A counterexample concerning the L_2-projector onto linear spline spaces

Peter Oswald

International University Bremen
e-mail: poswald@iu-bremen.de

Abstract

For the L_2-orthogonal projection P_V onto spaces of linear splines over simplicial partitions in polyhedral domains in $\mathbb{R}^d$, $d > 1$, we show that in contrast to the one-dimensional case, where $\|P_V\|_{L_\infty \to L_\infty} \le 3$ independently of the nature of the partition, in higher dimensions the L_∞-norm of P_V cannot be bounded uniformly with respect to the partition. This fact is folklore among specialists in finite element methods and approximation theory but seemingly has never been formally proved.

1 Introduction

Variational methods based on piecewise polynomial approximations are a workhorse in numerical methods for PDEs and data analysis. In particular, least-squares methods lead to the study of the L_2-orthogonal projection operator $P_V : L_2(\Omega) \to V$ onto a given spline space V defined on a domain $\Omega \in \mathbb{R}^d$. A question of considerable interest is the uniform boundedness of the L_∞-norm

$$\|P_V\|_{L_\infty \to L_\infty} := \max_{f \in L_\infty(\Omega):\, \|f\|_{L_\infty}=1} \|P_V f\|_{L_\infty}$$

of P_V with respect to families of spline spaces V. If Ω is a bounded interval in $\mathbb{R}^1$, then the question has been intensively studied for the family of spaces of smooth splines of fixed degree r over arbitrary partitions, where Shadrin [7] has recently established that

$$\|P_V\|_{L_\infty \to L_\infty} \le C(r) < \infty, \qquad r \ge 1, \tag{1}$$

for any partition. This result was known for a long time for small values of r, e.g., Ciesielski [3] proved that for linear splines one can take $C(1) = 3$, while de Boor [2] solved the case $r \le 4$. The estimate (1) plays an important role in numerical analysis and for the investigation of orthonormal spline systems such as the Franklin system in L_p-based scales of function spaces, $1 \le p \le \infty$.

In higher dimensions, the study of the L_∞-norm of P_V arose mostly in the context of obtaining L_p error estimates in the finite element Galerkin method [5, 6], where sufficient conditions on the underlying partition and nodal basis $\{\phi_i\}$ of a finite element space V are formulated under which the norms $\|P_V\|_{L_\infty \to L_\infty}$ are bounded by a certain finite constant. Interestingly enough, these results suggest that such conditions on partitions resp. finite element type are essential for obtaining uniform bounds but formal proof of their *necessity* was not given.

Similarly, in the theory of multivariate splines final results on the uniform boundedness of $\|P_V\|_{L_\infty \to L_\infty}$ could not be localized. Recently, Ciesielski [4] asked about the extension of his result for linear splines [3] to the higher-dimensional case, and the unanimous opinion of the audience was that in higher dimensions a similar result cannot hold. However, other than a vague reference to unpublished work by A. A. Privalov, no concrete proof could be found.

It is the aim of this note to provide an elementary example of triangulations $\mathcal{T}_J$ of a square $\Omega \subset \mathbb{R}^2$ into $\mathrm{O}(J)$ triangles for which

$$\|P_{V(\mathcal{T}_J)}\|_{L_\infty \to L_\infty} \geq J, \qquad J \geq 1, \tag{2}$$

where $V(\mathcal{T}_J)$ is the space of linear C^0 splines (or finite element functions) on $\mathcal{T}_J$; see Theorem 1 below. As $J \to \infty$, the triangulations $\mathcal{T}_J$ will not satisfy the minimum angle condition, which is natural since for these types of triangulations a uniform bound can easily be established. However, they satisfy the maximum angle condition, [1], and do not possess vertices of high valence. The example can easily be extended to $d \geq 3$, and implies that $C(1) = \infty$ for all $d > 1$.

2 Notation and Result

We concentrate on $d = 2$, the case $d \geq 3$ is mentioned in Section 3. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain equipped with a finite triangulation $\mathcal{T}$ into non-degenerate closed triangles Δ satisfying the usual regularity condition that two different triangles may intersect at a common vertex resp. edge only. The set of vertices of $\mathcal{T}$ is denoted by $\mathcal{V}_\mathcal{T}$. Let $|E|$ denote the Lebesgue measure of a measurable set $E \subset \mathbb{R}^2$. By $0 < \underline{\alpha}_\mathcal{T} \leq \bar{\alpha}_\mathcal{T} < \pi$ we denote the minimal and maximal interior angle of all triangles in $\mathcal{T}$.

