L_∞-bounds for the L_2-projection onto linear spline spaces

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Dedicated to Kostja Oskolkov

Abstract

In the univariate case, the L_2-orthogonal projection P_V onto a spline space V of degree k is bounded as an operator in L_∞ by a constant $C(k)$ depending on the degree k but independent of the knot sequence [14]. By earlier results [3, 8, 9], in the case of linear splines the sharp bound is $\|P_V\|_{L_\infty \to l_\infty} < 3$. As was shown more recently [10], the L_2-orthogonal projection P_V onto spaces $V = V(\mathcal{T})$ of linear splines over triangulations $\mathcal{T}$ of a bounded polygonal domain in $\mathbb{R}^2$ cannot be bounded in L_∞ by a constant that is independent of the underlying triangulation. Similar counterexamples show this for higher dimensions as well.

In this note we state a new geometric condition on families of triangulations under which uniform boundedness of $\|P_V\|_{L_\infty \to L_\infty}$ can be guaranteed. It covers certain families of triangular meshes of practical interest, such as Shishkin and Bakhvalov meshes. On the other hand, we show that even for type-I triangulations of a rectangular domain uniform boundedness of P_V in L_∞ cannot be established.

Keywords Linear finite elements, splines, least-squares methods, L_2-orthogonal projection, L_∞ bounds, Shishkin and Bakhvalov meshes

Mathematics Subject Classification (2000) 65N30, 41A15

1 Introduction

The study of the boundedness of projections onto spline spaces V with respect to various norms has been motivated by applications to the numerical analysis of the finite element method and by investigations on spline bases in function spaces. A particular problem is to find bounds for L_2-orthogonal projections P_V in the L_∞ norm. In the univariate case, A. Shadrin [14] has proved that

$$\|P_V\|_{L_\infty \to L_\infty} \le C(k) < \infty \qquad (1)$$

for any space of splines of degree k, answering a longstanding conjecture by C. de Boor (see, e.g., [2]). As far as we know, only for $k = 1$ the exact value of $C(k)$ is known: For linear spline spaces V and any knot sequence, we have $\|P_v\|_{L_\infty \to L_\infty} < C(1) := 3$ by a result of Z. Ciesielski [3], and the sharpness of this bound was independently shown by K. I. Oskolkov [8] and the author [9].

In higher dimensions, boundedness of projections onto spline and finite element spaces was studied under various geometric conditions on the underlying partitions. We refer to [1, 4, 5] for conditions based on maximum/minimum angle and other regularity assumptions. It was folklore that a bound (1) cannot hold in $\mathbb{R}^d$, $d > 1$, but formal proof was given only recently. In [10], we constructed a sequence of triangulations $\mathcal{T}_J$ of a square into $\mathrm{O}(J)$ such that the L_2-orthogonal projection operator $P_V : L_2(\Omega) \to V$ onto the linear spline space $V = V(\mathcal{T})$ defined over a finite triangulation $\mathcal{T}$ of a bounded polygonal domain $\Omega \in \mathbb{R}^2$ satisfies the lower bound

$$\|P_{V(\mathcal{T}_J)}\|_{L_\infty \to L_\infty} \geq J, \quad J \geq 1. \tag{2}$$

We also mentioned in [10] unpublished work by A. A. Privalov who (according to A. Shadrin and Yu. N. Subbotin) stated examples of a similar flavor based on type-I triangulations of a rectangle in a conference talk around 1984. At the time of the preparation of [10] we were not able to verify this information, and claimed that the methods leading to (2) cannot be used to give similar lower bounds for tensor-product type-I triangulations. In Section 3, we correct this wrong statement by showing a bound similar to (2) for a sequence of type-II triangulations $\mathcal{T}_J$ partitioning a square into $\mathrm{O}(J^2)$ triangles.

The reason for returning to this question was, however, a different one: Triangulations obtained by bisecting the rectangles of a tensor-product rectangular partition are popular in certain finite element applications, e.g., as Shishkin- or Bakhvalov-type meshes for boundary layer treatment in convection-dominated second order elliptic problems (see [11] for some survey and further references), and having L_∞ bounds for P_V for such triangulations could be helpful in the analysis of these problems. In Section 2, Theorem 1, we provide a uniform upper bound in terms of certain geometric parameters (such as maximal valence of vertices and growth of local triangle area ratios) that is applicable to these classes of meshes. Using these techniques, we establish that for triangulations $\mathcal{T}$ obtained from an arbitrary tensor-product partition $\mathcal{P}$ of a rectangle by bisecting the rectangular cells in $\mathcal{P}$ the bisection pattern can always be chosen such that $\|P_{V(\mathcal{T})}\|_\infty$ remains uniformly bounded, independently of $\mathcal{P}$ (see Theorem 3 in Section 3). The above mentioned counterexample shows that such a result cannot hold for all bisection patterns.

The author is indebted to H.-G. Roos for attracting his attention to Shishkin meshes which triggered work on this note. He also acknowledges the helpful information exchange with A. Shadrin and Yu. N. Subbotin about the unaccessible earlier work by A. A. Privalov.

