Criteria for Hierarchical Bases in Sobolev Spaces

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Abstract

Several approaches to solving elliptic problems numerically are based on hierarchical Riesz bases in Sobolev spaces. We are interested in determining the exact range of Sobolev exponents for which a system of compactly supported functions derived from a multiresolution analysis does form such a Riesz basis. This involves determining the smoothness of the dual system. The elements of the dual system typically consist of non-compactly supported functions, whose smoothness can be treated by extending the results of [7, 22, 9]. We show how to determine the exact range of Sobolev exponents in the multivariate case, both theoretically and numerically, from spectral properties of transfer operators. This technique is applied to several bases deriving from linear finite elements which have been proposed in the literature. For Stevenson's hierarchical basis [29], we find that it forms a Riesz basis in $H^s(\mathbb{R}^d)$ for $-0.990236... < s < 3/2$.

1 Introduction

The object of this paper is to determine, both theoretically and numerically, the range of s for which a hierarchical system

$$\mathbf{\Psi} := \{\phi(x-\alpha),\ 2^{jd/2}\psi^\lambda(2^jx-\alpha)\ :\ \alpha\in\mathbb{Z}^d,\ j=0,1,\dots,\ \lambda=1,\dots,2^d-1\} \quad (1)$$

forms a Riesz basis for the Sobolev space $H^s(\mathbb{R}^d)$. This system will be derived from a dyadic multiresolution analysis (MRA)

$$V_0 \subset V_1 \subset \dots \subset V_j \subset \dots \quad (2)$$

of closed subspaces of $L_2(\mathbb{R}^d)$ with scaling function $\phi \in V_0$ of compact support by specifying $2^d - 1$ functions $\psi^\lambda \in V_1$, $\lambda = 1, \ldots, 2^d - 1$, of compact support.

It turns out that the range of such s is determined by the Sobolev regularity s_ϕ of ϕ and the Sobolev regularity $s_{\tilde{\phi}}$ of the scaling function $\tilde{\phi}$ of the dual MRA. $\tilde{\phi}$ is determined by both ϕ and the ψ^λ. It is known (see e.g., [13, Theorem 5.1.1, (5.1.10)]) that if ϕ and $\tilde{\phi}$ are of compact support, then the system $\mathbf{\Psi}$ is a Riesz basis for $H^s(\mathbb{R}^d)$ for all s with $-s_{\tilde{\phi}} < s < s_\phi$ and that this interval is sharp.

Usually, however, $\tilde{\phi}$ is not of compact support. It may even happen that $\tilde{\phi} \notin L_2(\mathbb{R}^d)$ so that $\tilde{\phi}$ only exists in the distributional sense. The symbol $\tilde{m}(\theta)$ associated with the refinement equation for $\tilde{\phi}$ is a rational trigonometric function (with no discontinuities on the torus) instead of being a trigonometric polynomial. This causes two difficulties: the theorem quoted above cannot be applied and most of the known methods for determining the Sobolev regularity of $\tilde{\phi}$ do not apply, particularly in the multivariate case.

Most of this paper is devoted to overcoming these difficulties. We show, in Theorem 17, that the theorem on the Riesz basis property of $\mathbf{\Psi}$ quoted above also holds if the symbol $\tilde{m}(\theta)$ is a rational trigonometric function (some more general cases can be covered as well). We also show that, in this case, the Sobolev regularity of $\tilde{\phi}$ can be computed as a limit of the Sobolev regularities of scaling functions implicitly defined by trigonometric polynomial approximants to $\tilde{m}(\theta)$, see Lemma 12. This result can be used independently of the other considerations. That is, we have obtained a method to determine the Sobolev regularity of certain non-compactly supported scaling functions.

It is a (fortuitous) coincidence that the best algorithm for computing the Sobolev regularity of $\tilde{\phi}$ is based on the proof of the theoretical result, namely by approximating $\tilde{m}(\theta)$ by trigonometric polynomials. These polynomials correspond to scaling functions of compact support. For such functions, one can use ideas from [22, Section 4] or [6] to determine their Sobolev regularities. In the numerical part, section 3, we report on the results of the computations for several known and new Riesz bases.

The determination of the H^s–Riesz basis properties of systems of the form (1) is of sufficient mathematical interest to be pursued for its own sake. Nevertheless, our motivation stems from recent research on multilevel finite element preconditioners. Typically, the numerical solution of elliptic partial differential equations by the finite element method results in a linear system with sparse coefficient matrix whose condition number grows strongly as the mesh size of the discretization decreases. Due to the large dimensions of these systems, iterative methods are used for their solution but, since the condition number degenerates, an unacceptably large number of iterations need to be performed in order to obtain a satisfactory approximation to the solution.

Hierarchical Riesz bases can be used to construct multilevel preconditioners for such discretization matrices. If a Riesz basis fulfills certain conditions, then the preconditioned matrix has a condition number which is independent of the mesh size of the discretization and the cost of performing one iteration is proportional to the number of unknowns N of the matrix. Consequently, the operation count of obtaining an error reduction by a factor ε is proportional to $N|\log\varepsilon|$. Finally, by using nested iteration, the computational cost of obtaining an approximation of the exact solution to within the discretization error of the finite element scheme may even remain proportional to N, which is in some sense optimal.

The main conditions a system $\mathbf{\Psi}$ of the form (1) must fulfill in order to finally lead to such an optimal multilevel preconditioning method are as follows.

- If the solution of the partial differential equation is defined by an H^s–elliptic variational problem, then $\mathbf{\Psi}$ must be an H^s–Riesz basis. For example, $s=1$ for second order elliptic partial differential equations.
- Since the functions ϕ and ψ^λ occuring in $\mathbf{\Psi}$ stem from a multiresolution analysis, they satisfy refinement equations

$$\phi(x) = \sum_\alpha a_\alpha \phi(2x-\alpha)\ , \qquad \psi^\lambda(x) = \sum_\alpha a_\alpha^\lambda \phi(2x-\alpha)\ . \tag{3}$$

  The number of nonzero coefficients in these equations must be finite, which is equivalent to ϕ and ψ^λ being of compact support. The cost of carrying out one application of the preconditioner grows linearly with this number.

The mask (a_α) in (3) is used to define prolongation/restriction matrices to exchange information between different levels while the (a_α^λ) define approximate solvers (smoothers) on each level. These are the operations needed in the implementation of a multilevel preconditioner (and of similar multigrid methods). Thus, in contrast to schemes using both wavelet decomposition and reconstruction, we do not require that the refinement equation for $\tilde{\phi}$ which is analogous to (3), has only a finite number of non-zero coefficients. See [27, 14], for a detailed exposition of these remarks.

Although realistic applications require the consideration of hierarchical bases on bounded domains Ω, discretization spaces V_j that are not shift-invariant, the inclusion of boundary conditions etc., in this paper we assume the shift-dilation invariant setting of an MRA (2) in $L_2(\mathbb{R}^d)$. The simplification allows us to obtain sharp results on the s-intervals for which a hierarchical system $\mathbf{\Psi}$ forms a Riesz basis in $H^s(\mathbb{R}^d)$) which is of certain value for the mathematical foundation of existing methods but also for the design of new methods. As a rule, establishing the H^s–Riesz basis property of $\mathbf{\Psi}$ guarantees the boundedness of the condition number of the preconditioned discretization matrices of any H^s-elliptic problem with respect to V_J if a multilevel preconditioner derived from the finite sections

$$\mathbf{\Psi}_J = \{\phi(x-\alpha),\ 2^{jd/2}\psi^\lambda(2^j x-\alpha)\ :\ k\in\mathbb{Z}^d,\ j=0,1,\ldots,J,\ \lambda=1,\ldots,2^d-1\}$$

of $\mathbf{\Psi}$ is used. Unfortunately, the present theory does not provide reasonably sharp estimates of the Riesz constants for these $\mathbf{\Psi}_J$ and their asymptotical behavior for $J\to\infty$, the knowledge of which is of practical interest to eventually compare the preconditioning potential of different $\mathbf{\Psi}$. It is expected (by some kind of interpolation argument) that the bounds for the actual condition numbers are better in the interior and become worse near the endpoints of s-intervals where we establish the H^s–Riesz basis property of (1). Numerical tests for the non-asymptotical range of J are therefore of interest, see [26] for some experiments.