Let $V(\mathcal{T})$ denote the linear space of all continuous functions g whose restriction to any of the triangles $\Delta \in \mathcal{T}$ is a linear polynomial. Any $g \in V(\mathcal{T})$ has a unique representation of the form

$$g = \sum_{P \in \mathcal{V}_\mathcal{T}} g(P)\phi_P, \tag{3}$$

where the Courant hat functions $\phi_P \in V(\mathcal{T})$ are characterized by the conditions $\phi_P(P) = 1$, and $\phi_P(Q) = 0$, where $Q \neq P$ is any of the remaining vertices of $\mathcal{T}$. Thus, $\dim V(\mathcal{T}) = \#\mathcal{V}_\mathcal{T}$, and

$$\operatorname{supp} \phi_P = \Omega_P := \cup_{\Delta \in \mathcal{T}:\, P \in \Delta} \Delta.$$

The set Ω_P corresponds to the 1-ring neighborhood of P in $\mathcal{T}$, and we denote by $\mathcal{V}_P = \{Q \in \Omega_P \cap \mathcal{V} :\ Q \neq P\}$ the set of all neighboring vertices to P.

The L_2-orthogonal projection of a function $f \in L_2(\Omega)$ onto $V(\mathcal{T})$ is given by the unique $g := P_{V(\mathcal{T})} f \in V(\mathcal{T})$ such that

$$(f - g, \phi_P) = 0 \qquad \forall\, P \in \mathcal{V}_{\mathcal{T}}.$$

Here and in the sequel, $(\cdot,\cdot)$ stands for the $L_2(\Omega)$ inner product. Using (3) with unknown nodal values $x_P = g(P)$ as ansatz, this is equivalent to the linear system

$$\sum_{Q \in \mathcal{V}} (\phi_Q, \phi_P) x_Q = (f, \phi_P) \qquad \forall\, P \in \mathcal{V}. \tag{4}$$

A simple calculation shows that

$$(\phi_P, \phi_P) = \frac{|\Omega_P|}{6}.$$

Similarly, if $Q \in \mathcal{V}_P$ is a neighbor of P then

$$(\phi_Q, \phi_P) = \sum_{\Delta \in \mathcal{T}:\, P,Q \in \Delta} \frac{|\Delta|}{12},$$

in all other cases we have $(\phi_Q, \phi_P) = 0$. This shows in particular that

$$(\phi_P, \phi_P) = \sum_{Q \neq P} (\phi_Q, \phi_P) = (1, \phi_P)/2.$$

I.e., if we normalize in (4) by (ϕ_P, ϕ_P) then (4) turns into a linear system

$$Ax = b, \qquad x := (x_Q :\ Q \in \mathcal{V}_{\mathcal{T}})^T, \quad b := \Big(b_P = \frac{(f, \phi_P)}{(\phi_P, \phi_P)} :\ P \in \mathcal{V}_{\mathcal{T}}\Big)^T, \tag{5}$$

where $\|b\|_\infty \le 2\|f\|_{L_\infty}$, and $A := (a_{PQ})$ satisfies

$$a_{PQ} = \begin{cases} 1, & Q = P, \\ \frac{(\phi_Q, \phi_P)}{(\phi_P, \phi_P)} > 0, & Q \in \mathcal{V}_P, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

and

$$1 = \sum_{Q \neq P} a_{PQ} = \sum_{Q \in \mathcal{V}_P} a_{PQ}, \qquad \forall\, P \in \mathcal{V}_{\mathcal{T}}. \tag{7}$$

Thus, the matrix A is only weakly diagonally dominant, and not strictly diagonally dominant as in the one-dimensional case. Otherwise, we could estimate $\|A^{-1}\|_\infty$ in a trivial way, and use the inequality

$$\|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} \le 2\|A^{-1}\|_\infty = 2 \max_{\|Ay\|_\infty \le 1} \|y\|_\infty, \tag{8}$$

which follows from the above, in conjunction with the obvious equality $\|P_{V(\mathcal{T})} f\|_{L_\infty} = \|x\|_\infty$. Let us mention without proof that (8) implies the following partial result.