2 Sufficient geometric conditions

We recall some simple facts and notation already used in [10]. As usual, we parameterize $g \in V(\mathcal{T})$ by its nodal values $x_P = g(P)$ at the vertices $P \in \mathcal{V}_\mathcal{T}$ of $\mathcal{T}$, i.e.,

$$g = \sum_{P \in \mathcal{V}_\mathcal{T}} x_P \phi_P, \tag{3}$$

where $\{\phi_P \in V(\mathcal{T})\}$ denotes the standard nodal basis in $V(\mathcal{T})$. The support set $\Omega_P := \operatorname{supp} \phi_P$ of the nodal basis function ϕ_P corresponds to the 1-ring neighborhood of P in $\mathcal{T}$, its area is denoted by A_P. We denote by $\mathcal{V}_P = \{Q \in \Omega_P \cap \mathcal{V}_\mathcal{T} : \ Q \neq P\}$ the set of all neighboring vertices to P, by $K_P = \#\mathcal{V}_P$ the valence of P in $\mathcal{T}$, and by A_{PQ} the sum of the areas of the triangles attached to the edge PQ, $Q \in \mathcal{V}_P$. The following lemma was proved in [10].

Lemma 1 *We have*

$$\|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} \asymp \|A^{-1}\|_\infty, \tag{4}$$

with constants independent of $\mathcal{T}$, where the matrix A is the diagonally scaled Gram matrix of the nodal basis, with entries a_{PQ} given by the formula

$$a_{PQ} = \begin{cases} 1, & Q = P, \\ \frac{(\phi_Q,\phi_P)}{(\phi_P,\phi_P)} = \frac{A_{PQ}}{2A_P} > 0, & Q \in \mathcal{V}_P, \\ 0, & \textit{otherwise}, \end{cases} . \tag{5}$$

In particular,

$$1 = \sum_{Q \neq P} a_{PQ} = \sum_{Q \in \mathcal{V}_P} a_{PQ}, \qquad \forall\, P \in \mathcal{V}_\mathcal{T}. \tag{6}$$

We will use this structure of A to give uniform bounds for $\|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty}$ in terms of the *maximal valence* $K_{\max}(\mathcal{T}) := \max_{P \in \mathcal{V}} K_P$ and another, a bit unusual geometric characteristics $L_{\max}(\mathcal{T}, r) := \max_{P \in \mathcal{V}} L_P(r)$ of $\mathcal{T}$ called *depth of local area growth* with ratio $r \geq 1$. To define it, we denote by Δ'_Q the area-largest triangle attached to $Q \in \mathcal{V}$, and by A'_Q its area (if there are several ones, we consider all of them one-by-one). The quantity $L_P(r)$ equals the length L of the *shortest* chain of vertices $P_0 = P, P_1, \ldots, P_L$ such that (a) the vertex $P_k \neq P_{k-1}$ belongs to $\Delta'_{P_{k-1}}$ for $k = 1, \ldots, L$, and (b) $A'_{P_L} \leq rA'_{P_{L-1}}$ (as a consequence of the requirement that this chain is the shortest among all chains with properties (a) and (b), we automatically have $A'_{P_k} \geq rA'_{P_{k-1}}$ for $k = 1, \ldots, L-1$). Obviously, $L_P(r)$ (and thus $L_{\max}(\mathcal{T}, r)$) are bounded by the number of triangles in $\mathcal{T}$. The counterexamples from [10] provide evidence that $L_{\max}(\mathcal{T}, r)$ can grow proportionally with the number of triangles.

Theorem 1 *Given positive integers K_0, L_0, and a real number $r_0 > 1$, there is a finite constant $C(K_0, L_0, r_0)$ such that for any triangulation $\mathcal{T}$ satisfying $K_{\max}(\mathcal{T}) \leq K_0$, and $L_{\max}(\mathcal{T}, r) \leq L_0$ for some $r \leq r_0$, we have the bound*

$$\|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} \leq C(K_0, L_0, r_0).$$

Proof. According to (4), it is enough to show that the solution of the system $Ax = b$ satisfies

$$\|x\|_\infty := \max_{P\in\mathcal{V}_\mathcal{T}} |x_P| \le C(K_0, L_0, r_0) \tag{7}$$

for any right-hand side b with $\|b\|_\infty \le 1$. Denote $M = \|x\|_\infty$, and find the vertex P with $|x_P| = M$. W.l.o.g., let $x_P = M$. Since by assumption $L_P(r) \le L_0$, we can find a sequence of points $P_0 = P, P_1, \ldots, P_L$, $L \le L_0$, with properties (a) and (b) stated above. Now, set $a_k := M - (-1)^k x_{P_k}$. Obviously, $a_0 = 0$, and $a_k \ge 0$ because $M \pm x_Q \ge 0$ for all Q (this latter fact will be used without further mentioning). By induction in $k = 1, \ldots, L$, we show an upper bound for a_k for $k \le L$. Indeed, by (a) $P_k \ne P_{k-1}$ is one of the vertices of $\Delta'_{P_{k-1}}$, and belongs to $\mathcal{V}_{P_{k-1}}$. Since $\Delta'_{P_{k-1}}$ is the largest in area triangle among the $K_{P_{k-1}} \le K_{\max}(\mathcal{T}) \le K_0$ triangles attached to P_{k-1}, it follows from the definition (5) that $a_{P_{k-1}P_k} \ge 1/(2K_0)$, and

