm defined in (4) to the norm in the trace space $H^{1/2}(\Gamma)$ for $H^1(\Omega)$ functions (intrinsic $H^{1/2}(\Gamma)$ norms are described, e.g., in [7, 16]). By definition of $\|\cdot\|_{\hat{V}}$, we obviously have $\|\hat{v}\|_{H^{1/2}(\Gamma)} \le \|\hat{v}\|_{\hat{V}}$. The proof of the opposite estimate uses the same techniques as were applied above to derive (22) from the standard BPX-splitting for the whole space $\{H^1(\Omega); (\cdot,\cdot)_{H^1}\}$. The necessary quasi-interpolant operators

$$\hat{P}_j\hat{v}(x) = \sum_i (\hat{v}, \hat{\lambda}_{j,i})_{L_2(\Gamma)} \hat{N}_{j,i}(x) \ , \tag{30}$$

and $\hat{Q}_j : \hat{V} \to \hat{V}_j$ are now defined w.r.t. the refinement process $\{\hat{\Omega}_j\}$ on Γ in exactly the same way as given before (see the definitions following (24)) for the refinement process $\{\Omega_j\}$ on Ω. We leave it upon the reader to repeat the argument (note the change from 2^{2j} to 2^j in the forefactors, this restricts the choice of ϵ in the above proof to the interval $0 < \epsilon < 1$).

Secondly, in order to define an extension operator $E : \hat{V} \to V$ along the lines of Corollary 2, we need a realization of (14) for which we propose to use the quasi-interpolants of the previous step. Indeed, we have

$$\hat{v} = \sum_{j=0}^{J} \hat{Q}_j \hat{v} \qquad \forall\, \hat{v} \in \hat{V} ,$$

with $\hat{Q}_j \hat{v} \in \hat{V}_j$ for all $0 \le j \le J$, and

$$\sum_{l=0}^{J} 2^l \|\hat{Q}_l \hat{v}\|^2_{L_2(\Gamma)} \le C \|\hat{v}\|^2_{H^{1/2}(\Gamma)}$$

from the argument in the first step. Let us denote by $\hat{Q}_{j,i}$ the resulting linear mappings from $\hat{V}$ into the one-dimensional $\hat{V}_{j,i}$ corresponding to the basis functions $\hat{N}_{j,i}$ (these are uniquely defined from the requirement $\hat{Q}_j = \sum_i \hat{Q}_{j,i}$).

By Corollary 2, we arrive at

Corollary 4 *Under the above assumptions on the refinement process, the extension operator*

$$E = \sum_{j=0}^{J} E_j \hat{Q}_j = \sum_{j=0}^{J} \sum_i E_{j,i} \hat{Q}_{j,i} \ : \ \hat{V} \to V$$

satisfies

$$\|E\hat{v}\|_{H^1(\Omega)} \le C \|\hat{v}\|_{H^{1/2}(\Gamma)} , \quad \hat{v} \in \hat{V} , \tag{31}$$

where the constant C is independent of the particular refinement process. The application of the operator E to an arbitrary $\hat{v} \in \hat{V}$ can be implemented with essentially $\mathrm{O}(\dim \hat{V})$ *arithmetical operations and storage, provided that appropriate representations are chosen in $\hat{V}$ and V.*

What concerns the statement about the efficient computation with this extension operator, we make the following comments. First of all, answers slightly depend on how finite element functions are represented in V and $\hat{V}$. In both cases, we could use the bases $\mathcal{N}$, $\hat{\mathcal{N}}$ mentioned above or (allowing for some non-uniqueness in the representation) slightly larger generating systems

$$\mathcal{R} = \{N_{j,i} \ : \ N_{j,i} \in V_j \, , \, j \le J\} \, , \quad \hat{\mathcal{R}} = \{\hat{N}_{j,i} \ : \ \hat{N}_{j,i} \in \hat{V}_j \, , \, j \le J\} \, .$$

It is easy to show that $\#\mathcal{R} \le C\#\mathcal{N}$ and $\#\hat{\mathcal{R}} \le \#\hat{\mathcal{N}}$, with C independent of the refinement process. The operator E naturally maps $\hat{v}$ (given in either $\hat{\mathcal{R}}$