Proposition 1 *If for any two neighboring vertices $P \neq Q$ from $\mathcal{V}_\mathcal{T}$ we have $a_{PQ} \geq c_0 > 0$, then*

$$\|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} \leq (1+2c_0)c_0^{-2}.$$

For triangulations satisfying the minimum angle condition uniformly, i.e., $\underline{\alpha}_\mathcal{T} \geq \underline{\alpha}_0 > 0$, this result is applicable with a c_0 determined solely by α_0, and thus covers the bounds considered in the finite element literature [5, 6]. The main result of this note is the following

Theorem 1 *For any $J \geq 1$, there is a triangulation $\mathcal{T}_J$ of a square into $8J+4$ triangles such that the norm of the L_2-projector $P_{V(\mathcal{T}_J)}$ satisfies*

$$\|P_{V(\mathcal{T}_J)}\|_{L_\infty \to L_\infty} \geq 2J.$$

Thus, for spatial dimension $d = 2$ we have

$$C(1) := \sup_\mathcal{T} \|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} = \infty.$$

We conjecture that in terms of the number of triangles our result is asymptotically sharp for bounded polygonal domains in $\mathbb{R}^2$, i.e.,

$$\sup_{\mathcal{T}:\, \#\mathcal{T} \leq N} \|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} \asymp N, \qquad N \to \infty. \tag{9}$$

That Ω is a square is not crucial. The examples given below can easily be modified to show $C(1) = \infty$ for simplicial partitions in higher dimensions as well.

3 Proof of Theorem 1

We will use the following notation. Let $S_a = [-a, a]^2$ be the square of side-length $2a$, $a > 0$, with center at the origin. Set $\Omega := S_1$, with vertices denoted (in clockwise direction) by $P_{0,i}$, $i = 1, \ldots, 4$, and fix the parameter $t \in (0, 1)$. The triangulation $\mathcal{T}_J$, $J \geq 1$, of Ω is obtained by inserting the squares S_t, S_{t^2},…, S_{t^J}, whose vertices will be denoted similarly by $P_{j,i}$, $i = 1, \ldots, 4$, $j = 1, \ldots, J$, placing an additional vertex P_{J+1} at the origin, connecting P_{J+1} by straight lines with the 4 vertices $P_{0,i}$ of S_1, and finally subdividing each of the remaining trapezoidal regions $P_{j-1,i-1}P_{j,i-1}P_{j,i}P_{j-1,i}$ into two triangles by connecting $P_{j-1,i-1}$ with $P_{j,i}$ in a consistent way. The outer rings of the resulting triangulation are shown in Fig. 1 a).

Let the function $f \in L_\infty(\Omega)$ with $\|f\|_{L_\infty} = 1$ be defined as follows:

$$f(x) = (-1)^j, \qquad x \in \Omega_j := \begin{cases} S_{t^{j-1}} \backslash S_{t^j}, & j = 1, \ldots, J, \\ S_{t^J}, & j = J+1. \end{cases}$$

We use the same notation as above, and consider the linear system (5) corresponding to the L_2-orthogonal projection $g = P_{V(\mathcal{T}_J)}f$ of this f onto $V(\mathcal{T}_J)$. Because of uniqueness of orthogonal projections and the rotational symmetry of $\mathcal{T}_J$ and f, the entries of the vector

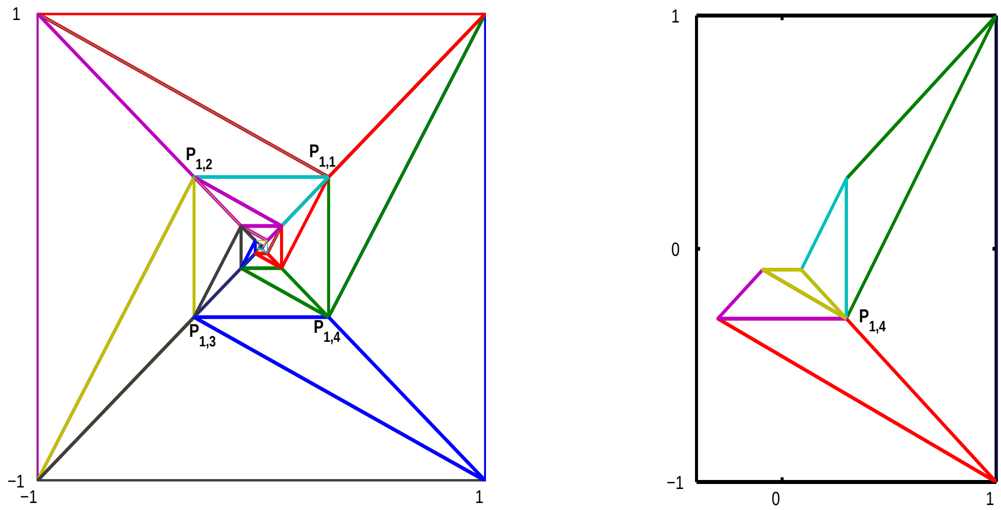

Figure 1: Triangulation $\mathcal{T}_J$ for $t = 0.3$ (left), and typical $\Omega_{P_{j,i}}$ (right)