The material is organized as follows. Section 2 contains criteria for the Riesz basis property in $H^s(\mathbb{R}^d)$ of a given hierarchical system (1). In subsection 2.1, the notation for the $\mathbb{R}^d$ case is fixed, and preliminary results on MRA's are collected. The basics of stable two-level splittings

$$V_j = V_{j-1} \dot{+} W_j$$

of V_j into V_{j-1} and a wavelet space W_j are introduced. The importance of approximation-theoretical properties of $\{V_j\}$ and its dual MRA and, consequently, of the Sobolev regularity of the dual scaling function for proving Riesz basis properties is pointed out (see [27, 12, 13, 14]). More material on L_2-Jackson-Bernstein inequalities for an MRA is given later in subsection 2.6. In subsection 2.2, we derive a formula for the symbol $\tilde{m}(\theta)$ associated with the refinement equation of the dual MRA, and discuss its properties. In subsection 2.3 we consider properties of the transfer operators associated with a refinement equation following [7, 18]. In subsection 2.4, we show that at least the Fourier transform of the dual scaling function exists, and how to compute the Sobolev regularity $s_{\tilde{\phi}}$ from it. In addition, we show that $s_{\tilde{\phi}}$ does not exceed the Strang-Fix order of $\tilde{m}(\theta)$, just as in the case of compactly supported scaling functions.

In subsections 2.5 and 2.6, we show that the system $\mathbf{\Psi}$ defined in (1) is an H^s–Riesz basis for all s with $-s_{\tilde{\phi}} < s < s_\phi$, again as in the compactly supported case. The material of these two sections is organized by treating the cases $\tilde{\phi} \in L_2(\mathbb{R}^d)$ (subsection 2.6) and $\tilde{\phi} \notin L_2(\mathbb{R}^d)$ (subsection 2.5). But in fact, if one is only interested in knowing when the system $\mathbf{\Psi}$ forms a Riesz basis in $H^s(\mathbb{R}^d)$ for $s > 0$, then the results of subsection 2.5 suffice for both cases and much of the material in the previous subsections is not needed. The main results are to be found in Theorem 13 of subsection 2.5 and Theorem 17 of subsection 2.6. Although these theorems are formulated for a compactly supported scaling function ϕ, they hold with the same proof if $m(\theta)$, the symbol associated with the refinement equation for ϕ, is a rational trigonometric function or, even more generally, when $m(\theta)$ and $\tilde{m}(\theta)$ are periodic functions with exponentially decaying Fourier coefficients (see condition A3 of subsection 2.3).

Although subsections 2.5 and 2.6 contain some theoretical improvements of existing results for wavelet regularity and Riesz basis property of biorthogonal systems in the case of infinite masks, we still view section 3 as the most interesting contribution of the paper. Here we consider a number of particular hierarchical systems $\mathbf{\Psi}$ based on box spline MRAs. Some of them have counterparts for sequences of finite element spaces on bounded domains or have been introduced in other papers. In subsection 3.1 we give a detailed treatment of the hierarchical system $\mathbf{\Psi}_{3HB}$ investigated by Stevenson in [29, 30, 31]. This example is based on piecewise linear splines on simplicial meshes, with ψ^λ–masks consisting of 3 coefficients, independently of the dimension d, and possesses surprisingly nice properties. More examples for the piecewise linear case are considered and compared in subsection 3.2 while subsection 3.3 is devoted to other box spline MRAs. In these subsections we also briefly discuss factorization techniques which are sometimes useful to enhance the performance of the numerical methods for approximating the exact regularity exponents.

Let us finally note that, in comparison to the preprint versions [24, 25] of this study, the exposition has changed substantially. We have corrected a number of misprints and inconsistencies, improved our numerical experiments and incorporated recent results on wavelet regularity. We would like to thank A. Cohen and R.-J. Jia for making available to us the manuscripts [9, 22], and the referees for their constructive criticism.

2 Theory: Riesz bases in $H^s(\mathbb{R}^d)$

2.1 General definitions and assumptions

By $\hat{f}$, we denote the Fourier transform of $f \in L_2(\mathbb{R}^d)$,

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(x) e^{-ix\theta} dx \ ,$$

and by $\|f\|_{H^s}$ the Sobolev norm of $f \in H^s(\mathbb{R}^d)$,

$$\|f\|_{H^s}^2 := (2\pi)^{-d} \int_{\mathbb{R}^d} (1+|\theta|^2)^s |\hat{f}(\theta)|^2 d\theta = (2\pi)^{-d} \int_{\mathcal{C}} g_{f,s}(\theta) d\theta \ ,$$

where

$$g_{f,s}(\theta) := \sum_{\alpha\in\mathbb{Z}^d} (1+|\theta+2\pi\alpha|^2)^s |\hat{f}(\theta+2\pi\alpha)|^2 \tag{4}$$

is a non-negative periodic function, and $\mathcal{C} := [-\pi,\pi]^d$ is the centered torus. For $s=0$, we use L_2 instead of H^0 in the notation, and drop s as an index.

Let V be a separable Hilbert space. An at most countable system $F := \{f_k\} \subset V$ will be called a *Riesz system* if there are two positive constants A, B such that

$$A \sum_k |c_k|^2 \|f_k\|_V^2 \le \|\sum c_k f_k\|_V^2 \le B \sum_k |c_k|^2 \|f_k\|_V^2$$

for ℓ_2–sequences (c_k). The optimal values for A, B are called *Riesz constants.* A system is a *Riesz basis* of V if it is a basis for V and a Riesz system. If F is a Riesz system then it is automatically a Riesz basis in its closed linear span. Moreover, any subsystem of a Riesz system is itself a Riesz system. Note that by including the norm squares $\|f_k\|_V^2$ into the definition, the Riesz basis property becomes scaling-invariant.

Thus a system $\{\psi_\alpha^\lambda := \psi^\lambda(\cdot-\alpha) \ : \ \alpha\in\mathbb{Z}^d,\ \lambda\in\Lambda\}$ generated by the integer shifts of finitely many functions $\psi^\lambda \in H^s(\mathbb{R}^d)$ (i.e., $\#\Lambda < \infty$) is a Riesz system in $H^s(\mathbb{R}^d)$ if

$$\|\sum_{\lambda\in\Lambda}\sum_{\alpha\in\mathbb{Z}^d} c_\alpha^\lambda \psi_\alpha^\lambda\|_{H^s}^2 \asymp \sum_{\lambda\in\Lambda} \|\psi^\lambda\|_{H^s}^2 \sum_{\alpha\in\mathbb{Z}^d} |c_\alpha^\lambda|^2 \asymp \sum_{\lambda\in\Lambda} \|\psi^\lambda\|_{H^s}^2 \|c^\lambda\|_{L_2(\mathcal{C})}^2 \tag{5}$$

for all $\ell_2(\mathbb{Z}^d)$–sequences (c_α^λ), $\lambda \in \Lambda$. Here and, if not defined differently, throughout the paper, we denote the periodic function

$$c(\theta) := \sum_{\alpha \in \mathbb{Z}^d} c_\alpha e^{-i\alpha\theta} \in L_2(\mathbb{T}^d) \ . \tag{6}$$

associated to an $\ell^2(\mathbb{Z}^d)$–sequence $c := (c_\alpha)$ by the same letter. The symbol $\asymp$ is used for two-sided inequalities that hold with positive constants independent of the parameters, functions, sequences etc. involved.

As a special case of (5), when the system is generated from a single function ϕ, by taking Fourier transforms we have that $\{\phi_\alpha := \phi(\cdot - \alpha) \ : \ \alpha \in \mathbb{Z}^d\}$ is an H^s–Riesz basis for its closed linear span if and only if

$$\|c\|^2_{L_2(\mathcal{C})} \|\phi\|^2_{H^s} \asymp \int_{\mathbb{R}^d} |c(\theta)|^2 (1 + |\theta|^2)^s |\hat{\phi}(\theta)|^2 d\theta = \int_{\mathcal{C}} |c(\theta)|^2 g_{\phi,s}(\theta) d\theta$$

for all $c \in L_2(\mathcal{C})$. But this is possible if and only if there are positive constants C_1, C_2 with

$$0 < C_1 \leq g_{\phi,s}(\theta) \leq C_2 < \infty \ , \quad \text{a.e. on } \mathcal{C} \ . \tag{7}$$

Of course, one obtains the usual condition for an L_2–Riesz basis on setting $s = 0$. Properties of the function $g_{\phi,s}$ play an important role throughout the paper.