$$\begin{aligned} a_k &\le 2K_0 a_{P_{k-1}P_k}(M + (-1)^k x_{P_k}) \le 2K_0 \left(\sum_{Q\in\mathcal{V}_{P_{k-1}}} a_{P_{k-1}Q}(M + (-1)^k x_Q) \right) \\ &= 2K_0 \left(M + (-1)^k \sum_{Q\in\mathcal{V}_{P_{k-1}}} a_{P_{k-1}Q} x_Q \right) \\ &= 2K_0(M + (-1)^k(b_{P_{k-1}} - x_{P_{k-1}})) \le 2K_0(a_{k-1} + 1), \end{aligned}$$

and this gives by induction

$$\max_{k=0,\ldots,L} a_k \le (2K_0)^L + \ldots + (2K_0)^2 + 2K_0 = 2K_0 \frac{(2K_0)^L - 1}{2K_0 - 1} \le 2(2K_0)^{L_0}.$$

Now consider $\Delta'_{P_{L-1}}$. By construction, it has vertices P_{L-1}, P_L, and a third one which we denote by R. Let $a_R = M - (-1)^L x_R$, and observe that by the same reasoning as above for $k = L - 1$, we also have $0 \le a_R \le 2(2K_0)^{L_0}$. I.e., substituting this information into the equation corresponding to P_k we have

$$\begin{aligned} (-1)^L b_{P_L} &= (-1)^L x_{P_L} + a_{P_L R}(-1)^L x_R + \sum_{Q\in\mathcal{V}_{P_L}, Q\ne R} a_{P_L Q}(-1)^L x_Q \\ &\ge (1 + a_{P_L R})(M - 2(2K_0)^{L_0}) - M \sum_{Q\in\mathcal{V}_{P_L}, Q\ne R} a_{P_L Q} \\ &\ge (1 + a_{P_L R})(M - 2(2K_0)^{L_0}) - (1 - a_{P_L R})M \ge 2a_{P_L R}M - 4(2K_0)^{L_0}. \end{aligned}$$

This gives

$$M = \|x\|_\infty \le \frac{4(2K_0)^{L_0} + 1}{2a_{P_L R}}.$$

It remains to observe that by property (b) of the chain $A'_{P_L} \le rA'_{P_{L-1}}$, and thus

$$a_{P_L R} \ge \frac{A'_{P_{L-1}}}{2A_{P_L}} \ge \frac{A'_{P_{L-1}}}{2K_{P_L} A'_{P_L}} \ge \frac{1}{2K_0 r_0}.$$

Putting the last two estimates together, we arrive at (7). This concludes the proof of Theorem 1.

Remark 1. The same bound holds if we consider the L_2-projection onto subspaces of $V \subset V(\mathcal{T})$ generated by any subset of nodal basis functions (e.g., subspaces of linear finite elements over $\mathcal{T}$ satisfying homogeneous Dirichlet boundary conditions). What changes is that equality in (6) is replaced by inequality if $\mathcal{V}_P$ contains an eliminated node. The modifications in the proof are minimal, and left to the reader.

Remark 2. Here are a couple of simpler conditions on $\mathcal{T}$ that imply bounds on $L_{\max}(\mathcal{T}, r)$. *Locally area-quasiuniform triangulations* are characterized by an uniform bound $\le r_0$ for the ratios of the areas of any two triangles sharing at least one vertex. According to our definition of the depth of local area growth, this implies $L_{\max}(\mathcal{T}, r_0) = 1$. *Regular triangulations* (a standard assumption in finite element applications) for which the ratio of radii of circumscribed to inscribed circle is bounded by γ for any triangle in $\mathcal{T}$ are obviously locally area-preserving for some properly chosen r_0 (and satisfy in addition $K(\mathcal{T}) \le K_0$ for some K_0 depending on γ only).

Another example is as follows. Suppose that the triangles in $\mathcal{T}$ can be sorted into L_0 classes where within each class any two triangles (not only those touching each other!) have an area ratio $\le r_0$. Then necessarily $L_{\max}(\mathcal{T}, r_0) \le L_0$. Indeed, choose $P \in \mathcal{V}$ such that $L_P(r_0) = L_{\max}(\mathcal{T}, r_0)$, and consider the associated shortest chain $P_0 = P, P_1, \ldots, P_{L_P(r_0)}$ satisfying conditions (a) and (b). Since $A'_{P_k} > r_0 A'_{P_{k-1}}$ for $k = 1, \ldots, L_P(r_0) - 1$, the associated triangles Δ'_{P_k}, $k = 0, 1, \ldots, L_P(r_0) - 1$, must belong to different classes. Thus, $L_{\max}(\mathcal{T}, r_0) = L_P(r_0) \le L_0$. Since Shishkin-type meshes are composed of a finite number of quasi-uniform meshes, this assures that for those uniform L_∞-boundedness of $P_{V(\mathcal{T})}$ holds as long as the number of different boundary/interior layers is fixed (often $L_{\max}(\mathcal{T}, r_0)$ is much smaller than the crude upper bound given by the number of layers). It is not hard to show that also Bakhvalov-type meshes [11, 12] have uniformly bounded $L_{\max}(\mathcal{T}, r)$ if $r > 1$ is properly chosen, and that layer-adapted partitions on general domains [6] can be treated as well.