x that represent the nodal values of g corresponding to the vertices $P_{j,i}$, $i = 1, \ldots, 4$, are equal, and will be denoted by x_j, $j = 0, \ldots, J$ (the value at the origin is denoted by x_{J+1}). Moreover, the 4 equations in (5) corresponding to the 4 vertices of a square S_{t^j}, $j = 0, \ldots, J$, can be replaced by one, thus turning the original system $Ax = b$ of dimension $4J + 5$ into a reduced tridiagonal system $\tilde{A}\tilde{x} = \tilde{b}$ of dimension $J + 2$ for the vector $\tilde{x} = (x_0, \ldots, x_{J+1})^T$. Since the equations in the system (5) are invariant under affine transformations, and because of the definition of $\mathcal{T}_J$ via squares S_{t^j} shrinking at a fixed geometric rate, it is easy to see that, with the exception of the first and last two, all equations in $\tilde{A}\tilde{x} = \tilde{b}$ have the same form:

$$\alpha x_{j-1} + \beta x_j + \gamma x_{j+1} = (-1)^j \delta, \qquad j = 1, \ldots, J-1. \tag{10}$$

The coefficients can be found from the triangular neighborhood $\Omega_{P_{j,i}}$ of any of the $P_{j,i}$, $j = 1, \ldots, J-1$ (see Fig. 1 for an illustration), and the definitions leading to (5). We do not need their exact coefficient expressions, just their limit behavior as $t \longrightarrow 0$, i.e., we will be looking for the entries of $\hat{A} := \lim_{t \to 0} \tilde{A}$ and $\hat{b} := \lim_{t \to 0} \tilde{b}$. Indeed, since for $t \longrightarrow 0$ the whole $\Omega_{P_{j,i}}$ is essentially covered by the single triangle with vertices $P_{j,i}$, $P_{j-1,i-1}$, $P_{j-1,i}$, and since $f(x) = (-1)^j$ on the latter, we have

$$\alpha = 1 + \mathrm{O}(t), \quad \beta = 1 + \mathrm{O}(t), \quad \gamma = \mathrm{O}(t^2), \quad \delta = 2 + \mathrm{O}(t).$$

From (10), we obtain in the limit $t \longrightarrow 0$ the equations

$$\hat{x}_{j-1} + \hat{x}_j = 2(-1)^j, \qquad j = 1, \ldots, J-1,$$

where $\hat{x}_j = \lim_{t \to \infty} x_j$ (the existence of these limits follows from the invertibility of the limit matrix $\hat{A}$, see below). Similar considerations for the first and the last two equations of $\tilde{A}\tilde{x} = \tilde{b}$ yield the remaining three equations of $\hat{A}\hat{x} = \hat{b}$ as follows:

$$\frac{3}{2}\hat{x}_0 + \frac{1}{2}\hat{x}_1 = -2, \qquad \hat{x}_{j-1} + \hat{x}_j = 2(-1)^j, \quad j = J, J+1.$$

The resulting matrix $\hat{A}$ is obviously invertible. After finding $\hat{x}_0 = \hat{x}_1 = -1$ from the first two equations, forward substitution gives $\hat{x}_j = (2j-1)(-1)^j$, $j = 2, \ldots, J+1$. This implies that for any $\epsilon > 0$ one can find a sufficiently small $t > 0$ such that

$$\|\tilde{x}\|_\infty \geq \|\hat{x}\|_\infty - \epsilon = 2J + 1 - \epsilon.$$

This proves Theorem 1. Note that the above reasoning does not work for type-I triangulations of a square obtained from a non-uniform rectangular tensor-product partition.

We conclude with the straightforward extension of the above example to arbitrary $d > 2$. Let e^m denote the m-th unit coordinate vector in $\mathbb{R}^d$, $m = 1, \ldots, d$. As Ω we take the convex polyhedral domain with vertices

$$P_{0,1} = e^1 + e^2, \quad P_{0,2} = e^1 - e^2, \quad P_{0,3} = -e^1 - e^2, \quad P_{0,4} = -e^1 + e^2,$$

and $P'_m = e^m$, $m = 3, \ldots, d$. For $d = 3$, this domain is a pyramid with square base in the xy-plane, and tip on the z-axis.