A *dyadic multiresolution analysis (MRA)* of $L_2(\mathbb{R}^d)$ is an increasing sequence (2) of closed subspaces $V_j \subset L_2(\mathbb{R}^d)$, $j \geq 0$, with the properties

- $v_j \in V_j \iff v_j(2\cdot) \in V_{j+1}$
- $\text{clos}_{L^2(\mathbb{R}^d)}(\cup_{j=0}^\infty V_j) = L_2(\mathbb{R}^d)$ (density)
- there exists a function $\phi \in V_0$ (the *scaling function*) such that $\{\phi_\alpha \ : \ \alpha \in \mathbb{Z}^d\}$ is a Riesz basis for V_0.

It follows that V_j is invariant with respect to shifts by $2^{-j}\alpha$, $\alpha \in \mathbb{Z}^d$. In the following we consistently use the notation

$$\phi_{j,\alpha} = 2^{jd/2} \phi(2^j \cdot - \alpha) \ , \quad \alpha \in \mathbb{Z}^d \ , \ \ j \geq 0 \ , \tag{8}$$

for the dyadic shifts and dilates of scaling functions. Due to the scaling factor in (8), we have $\|\phi_{j,\alpha}\|_{L_2(\mathbb{R}^d)} = \|\phi\|_{L_2(\mathbb{R}^d)}$. We call

$$\Phi_j := \{\phi_{j,\alpha} \ : \ \alpha \in \mathbb{Z}^d\} \tag{9}$$

the *standard basis* of V_j. Since the single function ϕ essentially determines the MRA, we will say for short that ϕ *generates a MRA* if the above properties are satisfied. We have chosen $j = 0$ as the coarsest level which is arbitrary but convenient.

Since $\phi \in V_0 \subset V_1$, and by the Riesz basis property of Φ_1 in V_1, the function ϕ must satisfy a *refinement equation*

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^d} a_\alpha \phi(2x - \alpha) \tag{10}$$

for some sequence $(a_\alpha) \in \ell_2(\mathbb{Z}^d)$. After taking the Fourier transform of both sides, we have

$$\hat{\phi}(\theta) = m(\theta/2)\hat{\phi}(\theta/2) \ , \tag{11}$$

where

$$m(\theta) := 2^{-d} \sum_{\alpha \in \mathbb{Z}^d} a_\alpha e^{-i\alpha\theta} \ . \tag{12}$$

The sequence (a_α) and $m(\theta)$ are called *mask* and *symbol* associated with the refinement equation (10), respectively (below we will also use the shorter expressions *ϕ–mask* and *symbol of ϕ*). Note the scaling factor 2^{-d} in (12) which will simplify some expressions below. We will now specialize further by assuming that the mask (a_α) in (10) is finite. Thus, $m(\theta)$ is a trigonometric polynomial written in complex form and, as is well known, it follows that the scaling function ϕ has compact support. This, in particular, implies that $\hat{\phi}$ is a C^∞ function, and that pointwise

$$\hat{\phi}(\theta) = \prod_{k=1}^{\infty} m(2^{-k}\theta) \ . \tag{13}$$

Here, and in what follows, the normalization $\hat{\phi}(0) = 1$ is assumed. Note that $m(0) = 1$ then follows. Formula (13) makes it possible to derive L_2 and Sobolev space properties of ϕ from knowledge about its symbol via Fourier transform techniques and is extensively used in the wavelet literature. E.g., one can prove that the lower inequality in (7) is valid if and only if the sequence $(\hat{\phi}(\theta+2\pi\alpha) \ : \ \alpha \in \mathbb{Z}^d)$ is different from the null sequence for all θ (thus, (13) is either satisfied for all s or for no s at all). The sets

$$\Lambda = \{0,1\}^d \ , \qquad \Lambda' = \Lambda \backslash \{0\} \ ,$$

of vertices of the unit cube in $\mathbb{R}^d$ will be fixed from now on. Note that Λ' has $2^d - 1$ elements.

We introduce the following two important numbers associated with ϕ:

- *Maximal Sobolev regularity.* The number

  $$s_\phi = \sup\{s \ : \ \phi \in H^s(\mathbb{R}^d)\} \ ,$$

  is called the *regularity exponent* of ϕ. Since so far we have assumed that $\phi \in L_2(\mathbb{R}^d)$ has compact support and generates a MRA, we necessarily have $s_\phi > 0$, compare [38, 22, 9]. From these sources, it also follows that the supremum cannot be replaced by the maximum since $\phi \notin H^{s_\phi}(\mathbb{R}^d)$. Below we discuss in more detail how to determine the numerical value of s_ϕ (or sufficiently accurate approximations to it) for a ϕ which is only defined implicitly by $m(\theta)$ as in (13).

- *Strang-Fix order.* If $m(\theta)$ is a sufficiently smooth periodic function with $m(0) \neq 0$ then its *Strang-Fix order* L_m will be defined as the largest non-negative integer L such that

  $$\partial_\beta m(\pi\lambda) = 0 \ , \quad \forall \, \lambda \in \Lambda' \ , \quad \forall \, \beta \in \mathbb{Z}_+^d \ : \ |\beta|_1 < L \tag{14}$$

(the 1-norm for vectors is defined by $|\beta|_1 = |\beta_1|+\ldots+|\beta_d|$). Condition (14) in terms of $m(\theta)$ is equivalent to the Strang-Fix conditions formulated in terms of $\hat{\phi}$, and to the fact that algebraic polynomials of total degree $< L$ can be represented locally (i.e., on any compact set in $\mathbb{R}^d$) by linear combinations of the basis functions in Φ_j ($j \geq 0$). It follows from [21] that if a compactly supported ϕ generates a MRA, the condition $\phi \in H^s(\mathbb{R}^d)$ (for some $s \geq 0$) necessarily implies $L_m > s$ (i.e., $s_\phi \leq L_m$, at least). As a consequence, for such ϕ we have at least $L_m \geq 1$, i.e., the relations

$$m(0) = 1 \; , \quad m(\lambda\pi) = 0 \; , \; \lambda \in \Lambda' \; , \tag{15}$$

have to be satisfied.

The standard way to define hierarchical systems $\mathbf{\Psi}$ associated with an MRA is by first finding a stable, direct sum decomposition

$$V_1 = V_0 \dot{+} W_1 \; , \tag{16}$$

where W_1 is a closed subspace of V_1 which is preserved under shifts by $\alpha \in \mathbb{Z}^d$. Moreover, the existence of $2^d - 1$ functions $\psi^\lambda \in V_1$, $\lambda \in \Lambda'$ (which we call *wavelets* for convenience) such that the set $\Psi_1 := \{\psi^\lambda_\alpha \; : \; \alpha \in \mathbb{Z}^d, \; \lambda \in \Lambda'\}$ forms a Riesz basis for W_1 is also assumed. The above conditions can be formulated by the single condition that the system $\Phi_0 \cup \Psi_1$ is an L_2-Riesz basis of V_1. Just as for the scaling function ϕ, we will require that

$$\psi^\lambda(x) = \sum_{\alpha \in \mathbb{Z}^d} a^\lambda_\alpha \phi(2x - \alpha) \; , \qquad \lambda \in \Lambda' \; , \tag{17}$$

holds with finite sequences (a^λ_α) which also implies compact support of the wavelets ψ^λ. These sequences are called ψ^λ*–masks*, the corresponding symbols

$$m^\lambda(\theta) = 2^{-d} \sum_{\alpha \in \mathbb{Z}^d} a^\lambda_\alpha e^{-i\alpha\theta} \tag{18}$$

are defined in analogy with (12).

We gather these considerations together into two general assumptions which will hold for the rest of the paper. Conditions for verifying that these assumptions hold with respect to the L_2–norm (or H^s–norms when applicable) using information about the refinement equations (10) and (17) (i.e., about the masks and symbols (a_α), $m(\theta)$, and (a^λ_α), $m^\lambda(\theta)$) are well-known and can be formulated in various terms (see [34, 23, 16]).