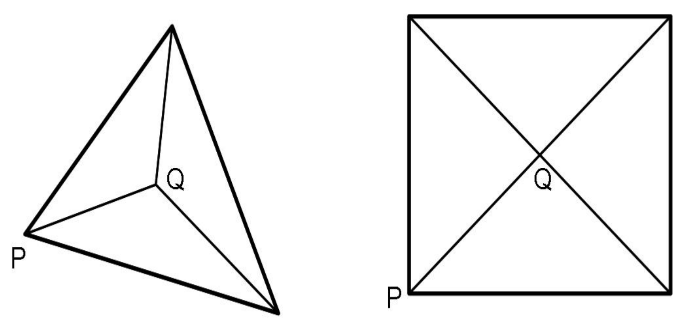

Figure 1: Clough-Tocher split of a triangle resp. criss-cross split of a rectangle

Last but not least, for any $\mathcal{T}$ we can obtain a refined triangulation $\tilde{\mathcal{T}}$ by inserting one new vertex into each triangle at its barycenter, and subdividing it into 3 triangles of equal area (Clough-Tocher split). Then

$$L_{\max}(\tilde{\mathcal{T}}, r) \le 2, \qquad r > 1.$$

Similarly, if we take any tensor-product partition $\mathcal{P}$ of a rectangle, and subdivide each of the rectangular cells in $\mathcal{P}$ into 4 triangles of equal area by inserting its two diagonals, then the resulting criss-cross triangulation $\mathcal{T}_{\mathcal{P}}$ also satisfies

$$L_{\max}(\mathcal{T}_{\mathcal{P}}, r) \le 2, \qquad r \ge 1,$$

while $K_{\max}(\mathcal{T}_{\mathcal{P}}) \le K_0 := 8$ holds automatically. Indeed, in both cases there are only two types of vertices: old vertices from $\mathcal{T}$ resp. $\mathcal{P}$, and newly inserted vertices (one per cell) that lead to $\tilde{\mathcal{T}}$ resp. $\mathcal{T}_{\mathcal{P}}$. By construction, for each old vertex P there are at least two neighboring area-largest triangles Δ'_P which share one new vertex Q, see Fig. 1. Obviously, all triangles attached to Q have the same area, thus $A'_Q = A'_P \le rA'_P$ for any $r \ge 1$, and thus $L_P(r) = 1$. If we start at a new vertex Q, and for any of the old vertices P of the cell containing Q we have $A'_P \le rA'_Q$ then again $L_Q(r) = 1$. Otherwise, we have $L_Q(r) = L_P(r) + 1 = 2$ by the already established fact for old vertices P. Note that with appropriately chosen $r > 1$, the result also holds if the cells in a triangulation or quadrilateral partition are subdivided into 3 or 4 triangles, respectively, of approximately the same area.

Remark 3. The above proof heavily relies on the special properties of the diagonally scaled Gram matrices A which are non-negative and weakly diagonally dominant, see the formulas (5) and (6). These properties do not carry over to higher dimensions nor to most of the other finite element and spline spaces in use.

3 Triangulations obtained from tensor-product rectangular partitions

This section deals with linear spline spaces V on triangulations of rectangles (resp. of domains composed of a few axis-parallel rectangles such as, e.g., a L-shaped domain) obtained from a tensor-product rectangular partition by bisecting each of its rectangular cells. The major observation is that the uniform boundedness of P_V crucially depends on the particular bisection pattern.

Our first result concerns the existence of tensor-product rectangular partitions $\mathcal{P}_J$ of a rectangle into $\mathrm{O}(J^2)$ rectangles such that for the associated type-II triangulations $\mathcal{T}_J$ the lower bound (2) holds. Such an example was, according to recollections by A. Shadrin and Yu. N. Subbotin, first given by A. A. Privalov in a conference talk around 1984 but no published record seems available. How $\mathcal{T}_J$ looks like is depicted in Fig. 2.

Theorem 2 *There exist rectangular partitions $\mathcal{P}_J$ and associated type-II triangulations $\mathcal{T}_J$ of a rectangular domain into $\mathrm{O}(J^2)$ rectangles and triangles, respectively, such that*