A suitable simplicial partition of Ω is obtained as follows. The base square with vertices $P_{0,i}$, $i = 1, \ldots, 4$, is triangulated into $\mathcal{T}_J$ which depends on the parameters $0 < t < 1$ and J as described for $d = 2$. The resulting triangulation (now embedded into $\mathbb{R}^d$) and its vertices are again denoted by $\mathcal{T}_J$ resp. by $P_{j,i}$, $i = 1, \ldots, 4$, $j = 0, \ldots, J$, and P_{J+1}. Then each simplex in the associated simplicial partition $\mathcal{P}_J$ of Ω is generated by the $d-2$ vertices P'_m, $m = 3, \ldots, d$, and the three vertices of a triangle in $\mathcal{T}_J$. The latter is called base triangle of the associated simplex. Obviously, the d-dimensional volume of each simplex is proportional to the 2-dimensional area of its base triangle, with proportionality constant $2/d!$. To obtain a suitable function $f \in L_\infty(\Omega)$ with $\|f\|_{L_\infty} = 1$ we prescribe values ± 1 on the simplices by inheritance from the values on the base triangles of the above 2-dimensional f.

From the symmetry properties of f and $\mathcal{P}_J$, it is obvious that the L_2-orthogonal projection $P_{V(\mathcal{P}_J)}f$ of f onto the linear spline space $V(\mathcal{P}_J)$ is characterized by its value x_{J+1} at the origin P_{J+1}, by values x_j taken at the vertices $P_{j,i}$, $i = 1, \ldots, 4$, where $j = 0, \ldots, J$, and a common value x' taken at the remaining vertices P'_m, $m = 3, \ldots, d$. To estimate these values which, in complete analogy to the 2-dimensional case, are represented by the solution vector $\tilde{x}$ of a certain linear system $\tilde{A}\tilde{x} = \tilde{b}$ (now of dimension $J+3$), we need the limit version $\hat{A}\hat{x} = \hat{b}$ of this linear system for $t \to 0$. We spare the reader the elementary calculations, and state it without proof:

$$\begin{aligned}
\hat{x}_j + \hat{x}_{j-1} + \frac{d-2}{2}\hat{x}' &= (-1)^j \frac{d+2}{2}, \qquad j = 1, \ldots, J+1, \\
\frac{3}{2}\hat{x}_0 + \frac{1}{2}\hat{x}_1 + \frac{d-2}{2}\hat{x}' &= -\frac{d+2}{2}, \\
\hat{x}_0 + \frac{1}{2}\hat{x}_1 + (1 + \frac{d-3}{2})\hat{x}' &= -\frac{d+2}{2}.
\end{aligned}$$

From this system one easily concludes that $\|\hat{x}\|_\infty \geq cJ$, and consequently $\|P_{V(\mathcal{P}_J)} f\|_{L_\infty} \geq cJ$ for a small enough $t > 0$ which implies the desired result. The lower bound cJ obtained does not seem to accurately reflect the possible growth of the projector norms in L_∞ as a function of the number of simplices, for $d \geq 3$ we would rather expect an exponential rate.

References

[1] I. Babuska, A. K. Aziz, On the angle condition in the finite element method, SIAM J. Numer. Anal. **13** (1976), 214–226.

[2] C. de Boor, On a max-norm bound for the least-squares spline approximant, in *Approximation and Function Spaces* (Gdansk, 1979), Z. Ciesielski (ed.), pp. 163–175, North-Holland, Amsterdam, 1981.

[3] Z. Ciesielski, Properties of the orthonormal Franklin system, Studia Math. **23** (1963), 141–157.

[4] Z. Ciesielski, Private communication, Int. Conf. *Approximation Theory and Probability*, Bedlowo, 2004.

[5] J. Desloux, On finite element matrices, SIAM J. Numer. Anal. **9**, 2 (1972), 260–265.

[6] J. Douglas, Jr., T. Dupont, L. Wahlbin, The stability in L^q of the L^2-projection into finite element function spaces, Numer. Math. **23** (1975), 193–197.

[7] A. Yu. Shadrin, The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture, Acta Math. **187** (2001), 59–137.