A1. The scaling function $\phi \in L_2(\mathbb{R}^d)$ is of compact support, and generates a MRA $\{V_j\}$. Its mask (a_α) defined by (10) is finite, the symbol $m(\theta)$ is a trigonometric polynomial.

A2. There are $2^d - 1$ wavelets $\psi^\lambda \in V_1$, $\lambda \in \Lambda'$, of compact support such that the system

$$\Phi_0 \cup \Psi_1 = \{\phi(\cdot - \alpha), \, \psi^\lambda(\cdot - \alpha) \; : \; \alpha \in \mathbb{Z}^d, \; \lambda \in \Lambda'\}$$

is a L_2–Riesz basis of V_1. Their masks (a^λ_α) are finite, the symbols $m^\lambda(\theta)$ are trigonometric polynomials.

We introduce the following notation for the dyadic shifts and dilates of wavelets:

$$\psi_{j,\alpha}^{\lambda} = 2^{(j-1)d/2}\psi^{\lambda}(2^{j-1}\cdot -\alpha)\ , \quad \alpha \in \mathbb{Z}^d \quad j \geq 1\ , \tag{19}$$

for $\lambda \in \Lambda'$. Again, $\|\psi_{j,\alpha}^{\lambda}\|_{L_2} = \|\psi^{\lambda}\|_{L_2}$. Note that the dilation for the wavelets $\psi_{j,\alpha}^{\lambda}$ in (19) differs from that of the scaling functions $\phi_{j,\alpha}$ in (8) with the same $j > 0$.

In subsections 2.5 and 2.6, we will investigate the hierarchical system

$$\mathbf{\Psi} := \cup_{j=0}^{\infty}\Psi_j\ , \qquad \Psi_j := \begin{cases} \Phi_0 = \{\phi_\alpha\ :\ \alpha \in \mathbb{Z}^d\} & ,\quad j = 0 \\ \{\psi_{j,\alpha}^{\lambda}\ :\ \alpha \in \mathbb{Z}^d,\ \lambda \in \Lambda'\} & ,\quad j \geq 1 \end{cases} \tag{20}$$

as a potential Riesz basis in $H^s(\mathbb{R}^d)$. This definition is consistent with (1). We define the *wavelet spaces*

$$W_j := \mathrm{clos}_{L_2}(\mathrm{span}\,\Psi_j)\ , \quad j \geq 1\ . \tag{21}$$

By dilation arguments, it follows that $V_j = V_{j-1}\dot{+}W_j$, and that Ψ_j is an L_2–Riesz basis in W_j for all $j \geq 1$.

Suppose that the system (20) is a Riesz basis of $L_2(\mathbb{R}^d)$, and suppose that $\tilde{\phi}$ and $\tilde{\psi}^{\lambda}$, $\lambda \in \Lambda'$, belonging to $L_2(\mathbb{R}^d)$ generate another Riesz basis in $L_2(\mathbb{R}^d)$:

$$\tilde{\mathbf{\Psi}} := \cup_{j=0}^{\infty}\tilde{\Psi}_j\ , \qquad \tilde{\Psi}_j := \begin{cases} \{\tilde{\phi}_\alpha\ :\ \alpha \in \mathbb{Z}^d\} & ,\quad j = 0 \\ \{\tilde{\psi}_{j,\alpha}^{\lambda}\ :\ \alpha \in \mathbb{Z}^d,\ \lambda \in \Lambda'\} & ,\quad j \geq 1\ . \end{cases} \tag{22}$$

Then $\tilde{\mathbf{\Psi}}$ is said to be the *dual system*, or biorthogonal system to $\mathbf{\Psi}$ if

$$\begin{aligned} &(\phi_\alpha, \tilde{\phi}_\beta)_{L_2} = \delta_{\alpha,\beta}\|\phi\|_{L_2}\|\tilde{\phi}\|_{L_2} & \\ &(\phi_\alpha, \tilde{\psi}_{j,\beta}^{\lambda})_{L_2} = 0\ , \quad (\psi_{j,\alpha}^{\lambda}, \tilde{\phi}_\beta)_{L_2} = 0 & j \geq 1 \\ &(\psi_{j,\alpha}^{\lambda}, \tilde{\psi}_{k,\beta}^{\eta})_{L_2} = \delta_{j,k}\delta_{\alpha,\beta}\delta_{\lambda,\eta}\|\psi^{\lambda}\|_{L_2}\|\tilde{\psi}^{\eta}\|_{L_2} & j,k \geq 1. \end{aligned} \tag{23}$$

for any $\alpha, \beta \in \mathbb{Z}^d$, and $\lambda, \eta \in \Lambda'$.

Under these circumstances, $\tilde{\phi}$ generates the *dual MRA*

$$\tilde{V}_j := \mathrm{clos}_{L_2}(\mathrm{span}\,\tilde{\Phi}_j)\ , \qquad \tilde{\Phi}_j := \{\tilde{\phi}_{j,\alpha}\ :\ \alpha \in \mathbb{Z}^d\}\ , \quad j \geq 0\ .$$

In analogy to (21), we define detail spaces $\tilde{W}_j := \mathrm{clos}_{L_2}(\mathrm{span}\,\tilde{\Psi}_j)$, with basis $\tilde{\Psi}_j$, and we have $\tilde{V}_j = \tilde{V}_{j-1}\dot{+}\tilde{W}_j$, $j \geq 1$. It is easy to show that the dual MRA satisfies the general assumptions A1 and A2, except for the compact support properties of $\tilde{\phi}$ and $\tilde{\psi}^{\lambda}$ which we do not assume below unless explicitly mentioned.

For compactly supported ϕ and $\tilde{\phi}$, the following theorem (see [13]) shows how to use the dual system to determine the s-interval for which a system (20) is a Riesz basis for $H^s(\mathbb{R}^d)$.

Theorem 1 *Assume that $\mathbf{\Psi}$, $\tilde{\mathbf{\Psi}}$ are dual Riesz bases in $L_2(\mathbb{R}^d)$ of the form (20), (22), respectively, which are associated with the dual MRA's $\{V_j\}$, $\{\tilde{V}_j\}$ satisfying A1. In particular, the symbols $m(\theta)$, $\tilde{m}(\theta)$ of the scaling functions $\phi, \tilde{\phi} \in L_2(\mathbb{R}^d)$ of* *the two MRA are trigonometric polynomials (i.e., are defined by finitely supported masks (a_α), $(\tilde{a}_\alpha)$). Then the regularity exponents of ϕ and $\tilde{\phi}$ are positive, and*

$$\mathbf{\Psi} \text{ is a Riesz basis in } H^s(\mathbb{R}^d) \iff -s_{\tilde{\phi}} < s < s_\phi \,, \tag{24}$$

respectively

$$\tilde{\mathbf{\Psi}} \text{ is a Riesz basis in } H^s(\mathbb{R}^d) \iff -s_\phi < s < s_{\tilde{\phi}} \,. \tag{25}$$

There are other versions of this theorem which replace some of the assumptions on dual systems with Jackson and Bernstein inequalities for the MRA's (see below). The reason that we cannot apply this theorem directly is that the dual scaling function $\tilde{\phi}$ is often not of compact support and, consequently, that $\tilde{m}(\theta)$ is not a trigonometric polynomial.