$$\|P_{V(\mathcal{T}_J)}\|_{L_\infty \to L_\infty} \ge J, \qquad J \ge 1.$$

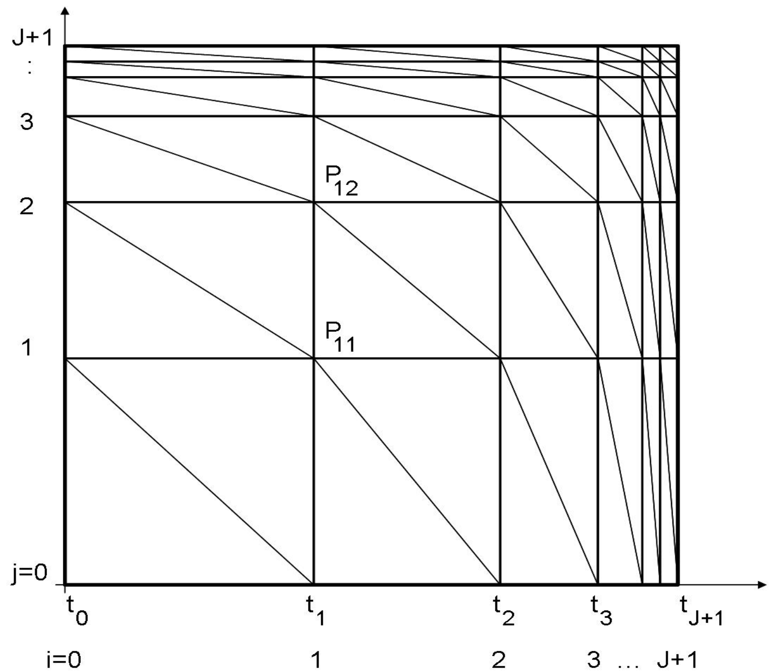

Figure 2: Sketch of a type-II triangulation $\mathcal{T}_J$ under consideration

Proof. The argument is similar to the one used in [10]. We temporarily fix J, and look at tensor products of the same univariate partition $0 = t_0 < t_1 < \ldots < t_J < t_{J+1} = 1$ to generate a family of rectangular partitions $\mathcal{P}$ and of the associated type-II triangulations $\mathcal{T}$ with $\mathrm{O}(J^2)$ rectangles and triangles, respectively. If we let the local mesh ratios $(t_{j-1} - t_j)/(t_j - t_{j+1})$, $j = 1, \ldots, J$, simultaneously tend to ∞, then the scaled Gram matrices A associated with $P_{V(\mathcal{T})}$ converge to an invertible limit matrix A_J satisfying the bound

$$\|A_J^{-1}\|_\infty \geq \frac{3}{2}J, \qquad J \geq 1. \tag{8}$$

This shows the claim of Theorem 2 since by continuity we can choose from the considered family of triangulations a type-II triangulation $\mathcal{T} =: \mathcal{T}_J$ into $2(J+1)^2$ triangles with $A \approx A_J$ and $A^{-1} \approx A_J^{-1}$ such that $\|A^{-1}\|_\infty \geq J$ (the constant $c > 0$ coming from (4) can easily be compensated for by choosing a slightly larger $J' \leq CJ$ in this construction).

To establish (8), we use pairs (i,j), $i,j = 0, \ldots, J+1$, to index the entries associated with the "virtual" vertex that corresponds to the limit of $P_{i,j} = (t_i, t_j)$ (see Fig. 1) of the column vectors b and x in the linear system $A_J x = b$ (note that in the "physical world" all these points P_{ij} collapse into one of the corners of the unit square in the limit). For the ease of proof, we use a modified checkerboard pattern for the right-hand side b given by

$$b_{i,j} = 0, \quad \min(i,j) \leq 1, \qquad b_{i,j} = (-1)^{i+j}, \quad \min(i,j) > 1.$$

This choice corresponds to an extremal function f on the square that takes value zero in the strips of rectangles attached to the axes, and value $(-1)^{i+j}$ in the remaining rectangles with P_{ij} as lower-left corner, $\min(i,j) \geq 1$.

Because of symmetry, the solution vector x satisfies $x_{i,j} = x_{j,i}$. Using the geometric definition (6) of the matrix elements, the limit system $A_J x = b$ can be explicitly given, and solved in a simple recursive fashion since it is essentially in lower triangular form if the variables are ordered properly. We use the above symmetry to reduce its size, the order we write down the system corresponds to marching in the rectangular mesh as follows: Start from the bottom-left square, move one-by-one to the right end in the first row of rectangles, then switch to the "diagonal" square in the second row, move to the right end in the second row, switch to the "diagonal" square in the third row, and so on. The three equations in $Ax = b$ associated with the bottom-left square are

$$x_{00} + x_{10} = 0; \qquad \frac{3}{2}x_{10} + \frac{1}{4}(x_{00} + x_{11}) = 0; \qquad x_{11} + x_{10} = 0,$$

from which we get $x_{ij} = 0$ for $i, j \leq 1$. We proceed with the remaining equations corresponding to the first row of rectangles:

$$\begin{aligned} x_{i0} + \frac{1}{4}x_{i1} &= -\frac{1}{4}x_{i-1,0} - \frac{1}{2}x_{i-1,1}, \\ \frac{1}{2}x_{i0} + x_{i1} &= -\frac{1}{2}x_{i-1,1}, \end{aligned}$$

which by recursion in $i = 2, \ldots, J+1$ gives $x_{i,j} = 0$ for all i, j with $\min(i,j) \leq 1$.