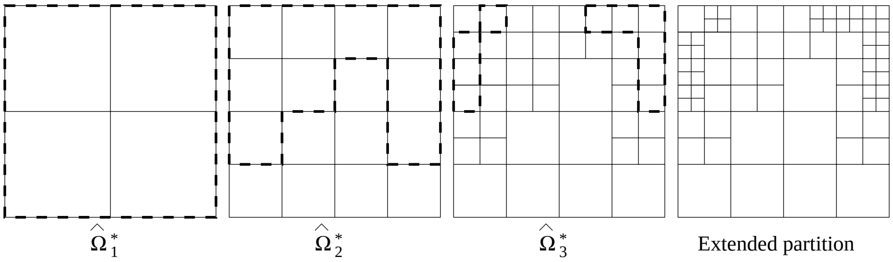

Figure 3: Definition of $\hat{\Omega}_j^*$ with $j = 1$ for the refinement process of Figure 1

or $\hat{\mathcal{N}}$ representation) into a coefficient vector w.r.t. a subset of $\mathcal{R}$ which is of the same size as $\hat{\mathcal{R}}$, and we will show that it can be implemented in $\mathrm{O}(\#\hat{\mathcal{R}})$ operations. Thus, except for the conversion of $E\hat{v}$ into a representation w.r.t. $\mathcal{N}$, storage and amount of work associated with E are proportional to $\dim \hat{V}$.

To substantiate this claim, we provide an efficient computational scheme for $\hat{Q}_j\hat{v}$. The basic idea is to switch to yet another representation of finite element functions, this time w.r.t. the standard basis of picewise linear (and not necessarily continous) functions w.r.t. the given nested refinement structure. Set $\hat{\mathcal{B}} = \cup_{j=0}^{J}\hat{\mathcal{B}}_j$, where $\hat{\mathcal{B}}_j$ consists of all piecewise linear functions $\hat{\psi}_{\Delta,P}$ with support in exactly one simplex $\Delta \in \hat{\mathcal{T}}_j$ from a yet to be defined set $\hat{\Omega}_j^* \supset \hat{\Omega}_j$, and having value 1 at the vertex P of Δ and vanishing at all other vertices. Minimally, a suitable $\hat{\Omega}_j^*$ should contain $\hat{\Omega}_j$ and possibly an additional layer of $\hat{\mathcal{T}}_{j-1}$ simplices containing the supports of all $\hat{N}_{j,i}$ whose supports intersect with $\hat{\Omega}_{j+1}$ (the reason for this will become clear below). Obviously, the size of $\hat{\mathcal{B}}$ is still within a constant multiple of $\dim \hat{V}$. We also need the subsets $\hat{\mathcal{B}}_j' \subset \hat{\mathcal{B}}_j$ of all those functions $\hat{\psi}_{\Delta,P}$ for which $\Delta \in \hat{\Omega}_j^*\backslash\hat{\Omega}_{j+1}^*$ (again, $\hat{\Omega}_{J+1}^* = \emptyset$, and $\Omega_0^* = \Omega$). Figure 3 shows the sets $\hat{\Omega}_j^*$ for the refinement of Figure 3 (assuming that this figure corresponds to a refinement process on Γ, i.e., shows the $\hat{\Omega}_j$ rather than Ω_j). The standard basis in the space of discontinuous, piecewise bilinear functions w.r.t. the final, extended partition shown in Figure 3 determines the basis $\mathcal{B}'$.

The unique coefficients of $\hat{v} \in \hat{V}$ w.r.t. $\hat{\mathcal{B}}' = \cup_{j=0}^{J}\hat{\mathcal{B}}_j'$ can be computed fast, from either the $\hat{\mathcal{R}}$ or the $\hat{\mathcal{N}}$ representation, by recursion in $j = 0, 1, \ldots, J$. After this, by recursion in $j = J, J-1, \ldots, 0$, all scalar products $\int_\Gamma \hat{v}\hat{\psi}_{\Delta,P}$ associated with functions $\hat{\psi}_{\Delta,P} \in \hat{\mathcal{B}}_j$ can be computed in an amount of operations which stays proportional to $\#\hat{\mathcal{B}}$. Indeed, for a level j simplex $\Delta \in \hat{\Omega}_j^*\backslash\hat{\Omega}_{j+1}^*$ this reduces to a multiplication with a local Gram matrix, while for $\Delta \subset \hat{\Omega}_{j+1}^*$ the refinability of the basis functions $\hat{\psi}_\Delta$ is used: each such function can be expressed as a linear combination of a fixed finite number of functions $\hat{\psi}_{\Delta',P'}$ associated with $\Delta' \in \hat{\mathcal{T}}_{j+1}$ in $\hat{\Omega}_{j+1}^*$ for which the corresponding integrals have