If the systems $\mathbf{\Psi}, \tilde{\mathbf{\Psi}}$ form a pair of dual Riesz bases, we have

$$\begin{aligned} u = &\textstyle\sum_{\alpha\in\mathbb{Z}^d} \frac{(u,\tilde{\phi}_\alpha)_{L_2}}{\|\phi\|_{L_2}\|\tilde{\phi}\|_{L_2}}\phi_\alpha \\ &+ \textstyle\sum_{j=1}^{\infty}\sum_{\lambda\in\Lambda'}\sum_{\alpha\in\mathbb{Z}^d} \frac{(u,\tilde{\psi}^\lambda_{j,\alpha})_{L_2}}{2^{-d(j-1)}\|\psi^\lambda\|_{L_2}\|\tilde{\psi}^\lambda\|_{L_2}}\psi^\lambda_{j,\alpha} \end{aligned} \tag{26}$$

for the unique L_2–converging representation of an arbitrary $u \in L_2(\mathbb{R}^d)$ with respect to the system $\mathbf{\Psi}$. Here no compact support assumptions are necessary. This decomposition allows us to define projections $Q_j : L_2(\mathbb{R}^d) \to V_j$, $j \geq 0$, the partial sum operators associated with $\mathbf{\Psi}$, by

$$\begin{aligned} Q_j u := &\textstyle\sum_{\alpha\in\mathbb{Z}^d} \frac{(u,\tilde{\phi}_\alpha)_{L_2}}{\|\phi\|_{L_2}\|\tilde{\phi}\|_{L_2}}\phi_\alpha \\ &+ \textstyle\sum_{k=1}^{j}\sum_{\lambda\in\Lambda'}\sum_{\alpha\in\mathbb{Z}^d} \frac{(u,\tilde{\psi}^\lambda_{k,\alpha})_{L_2}}{2^{-d(k-1)}\|\psi^\lambda\|_{L_2}\|\tilde{\psi}^\lambda\|_{L_2}}\psi^\lambda_{k,\alpha} \,. \end{aligned} \tag{27}$$

Similar formulas hold for the decomposition with respect to the system $\tilde{\mathbf{\Psi}}$ and for the projections $\tilde{Q}_j$, which are the L_2–adjoints of the Q_j. Obviously,

$$Q_j Q_k = Q_j \,, \qquad 0 \leq j < k < \infty \,. \tag{28}$$

We can give an alternative definition of these projections which only involves the associated scaling functions. Let $\phi, \tilde{\phi} \in L_2(\mathbb{R}^d)$ be the scaling functions of the dual MRA's $\{V_j\}$, $\{\tilde{V}_j\}$, respectively. From (23) we have the biorthogonality relation

$$(\phi_{j,\alpha}, \tilde{\phi}_{j,\beta})_{L_2} = \delta_{\alpha,\beta}\|\phi_{j,\alpha}\|_{L_2}\|\tilde{\phi}_{j,\beta}\|_{L_2} = 2^{-jd}\delta_{\alpha,\beta}\|\phi\|_{L_2}\|\tilde{\phi}\|_{L_2} \,. \tag{29}$$

Then

$$Q_j u = c_j^{-1}\sum_{\alpha\in\mathbb{Z}^d}(u,\tilde{\phi}_{j,\alpha})_{L_2}\phi_{j,\alpha} \,, \qquad u \in L_2(\mathbb{R}^d) \,, \tag{30}$$

where $c_j = 2^{-jd}\|\phi\|_{L_2}\|\tilde{\phi}\|_{L_2}$ is the appropriate normalization factor. For this reason, we say that two MRA, now without reference to any Riesz bases, generated by ϕ and $\tilde{\phi}$, are dual or biorthogonal if (29) holds. Then Q_j is defined by (30) and $\tilde{Q}_j$ is its adjoint.

Following [12, 13], we aim at establishing the norm equivalence

$$\|u\|_{H^s}^2 \asymp \|u\|_{Q,s}^2 := \|Q_0 u\|_{L_2}^2 + \sum_{j=1}^{\infty} 2^{2sj} \|Q_j u - Q_{j-1} u\|_{L_2}^2 \tag{31}$$

between the H^s–norm and a decomposition norm determined by $\{Q_j\}$. If (28) holds and the range of $Q_j - Q_{j-1}$ coincides with W_j (the closed span of its L_2–Riesz basis Ψ_j) which would be the case under the assumptions made so far, we see that (31) is equivalent to the H^s–Riesz basis property of $\mathbf{\Psi}$ (analogous statements hold for $\{\tilde{Q}_j\}$ and $\tilde{\mathbf{\Psi}}$). Indeed, consider any (finite) linear combination

$$u \;=\; \sum_{\alpha \in \mathbb{Z}^d} a_\alpha \phi_\alpha + \sum_{j=1}^{\infty} \sum_{\lambda \in \Lambda'} \sum_{\alpha \in \mathbb{Z}^d} a_{j,\alpha}^{\lambda} \psi_{j,\alpha}^{\lambda} \; .$$

Then, by (28),

$$Q_0 u = \sum_{\alpha \in \mathbb{Z}^d} a_\alpha \phi_\alpha \; , \quad Q_j u - Q_{j-1} u = \sum_{\lambda \in \Lambda'} \sum_{\alpha \in \mathbb{Z}^d} a_{j,\alpha}^{\lambda} \psi_{j,\alpha}^{\lambda} \; , \quad j \geq 1 \; ,$$

and, consequently, by (31),

$$\|u\|_{H^s}^2 \asymp \| \sum_{\alpha \in \mathbb{Z}^d} a_\alpha \phi_\alpha \|_{L_2}^2 + \sum_{j=1}^{\infty} 2^{2sj} \| \sum_{\lambda \in \Lambda'} \sum_{\alpha \in \mathbb{Z}^d} a_{j,\alpha}^{\lambda} \psi_{j,\alpha}^{\lambda} \|_{L_2}^2 \; .$$

But $\|\phi_\alpha\|_{H^s}^2 \asymp \|\phi_\alpha\|_{L_2}^2 \asymp 1$ and $\|\psi_{j,\alpha}^{\lambda}\|_{H^s}^2 \asymp 2^{2js} \|\psi_{j,\alpha}^{\lambda}\|_{L_2}^2 \asymp 2^{2js}$, with constants depending on the finitely many functions ϕ and ψ^{λ}. Since Φ_0, Ψ_j are L_2–Riesz bases for V_0, W_j $(j \geq 1)$, respectively, we obtain

$$\begin{aligned} \|u\|_{H^s}^2 \;&\asymp\; \sum_{\alpha \in \mathbb{Z}^d} |a_\alpha|^2 + \sum_{j=1}^{\infty} 2^{2sj} \sum_{\lambda \in \Lambda'} \sum_{\alpha \in \mathbb{Z}^d} |a_{j,\alpha}^{\lambda}|^2 \\ &\asymp\; \sum_{\alpha \in \mathbb{Z}^d} \|\phi_\alpha\|_{H^s}^2 |a_\alpha|^2 + \sum_{j=1}^{\infty} \sum_{\lambda \in \Lambda'} \sum_{\alpha \in \mathbb{Z}^d} \|\psi_{j,\alpha}^{\lambda}\|_{H^s}^2 |a_{j,\alpha}^{\lambda}|^2 \; , \end{aligned}$$

which is the H^s–Riesz basis property of $\mathbf{\Psi}$. The argument for the opposite direction is completely analogous.

It turns out that norm equivalences such as (31) follow from the basic approximation-theoretic properties of $\{V_j\}$ and $\{\tilde{V}_j\}$ (see [13, Section 5.1]). We say that $\{V_j\}$ satisfies a Jackson respectively a Bernstein inequality of some order $s > 0$ if, with constants independent of $j \geq 0$,

$$\inf_{v_j \in V_j} \|u - v_j\|_{L_2} \leq C 2^{-js} \|u\|_{H^s} \qquad \forall \, u \in H^s(\mathbb{R}^d) \; , \tag{32}$$

respectively

$$\|v_j\|_{H^s} \leq C 2^{js} \|v_j\|_{L_2} \qquad \forall \, v_j \in V_j \; . \tag{33}$$

Theorem 2 *Let $\phi, \tilde{\phi} \in L_2(\mathbb{R}^d)$ generate the dual MRA $\{V_j\}$ and $\{\tilde{V}_j\}$. In addition, let $\{V_j\}$ and $\{\tilde{V}_j\}$ satisfy Jackson-Bernstein inequalities of order $\gamma > 0$ and $\tilde{\gamma} > 0$, respectively. Then (31) holds if $-\tilde{\gamma} < s < \gamma$. If, in addition, the ranges of $Q_j - Q_{j-1}$, $j \geq 1$, possess L_2–Riesz bases Ψ_j then $\mathbf{\Psi}$ is a Riesz basis in $H^s(\mathbb{R}^d)$ for the same s–range. An analogous statement holds for $\tilde{\mathbf{\Psi}}$.*

The more general form of Theorem 2 contained in [12, 13] makes the additional assumptions that (28) holds and that the Q_j, $\tilde{Q}_j$ are uniformly L_2-bounded. However, both of these follow from our assumption that ϕ and $\tilde{\phi}$ generate dual MRA.