The remaining equations (one for each (i,j) with $\min(i,j) > 1$) can be written in the form

$$x_{ij} = (-1)^{i+j} - \frac{1}{2}(x_{i-1,j} + x_{i,j-1}) \quad \Longleftrightarrow \quad y_{ij} = \frac{1}{2}(y_{i-1,j} + y_{i,j-1}) + 1,$$

where $y_{ij} = (-1)^{i+j} x_{ij}$. We start with $j = 2$, and obtain by recursion in $i = 2, \ldots, J+1$ that $y_{22} = 1$, $y_{32} = 3/2$, and generally $y_{i2} \geq 3/2$ for $i \geq 3$. Now, switch to $j = 3$. Looking at the equation for $i = 3$ we get $y_{33} = y_{32} + 1 = 5/2$, and for $i = 4$ we obtain $y_{43} = \frac{1}{2}(y_{33} + y_{42}) + 1 \geq 3$ and more generally $y_{i3} \geq 3$ for all $i = 4, \ldots, J+1$. By induction in j, it is easy to see that this pattern generalizes to

$$y_{jj} \geq (j-1)\frac{3}{2} - \frac{1}{2}, \qquad y_{ij} \geq (j-1)\frac{3}{2}, \quad i = j+1, \ldots, J+1,$$

for all $j = 2, \ldots, J+1$. Thus, taking $j = J+1$, we get

$$\|A_J^{-1}\|_\infty \geq \|x\|_\infty \geq |y_{J+1,J+1}| \geq \frac{3}{2}J$$

as was claimed above. The statement is proved.

Remark 4. It is not hard to show that $\|A_J^{-1}\|_\infty \asymp J$ as $J \to \infty$ in the above example. Thus, as to the asymptotic growth of the L_∞ norm of $P_{V(\mathcal{T})}$ with respect to the size of $\mathcal{T}$, the above example is less effective than the one provided in [10] since the latter provides the same lower estimate for triangulations into $\mathrm{O}(J)$ triangles. Recall that we conjectured in [10] that a matching upper bound of

$$\|P_{V(\mathcal{T})}\|_\infty = \mathrm{O}(\#\mathcal{T})$$

is to be expected. This conjecture remains open.

Remark 5. The above counterexample should be contrasted with the situation for L_2-orthogonal projections onto the space of bilinear Q1 elements over tensor-product rectangular partitions $\mathcal{P}$, where a uniform boundedness result holds, with no restrictions on the univariate partitions that generate $\mathcal{P}$. The proof is an easy exercise on tensor-product approximation. Also note that if we allow criss-crossing of all rectangles into 4 (rather than bisecting into 2) equal-area triangles then, according to the last example of Remark 2, Theorem 1 yields uniform boundedness of $P_{V(\mathcal{T}_\mathcal{P})}$ independently of $\mathcal{P}$.

In other words, cutting each rectangle in $\mathcal{P}$ into two triangles, and switching from rectangular Q1 element to triangular P1 element spaces makes a difference. A natural question is whether this effect depends on how we bisect the rectangles. E.g., if we flip all bisection lines in the above example, and consider the limit case of the resulting type-I triangulations then $\|A_J^{-1}\|_\infty$ is uniformly bounded (this is a difficulty in coming up with the counterexamples, and partly explains our misconception concerning tensor-product meshes that entered [10]). Since for both the type-I and the type-II triangulations the depth of local are growth with ratio r equals $J+1$ in the limit (to this end, consider the upper-right corner of the rectangle as P, and check the definition of $L_P(r)$ as $\min_{j=1,\dots,J}\,(t_{j-1}-t_j)/(t_j-t_{j+1}) \to \infty$), we also see that the boundedness of $L_{\max}(\mathcal{T},r)$ for some $r \geq 1$ required in Theorem 1 is only sufficient but not necessary for getting L_∞ bounds. A similar comment is true for the assumption on $K_{\max}(\mathcal{T})$ in Theorem 1, as can be seen from examining the case of 1-ring triangulations around a single interior vertex of high valence.

In light of the previous Remark 5, it is natural to ask if, for any tensor-product rectangular partition $\mathcal{P}$ of a rectangle, there exists a bisection pattern such that $\|P_{V(\mathcal{T})}\|_\infty$ is bounded from above by a universal constant. The answer is indeed yes if the following bisection algorithm is adopted: For any ordering of the vertices in $\mathcal{P}$, loop through the vertices. If P is the current vertex, consider its neighboring rectangles, and mark the area-largest ones among them. Marked rectangles that are not yet bisected should be bisected using the diagonal emanating from P. For marked but already bisected rectangles no action is necessary. Then proceed with the next vertex in the list. After all vertices are processed, a few rectangles may be left undivided. Those can be bisected by choosing a diagonal randomly. This algorithm is illustrated in Fig. 3.

Theorem 3 *For any tensor-product partition $\mathcal{P}$ of a rectangular domain, the above bisection algorithm yields a triangulation $\mathcal{T}$ such that the norms of the associated projectors $P_{V(\mathcal{T})}$ remain uniformly bounded. In other words,*