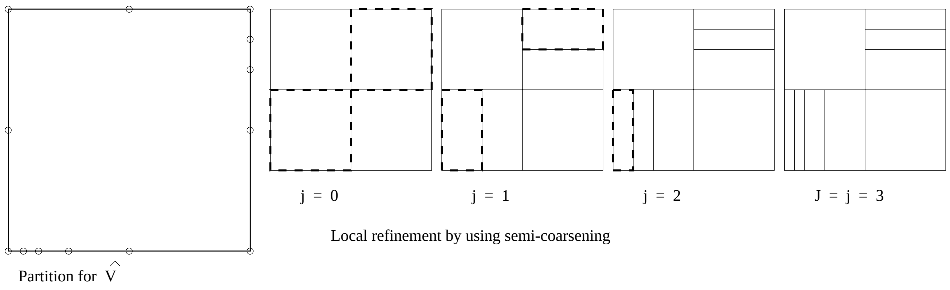

Figure 4: Nested anisotropic refinement

been computed before. Finally, by formula (30) this gives the coefficient vector of $\hat{Q}_j\hat{v} = \hat{P}_j\hat{v} - \hat{P}_{j-1}\hat{v}$ in the standard basis of $\hat{V}$ if we express the $\hat{N}_{j,i}$ as the corresponding finite linear combination of $\hat{\psi}_\Delta$ and recall that due to $\hat{P}_j\hat{v} = \hat{P}_{j-1}\hat{v}$ outside $\hat{\Omega}_j$ only terms with $\operatorname{supp}\hat{N}_{j,i} \subset \hat{\Omega}_j$ and $\hat{N}_{j-1,i} \not\equiv 0$ on $\hat{\Omega}_j$ are to be considered (here, the special definition of $\hat{\Omega}_j^*$ is used). Concatenating all these vectors results in the coefficients of $E\hat{v}$ w.r.t. $\mathcal{R}$ (coefficients corresponding to functions $N_{j,i} \in \mathcal{R}$ with $N_{j,i}|_\Gamma \equiv 0$ are set to 0).

We conclude with the following remarks.

- Not all possible boundary partitions fit into the above scheme. E.g., Figure 4 a) shows an *anisotropic refinement* on a rectangle ($d = 2$) which cannot be represented as the trace of a refinement process on the rectangle as allowed in our assumptions (to see this, observe that the refinement pattern near the lower-left corner contradicts the regularity assumptions for the underlying $\mathcal{T}_j$). However, interface preconditioners and discrete harmonic extension operators can still be derived if the underlying BPX-splitting (23) is replaced by splittings with respect to a hierarchy of spaces obtained by both full and semi-coarsening (see [3]). The anisotropic refinement in Ω corresponding to the boundary refinement on Γ of Figure 4 a) is indicated in Figure 4 b).

- Essentially the same techniques apply to other trace operators and extension problems. E.g., using the stability of multilevel splittings for C^1 finite element spaces based on the piecewise quadratic Powell-Sabin macroelement [9] or an analogous piecewise cubic C^1-macroelement [1], interface problems associated with plate bending can be treated. The case of the rectangular Bogner-Fox-Schmit element has been considered in [15].

- Using the link between the biharmonic problem and the two-dimensional Stokes problem via stream functions, extension operators into spaces of discretely divergence-free vector fields can also be obtained. For the above mentioned Powell-Sabin element and certain velocity spaces using conforming or nonconforming piecewise linear finite elements, one could follow [13]. For an alternative approach, see [6]. It is intriguing to look at the consequences of our approach in connection with the multilevel splittings for **H**(div) and **H**(**curl**) spaces proposed by Hiptmair [4, 5].

- In section 1, we have included a general set of prolongations (beyond the use of natural embeddings) into the construction. This might be of interest for nonconforming finite element spaces and other situations where the auxiliary spaces of the space splittings are not subspaces of the computational space under consideration. We leave this option open for further research.

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