2.2 The dual symbol

The goal of this subsection is to derive properties of the dual symbol. In principle, the dual symbol might not even be defined since we do not know whether our candidate system $\mathbf{\Psi}$ of the form (20) is a L_2–Riesz basis and, therefore, we do not know whether a dual system exists.

But we can find a formula for the dual symbol, were it to exist, directly from our assumptions A1 and A2 without reference to a dual system in the following way. Using the notation (12) and (18), we introduce the matrix function of size 2^d

$$L(\theta) := ((\, m^{\lambda'}(\theta + \pi\lambda)\,))_{\lambda,\lambda' \in \Lambda} \tag{34}$$

whose entries consist of trigonometric polynomials. We use the notation $m^0(\theta) := m(\theta)$, $(a^0_\alpha) := (a_\alpha)$ if $\lambda' = 0$. Then, by [34, Theorem 13],

$$\begin{array}{l} \Phi_0 \cup \Psi_1 \text{ is a } L_2\text{–Riesz basis in } V_1 \text{ if and only if} \\ L(\theta) \text{ is invertible for all } \theta \in \mathcal{C}. \end{array} \tag{35}$$

Equivalently, this could be expressed in terms of the matrix of *subsymbols*

$$M(\theta) = ((\, m^{\lambda'}_{\lambda}(\theta)\,))_{\lambda,\lambda' \in \Lambda} = 2^{-d} U L(\theta)\ ,$$

where the real transformation matrix $U = ((\,(-1)^{\lambda\lambda'}\,))_{\lambda,\lambda' \in \Lambda}$ satisfies $U = U^T = 2^d U^{-1}$, i.e., is orthogonal up to scaling. The entries $m^{\lambda'}_{\lambda}(\theta)$, $\lambda \in \Lambda$ in $M(\theta)$ represent the *subsymbols* of $m^{\lambda'}(\theta)$ $(\lambda' \in \Lambda)$. To define them, set

$$c_\lambda(\theta) := \sum_{\beta \in \mathbb{Z}^d} c_{2\beta+\lambda} e^{-i(2\beta+\lambda)\theta}\ , \quad \lambda \in \Lambda\ ,$$

for any $c(\theta) = \sum_{\alpha \in \mathbb{Z}^d} c_\alpha e^{-i\alpha\theta}$ and apply this notation to the symbols $m^{\lambda'}(\theta)$.

A formula for the dual symbol may be obtained by considering the unique decomposition of $v_1 \in V_1$ associated with (16):

$$v_1 := \sum_{\alpha \in \mathbb{Z}^d} c_\alpha \phi_{1,\alpha} = v_0 + w_1\ , \tag{36}$$

where

$$v_0 := \sum_{\beta \in \mathbb{Z}^d} d_\beta \phi_{0,\beta} \in V_0\ , \qquad w_1 := \sum_{\lambda \in \Lambda'} \sum_{\beta \in \mathbb{Z}^d} d^\lambda_\beta \psi^\lambda_{0,\beta} \in W_1\ . \tag{37}$$

Using the refinement equations (10) and (17),

$$v_0 + w_1 = \sum_{\alpha\in\mathbb{Z}^d} (\sum_{\lambda'\in\Lambda} \sum_{\gamma\in\mathbb{Z}^d} a^{\lambda'}_{\alpha-2\gamma} d^{\lambda'}_\gamma)\phi_{1,\alpha} \ ,$$

where, similarly, $(d^0_\beta) := (d_\beta)$. Thus,

$$c_\alpha = \sum_{\lambda'\in\Lambda} \sum_{\gamma\in\mathbb{Z}^d} a^{\lambda'}_{\alpha-2\gamma} d^{\lambda'}_\gamma \tag{38}$$

or, using (6),

$$c(\theta) = 2^d \sum_{\lambda'\in\Lambda} m^{\lambda'}(\theta) d^{\lambda'}(2\theta) \ . \tag{39}$$

Substituting the arguments $\theta+\lambda\pi$, $\lambda\in\Lambda$, we obtain the linear system $\mathbf{c}(\theta) = 2^d L(\theta)\mathbf{d}(2\theta)$ for determining the vector function $\mathbf{d}(\theta) = (\,d^\lambda(\theta)\,)^T_{\lambda\in\Lambda}$ from the vector function $\mathbf{c}(\theta) = (\,c(\theta+\pi\lambda)\,)^T_{\lambda\in\Lambda}$:

$$\mathbf{d}(2\theta) = 2^{-d} L^{-1}(\theta)\mathbf{c}(\theta) \ . \tag{40}$$

The invertibility of $L(\theta)$ is guaranteed by (35). In particular, the first component $d(\theta) = d(\theta)^0$ of $\mathbf{d}(\theta)$ is given by

$$d(2\theta) = 2^{-d} \sum_{\lambda\in\Lambda} \tilde{m}(\theta+\pi\lambda)^* c(\theta+\pi\lambda) \ = \sum_{\lambda\in\Lambda} \tilde{m}_\lambda(\theta)^* c_\lambda(\theta) \ , \tag{41}$$

where

$$\tilde{m}(\theta) := \frac{N(\theta)^*}{D(\theta)^*} \ . \tag{42}$$

Here, $D(\theta) = \det L(\theta)$, and $N(\theta) = \det L^0(\theta)$ where the matrix $L^0(\theta)$ is obtained from $L(\theta)$ by replacing the first column $(\,m(\theta{+}\pi\lambda)\,)^T$ (corresponding to the symbol of ϕ, and $\lambda' = 0$) by the unit vector $(1,0,\ldots,0)^T$. Here, and in the following, * denotes complex conjugation. Formula (42) follows by Cramer's rule which gives

$$2^d D(\theta) d(2\theta) = \det L^c(\theta) = \sum_{\lambda\in\Lambda} c(\theta+\pi\lambda) \det L^\lambda(\theta) \ .$$

Here $L^c(\theta)$ and $L^\lambda(\theta)$ are the matrices obtained by replacing the first column of $L(\theta)$ by $\mathbf{c}(\theta)$ and unit vectors (with the entry 1 at the position with row index λ), respectively. To conclude, observe that $\det L^\lambda(\theta) = \det L^0(\theta+\pi\lambda)$, $\lambda\in\Lambda$.

The function $\tilde{m}(\theta)$ defined by (42) is a rational trigonometric function, i.e., the quotient of two trigonometric polynomials $N(\theta)^*$ and $D(\theta)^* \neq 0$ (with real coefficients if the masks $(a^{\lambda'}_\alpha)$ are real-valued). To see that $\tilde{m}(0) = \tilde{m}(0)^* = 1$, observe that the matrices $L(\theta)$ and $L^0(\theta)$ coincide for $\theta = 0$ since from (15) we have $m(0) = 1$ und $m(\pi\lambda) = 0$ for $\lambda\in\Lambda'$. Obviously, since $D(\theta)\neq 0$, $\tilde{m}(\theta)$ is a C^∞–function. In particular, it is Lipschitz continuous at 0, i.e., we have

$$|1-\tilde{m}(\theta)| \ \leq \ C|\theta| \tag{43}$$

for all θ sufficiently small. Also, the Strang-Fix order $L_{\tilde{m}}$ is well-defined by (14).