$$\sup_{\mathcal{P}} \|P_{V(\mathcal{T})}\|_\infty < \infty.$$

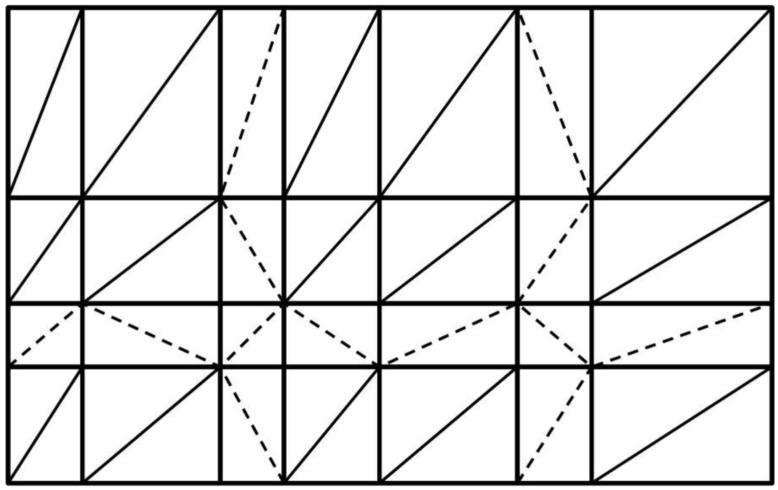

Figure 3: Bisection algorithm for a sample $\mathcal{P}$ (shown on the left). The ordering of vertices is row-by-row from left to right, starting with the bottom row. Dashed bisection lines indicate rectangles that have been randomly bisected after the loop over all vertices was finished.

Proof. Assume that x is the solution with maximal max-norm $M = \|x\|_\infty$ of the system $Ax = b$ with right-hand side b of unit norm $\|b\|_\infty = 1$. Without loss of generality, let P be the vertex of $\mathcal{T}$ for which $x_P = M$. Since the vertex set of $\mathcal{T}$ coincides with the vertex set of $\mathcal{P}$, this P has between one to four neighboring rectangles, the area-largest one of which is denoted by R'_P, and its area by A'_P (if there are several area-largest rectangles attached to P, we pick one of them). Our algorithm guarantees that R'_P or has been bisected with the diagonal emanating from P (e.g., when P was processed in the loop of the algorithm), or with the other diagonal when one of the two vertices of this rectangle neighboring with P were processed (and led to bisecting R'_P) earlier in the algorithm. The two possibilities are shown in Fig. 4, the situation on the left occurs if one of the neighbor-vertices Q of P in R'_P has triggered the bisection of R_P, the situation on the right occurs if the examination of either P or the vertex Q opposite to P led to the bisection.

We first give the complete argument for the situation depicted in Fig. 4 on the right, the situation on the left is similar and even simpler to deal with, details are given at the end of the proof. Since R_P is the area-largest rectangle attached to P, the coefficients a_{PS}, $S = Q, Q_1, Q_2$, are safely bounded away from zero. The following crude estimates suffice. We have $A_P \leq 4R'_P$, and

$$A_{PQ_1}, A_{PQ_2} \geq A'_P/2, \qquad A_{PQ} = A'_P,$$

which yields

$$a_{PQ_1},\, a_{PQ_2} \geq \frac{1}{16}, \qquad a_{PQ} \geq \frac{1}{8}.$$

As in the proof of Theorem 1, from the equation in $Ax = b$ corresponding to P and

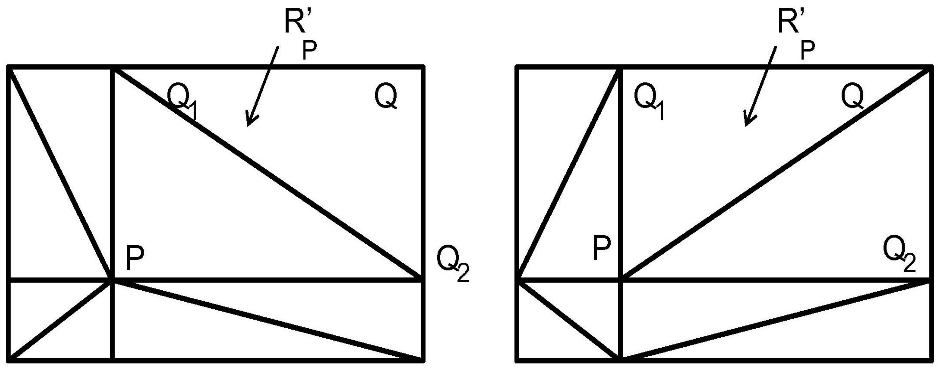

Figure 4: Two possible situations of bisection of R'_P

the assumptions $\|b\|_\infty = 1$ and $M = x_P = \|x\|_\infty$ we obtain upper bounds for the quantities $y_S := x_S + M \geq 0$ for the vertices $S = Q, Q_1, Q_2$ of R'_P (suggesting that the corresponding entries x_S are close to $-M$). Indeed,

$$1 \geq b_P = x_P + \sum_{S \neq P} a_{PS} x_S = \sum_{S \neq P} a_{PS} y_S \geq a_{PS} y_S, \quad S \neq P,$$

implies

$$y_Q \leq 1/a_{PQ} \leq 8, \qquad y_{Q_i} \leq 1/a_{PQ_i} \leq 16, \quad i = 1, 2. \tag{9}$$