We will now see that this function coincides with the symbol of the dual scaling function $\tilde{\phi}$ if $\mathbf{\Psi}$ really forms a Riesz basis. In that case, representing $\tilde{\phi} \in \tilde{V}_0 \subset \tilde{V}_1$ with respect to the basis $\tilde{\Phi}_1$, we obtain the *dual refinement equation*

$$\tilde{\phi}(x) = \sum_{\alpha \in \mathbb{Z}^d} \tilde{a}_\alpha \tilde{\phi}(2x - \alpha) \tag{44}$$

which holds with a (not necessarily finite) mask $(\tilde{a}_\alpha) \in \ell_2(\mathbb{Z}^d)$. As before, its symbol is introduced by

$$\tilde{m}(\theta) = 2^{-d} \sum_{\alpha \in \mathbb{Z}^d} \tilde{a}_\alpha e^{-i\alpha\theta} \ . \tag{45}$$

Returning to the additive decomposition of an arbitrary $v_1 = \sum_{\beta \in \mathbb{Z}^d} c_\beta \phi_{1,\beta} \in V_1$ into $v_0 = \sum_{\alpha \in \mathbb{Z}^d} d_\alpha \phi_\alpha \in V_0$ and $w_1 \in W_1$ as in (36), we use (26) and (44):

$$\begin{aligned} d_\alpha &= \frac{(v_1, \tilde{\phi}_\alpha)_{L_2}}{\|\phi\|_{L_2} \|\tilde{\phi}\|_{L_2}} = \frac{(\sum_{\beta \in \mathbb{Z}^d} c_\beta \phi_{1,\beta}, \sum_{\gamma \in \mathbb{Z}^d} \tilde{a}_\gamma \tilde{\phi}_{1,2\alpha+\gamma})_{L_2}}{\|\phi\|_{L_2} \|\tilde{\phi}\|_{L_2}} \\ &= \sum_{\beta \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}^d} \delta_{\beta, 2\alpha+\gamma} \frac{\|\phi_{1,\beta}\|_{L_2} \|\tilde{\phi}_{1,2\alpha+\gamma}\|_{L_2}}{\|\phi\|_{L_2} \|\tilde{\phi}\|_{L_2}} c_\beta \tilde{a}_\gamma^* = 2^{-d} \sum_{\beta \in \mathbb{Z}^d} c_\beta \tilde{a}_{\beta - 2\alpha}^* \ . \end{aligned}$$

Note that the biorthogonality and dilation-translation properties of the systems yield

$$(\phi_{j,\alpha}, \tilde{\phi}_{j,\beta})_{L_2} = \delta_{\alpha,\beta} \|\phi_{j,\alpha}\|_{L_2} \|\tilde{\phi}_{j,\beta}\|_{L_2} = 2^{-jd} \delta_{\alpha,\beta} \|\phi\|_{L_2} \|\tilde{\phi}\|_{L_2} \ ,$$

which was used for $j = 1$. Turning to the symbols, we get

$$\begin{aligned} d(2\theta) &= \sum_{\alpha \in \mathbb{Z}^d} d_\alpha e^{-2i\alpha\theta} = 2^{-d} \sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \in \mathbb{Z}^d} (c_\beta e^{-i\beta\theta})(\tilde{a}_{\beta-2\alpha} e^{-i(\beta-2\alpha)\theta})^* \\ &= \sum_{\lambda \in \Lambda} \tilde{m}_\lambda(\theta)^* c_\lambda(\theta) = 2^{-d} \sum_{\lambda \in \Lambda} \tilde{m}(\theta + \pi\lambda)^* c(\theta + \pi\lambda) \ . \end{aligned}$$

It remains to compare this formula with (41) and to observe that such representations are unique.

We can use this formula to define $\tilde{\phi}$ via

$$\hat{\tilde{\phi}}(\theta) = \prod_{k=1}^{\infty} \tilde{m}(2^{-k}\theta) \ . \tag{46}$$

Note that the function $\tilde{q}$ defined by

$$\tilde{q}(\theta) = |\tilde{m}(\theta)|^2 = \frac{|N(\theta)|^2}{|D(\theta)|^2} = \sum_{\alpha \in \mathbb{Z}^d} \tilde{q}_\alpha e^{-i\alpha\theta} \tag{47}$$

is a real-valued, nonnegative rational trigonometric function. It has a Fourier series with coefficients $\tilde{q}_\alpha$ satisfying

$$|\tilde{q}_\alpha| \le C r_0^{|\alpha|_1} \ , \ \ \tilde{q}_{-\alpha} = \tilde{q}_\alpha^* \ , \ \alpha \in \mathbb{Z}^d \ , \tag{48}$$

where $0 < r_0 < 1$ depends on the distance of the zero set of the denominator polynomial to the torus interpreted as subset of $\mathbb{C}^d$. If the masks (a_α), (a_α^λ) are real-valued then so are the coefficients $\tilde{q}_\alpha$. Note that $\tilde{m}(0) = 1$ implies

$$\tilde{q}(0) = \sum_{\alpha \in \mathbb{Z}^d} \tilde{q}_\alpha = 1 \ . \tag{49}$$

The factorization

$$\tilde{q}(\theta) = p(\theta) q(\theta) \ , \quad p(\theta) = |c_0 N(\theta)|^2 \ , \ q(\theta) = |c_0 D(\theta)|^{-2} > 0 \tag{50}$$

into a nonnegative trigonometric polynomial $p(\theta)$ with $p(0) = 1$, and a positive periodic C^∞ function $q(\theta)$ with $q(0) = 1$ and exponentially decaying Fourier coefficients q_α (as in (48)) will be extensively used below. The normalization factor c_0 has been fixed such that $c_0 N(0) = c_0 D(0) = 1$.

2.3 Properties of transfer operators

The *transfer operator* $\mathcal{L}_r$ associated with the 2π-periodic function r is defined by

$$(\mathcal{L}_r c)(\theta) = \sum_{\lambda \in \Lambda} r(\theta/2 + \pi\lambda) c(\theta/2 + \pi\lambda) \ . \tag{51}$$

Since for an arbitrary 2π-periodic function $f(\theta)$

$$2^{-d} \sum_{\lambda \in \Lambda} f(\theta/2 + \pi\lambda) = \sum_\beta f_{2\beta} e^{-i\beta\theta}$$

is again a 2π-periodic function, the operator $\mathcal{L}_r$ acts in spaces of periodic functions. Since its formal adjoint in $L_2(\mathbb{R}^d)$ is a dyadic subdivision operator, it naturally appears in a number of applications. Properties of $\mathcal{L}_r$ have been studied extensively [11, 20, 18, 7, 9, 22], especially for the univariate case respectively for r generated from multivariate trigonometric polynomials. In particular, we have

Theorem 3 *If $\phi \in L_2(\mathbb{R}^d)$ is compactly supported and generates a MRA, with symbol $m(\theta)$ defined by (12), then its regularity exponent can be computed as follows:*

$$s_\phi = -\frac{1}{2} \log_2 \rho_\phi \ , \tag{52}$$

where

$$\rho_\phi = \rho(\mathcal{L}_{|m|^2}, V_{|m|^2, z_L})$$

denotes the spectral radius of the restriction of $\mathcal{L}_{|m|^2}$ to the subspace

$$V_{|m|^2, z_L} = \mathrm{span}\{\mathcal{L}_{|m|^2}^k z_L \ : \ k \geq 0\} \tag{53}$$

generated from the trigonometric polynomial

$$z_L(\theta) = (\sum_{l=1}^d \sin^2 \frac{\theta_l}{2})^L \ . \tag{54}$$

An appropriate choice for L is $L = L_m$ or any (smaller) integer such that $s_\phi < L$. The subspace $V_{|m|^2,z_L}$ is finite-dimensional and consists of trigonometric polynomials for any L.

There are alternative formulations of Theorem 3 characterizing ρ_ϕ as the largest eigenvalue of $\mathcal{L}_{|m|^2}$ on a certain finite-dimensional invariant subspace consisting of trigonometric polynomials, the definition of which involves the Strang-Fix conditions for $m(\theta)$, see [17, 6, 22]. Theorem 3, in conjunction with Theorem 1, shows again how essential the operators $\mathcal{L}_{|m|^2}$ respectively $\mathcal{L}_{|\tilde{m}|^2}$ are, at least under the assumption of compactly supported masks. Part of the efforts in this and the following sections will be directed to overcoming this restriction by approximation arguments.

As was outlined in subsection 2.1, a key tool is the investigation of the projections Q_j in connection with norm equivalences such as (31). This can be done by considering their discrete counterparts $Q_j^{j+k} := Q_j|_{V_{j+k}} : V_{j+k} \to V_j$, $k > 0$, together with a limiting procedure $k \to \infty$. The advantage is that these projections can be defined without assuming any Riesz basis properties of the candidate system $\mathbf{\Psi}$, by just k times iterating the definition of the *two-level projection* $Q_0^1 := Q : V_1 \to V_0$ associated with the splitting (16). More precisely, using the formula (36) we define Q by

$$Qv_1 = v_0 \ , \qquad v_1 \in V_1 \ . \tag{55}$$

If the dual MRA exists, then the two-level projector can be written as

$$Qv_1 = \sum_{\alpha \in \mathbb{Z}^d} \frac{(v_1, \tilde{\phi}_\alpha)_{L_2}}{\|\phi\|_{L_2} \|\tilde{\phi}\|_{L_2}} \phi_\alpha \qquad \forall\, v_1 \in V_1 \ . \tag{56}$$

By a dilation argument, we define $Q_j^{j+1} v_{j+1} := Q_0^1(v_{j+1}(2^{-j}\cdot))(2^j\cdot)$ for all $v_{j+1} \in V_j$, and, finally, $Q_j^{j+k} := Q_j^{j+1} Q_{j+1}^{j+2} \ldots Q_{j+k-1}^{j+k}$.