Now consider without loss of generality the neighborhood of Q_2 (we could have equally chosen to proceed with neighborhood of Q_1). There are two cases: Or $R'_{Q_2} = R'_P$ (i.e., R'_P is also the area-largest rectangle attached to Q_2), or the rectangle $R'_{Q_2} := QQ_2Q_3Q_4$ has larger area than R'_P. In the first case, we have by a similar reasoning

$$A_{Q_2Q}, A_{Q_2P} \geq A'_P/2, \qquad A_{Q_2} \leq 7A'_P/2,$$

and thus

$$a_{Q_2Q},\ a_{Q_2P} \geq \frac{1}{14}.$$

Using (9) and inspecting the equation in $Ax = b$ corresponding to Q_2 results in

$$\begin{aligned} -1 \leq b_{Q_2} &= x_{Q_2} + \sum_{S \neq Q_2} a_{Q_2S} x_S = y_{Q_2} + \sum_{S \neq Q_2} a_{Q_2S}(x_S - M) \\ &\leq 16 + a_{Q_2Q}(x_Q - M) \leq 16 + (y_Q - 2M)/14 \leq 16 + 4/7 - M/7, \end{aligned}$$

since $a_{Q_2S} \geq 0$ and $x_S - M \leq 0$ for all $S \neq Q_2$. Consequently, we arrive at the crude upper bound $M \leq 123$.

In the second case, it depends on how R'_{Q_2} is bisected. The two situations are depicted in Fig. 5. In either case, we find as above that

$$A_{Q_2} \leq 4A'_{Q_2}, \quad A_{Q_2Q} \geq A'_{Q_2}/2 \quad \Longrightarrow \quad a_{Q_2Q} \geq \frac{1}{16}.$$

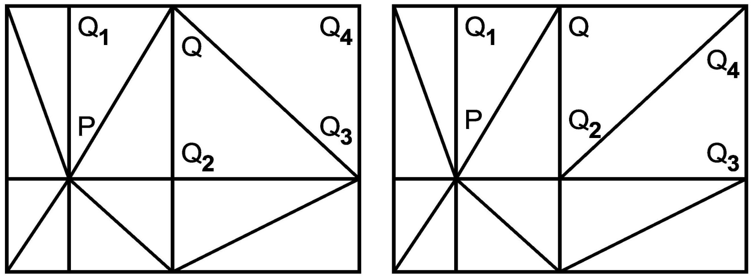

Figure 5: Two bisection situations for R'_{Q_2}

Thus, as above

$$-1 \le b_{Q_2} \le y_{Q_2} + a_{Q_2Q}(x_Q - M) \le 16 + (y_Q - 2M)/16 \le 16 + 1/2 - M/8,$$

and $M \le 140$. This concludes the proof of Theorem 3 for the situation shown in Fig. 4 on the right.

The remaining situation on the left in Figure 4 is even easier to deal with. In this case, the bisection of R'_P has been triggered by either Q_1 or Q_2, without loss of generality assume that it was Q_2. Thus, R'_P is the area-largest among the rectangles attached to both P and Q_2. As above, we obtain in a first step

$$A_P \le 7A'_P/2, \quad A_{PQ_1}, A_{PQ_2} \ge A'_P/2 \quad \Longrightarrow \quad a_{PQ_1}, a_{PQ_2} \ge \frac{1}{14},$$

and consequently

$$y_{Q_1}, y_{Q_2} \le 14.$$

In the second step, we examine the equation for Q_2, and get

$$A_{Q_2} \le 4A'_P, \quad A_{Q_2Q_1} = A'_P \quad \Longrightarrow \quad a_{Q_2Q_1} \ge \frac{1}{8},$$

which leads to

$$-1 \le b_{Q_2} \le y_{Q_2} + a_{Q_2Q_1}(x_{Q_1} - M) \le 14 + (y_{Q_1} - 2M)/8 \le 14 + 7/4 - M/4,$$

and eventually to $M \le 67$. The proof of Theorem 3 is complete.

Remark 6. The above obtained numerical upper bound of

$$\|P_{V(\mathcal{T})}\|_\infty \asymp \|A^{-1}\|_\infty \le 140$$

is certainly too pessimistic. Note that for the case of bilinear finite element spaces over tensor-product partitions $\mathcal{P}$ one has the sharp upper bound of

$$\|P_{V(\mathcal{P})}\|_\infty < 9,$$

which follows from Ciesielski's result [3] for the univariate case.

Remark 7. If one applies the bisection algorithm underlying Theorem 3 to the family of $\mathcal{P}$ used for establishing Theorem 2 (recall that the counterexamples require the partitions to collapse into the upper-right corner of the rectangle depicted in Fig. 2) then one ends up with essentially a type-I triangulation. To see this, start the loop through the vertex set of $\mathcal{P}$ from the upper-right corner and move row-by-row from right to left. This confirms the observation reported in Remark 5 for this type of $\mathcal{P}$.

Remark 8. The result of Theorem 3 applies to the two-dimensional examples of Shishkin- and Bakhvalov-type meshes for the treatment of elliptic and parabolic boundary layers in diffusion-convection problems considered in [12, 2.4.1-2], and can be carried over to the more general h-FEM meshes discussed in [6, Section 2.6]. Note the difference with applying Theorem 1 to these families of meshes: While Theorem 1 guarantees a bound independent of the bisection pattern but formally depending on the number of layers with different mesh-sizes, Theorem 3 gives a bound for a specific bisection pattern, independently of the number of layers with different mesh-sizes!

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