Equation (41) can be used to express the L_2-norm of $Q : V_1 \to V_0$ and, more generally, of the projections $Q_j^{j+k} : V_{j+k} \to V_j$.

Theorem 4 *Under the assumptions A1 and A2 on ϕ and ψ^λ, $\lambda \in \Lambda'$, the norms of the projection operators $Q_j^{j+k} : V_{j+k} \to V_j$ satisfy*

$$\|Q_j^{j+k}\|_{L_2}^2 = \|Q_0^k\|_{L_2}^2 = \|\mathcal{L}_{\tilde{q}}^k 1\|_{L_\infty(\mathcal{C})} \ , \quad 0 \le j < j+k < \infty \ , \tag{57}$$

where the transfer operator $\mathcal{L}_{\tilde{q}}$ is defined by (51) and (47).

Proof. Using the definition (55) of Q and the notation introduced in (36), (37), we have

$$\|Q\|_{L_2}^2 = \sup_{v_1 \neq 0} \frac{\|Qv_1\|_{L_2}^2}{\|v_1\|_{L_2}^2} = 2^d \sup_{c \neq 0} \frac{\|d\|_{L_2(\mathcal{C})}^2}{\|c\|_{L_2(\mathcal{C})}^2} \ . \tag{58}$$

Using (41), we estimate

$$
\begin{aligned}
2^d \|d\|^2_{L_2(\mathcal{C})} &= 2^{2d}\|d(2\cdot)\|^2_{L_2(\frac{1}{2}\mathcal{C})} = \int_{\frac{1}{2}\mathcal{C}} |\sum_{\lambda\in\Lambda} \tilde{m}(\theta+\pi\lambda)^* c(\theta+\pi\lambda)|^2\, d\theta \\
&\leq \int_{\frac{1}{2}\mathcal{C}} (\sum_{\lambda\in\Lambda} |\tilde{m}(\theta+\pi\lambda|^2)(\sum_{\lambda\in\Lambda} |c(\theta+\pi\lambda)|^2)\, d\theta \\
&\leq \|\sum_{\lambda\in\Lambda} \tilde{q}(\theta+\pi\lambda)\|_{L_\infty(\frac{1}{2}\mathcal{C})} \int_{\frac{1}{2}\mathcal{C}} \sum_{\lambda\in\Lambda} |c(\theta+\pi\lambda)|^2\, d\theta \\
&= \|\sum_{\lambda\in\Lambda} \tilde{q}(\theta/2+\pi\lambda)\|_{L_\infty(\mathcal{C})} \|c\|^2_{L_2(\mathcal{C})} \,.
\end{aligned}
$$

Since this estimate is sharp (with respect to all of $L_2(\mathcal{C})$), we see that

$$
\|Q_0^1\|^2_{L_2} = \|Q\|^2_{L_2} = \|\sum_{\lambda\in\Lambda} \tilde{q}(\theta/2+\pi\lambda)\|_{L_\infty(\mathcal{C})} = \|\mathcal{L}_{\tilde{q}} 1\|_{L_\infty(\mathcal{C})} \,. \tag{59}
$$

In completely the same way, by iterating (41), one obtains

$$
\|Q_j^{j+k}\|^2_{L_2} = \|\sum_{\lambda_1,\ldots,\lambda_k\in\Lambda} \prod_{l=1}^{k} \tilde{q}(\frac{\theta}{2^l} + \frac{\pi\lambda_1}{2^{l-1}} + \ldots + \pi\lambda_l)\|_{L_\infty(\mathcal{C})} = \|\mathcal{L}^k_{\tilde{q}} 1\|_{L_\infty(\mathcal{C})} \,. \tag{60}
$$

This completes the proof.

We will now impose the following restrictions on $r(\theta)$ which are, in view of the considerations in subsection 2.2, sufficient for the applications we have in mind.

A3. We assume the factorization $r(\theta) = p(\theta)q(\theta)$ where

$$
p(\theta) = \sum_{|\alpha|_\infty \leq n} p_\alpha e^{-i\alpha\theta} \geq 0 \,, \quad p_{-\alpha} = p_\alpha^* \,,
$$

is a non-negative trigonometric polynomial of degree $\leq n$, and where

$$
q(\theta) = \sum_{\alpha\in\mathbb{Z}^d} q_\alpha e^{-i\alpha\theta} > 0 \,, \quad q_{-\alpha} = q_\alpha^* \,, \quad q(0) = 1 \,,
$$

is a positive C^∞ function with exponentially decaying Fourier coefficients, i.e., there is a positive $r_0 < 1$ and a constant $C < \infty$ such that

$$
|q_\alpha| \leq C r_0^{|\alpha|_1} \,, \quad \alpha \in \mathbb{Z}^d \,.
$$

Note that the notation introduced in condition A3 is consistent with (48)-(50). Thus, assumptions A1 and A2 imply all properties in A3 for $r(\theta) = |\tilde{m}(\theta)|^2$. In some places, which will be mentioned below, we need also $p(0) = 1$ which is not required by A3. The special case $q(\theta) \equiv 1$ covers the case of trigonometric symbols. A consequence of A3 is that the Fourier coefficients of $r(\theta)$ satisfy

$$
|r_\alpha| \leq C r_0^{|\alpha|_1} \,, \qquad \alpha \in \mathbb{Z}^d \,. \tag{61}
$$

Thus, we are a bit more specific compared to [18, 7] where only the latter condition has been imposed.

We next list some properties of $\mathcal{L}_r$, following the paper [7] (be aware of differences in the notation). Let the Hilbert spaces $E^t \subset L_2(\mathbb{T}^d)$, $0 < t < 1$, be induced by the scalar product

$$(f,g)_{E^t} = \sum_{\alpha \in \mathbb{Z}^d} t^{-2|\alpha|_1} f_\alpha g_\alpha^* \,. \tag{62}$$

The norm in this space is given by

$$\|f\|_{E^t} := \|(f_\alpha t^{-|\alpha|_1})\|_{\ell^2(\mathbb{Z}^d)} \,.$$

Let E^t_+ be the cone of non-negative functions in E^t. Note that these are spaces of C^∞ functions with rapidly decreasing (complex) Fourier coefficients, with the exponential decay controlled by the parameter t. Concerning the Banach space $\sigma_1(H)$ of compact trace class operators acting on a Hilbert space H, we refer to [19, 28] for generalities, and to [18, 7] for a treatment of $\sigma_1(E^t)$.

The following theorem on the properties of $\mathcal{L}_r$ under assumption A3 was proved in [7].

Theorem 5 *Let $r(\theta)$ satisfy (61). Then the transfer operator $\mathcal{L}_r$ is a bounded linear operator in E_t if $r_0^2 < t < 1$. Moreover, it is a compact trace class operator for the same range of parameters t. It is also positive: $\mathcal{L}_r(E^t_+) \subset E^t_+$. The spectral radius $\rho(\mathcal{L}_r, E^t)$ coincides with the largest positive eigenvalue of $\mathcal{L}_r$, the associated eigenfunction belongs to E^t_+.*

We come to the effect of approximating $r(\theta)$ on spectral properties of $\mathcal{L}_r$. We start with a simple lemma on approximating functions with exponentially fast decaying Fourier coefficients by trigonometric polynomials.

Lemma 6 *Let $q(\theta)$ be as specified in A3. For all sufficiently large N, there exist positive trigonometric polynomials*

$$q_N(\theta) = \sum_{|\alpha|_\infty \le N} q_{N,\alpha} e^{-i\alpha\theta} \;, \quad q_{N,-\alpha} = q^*_{N,\alpha} \;, \tag{63}$$

such that

$$\|q - q_N\|_{L_\infty(\mathcal{C})} \le C N^l r_0^N \;, \qquad q_N(0) = 1 \;, \tag{64}$$

and, consequently,

$$|q_\alpha - q_{N,\alpha}| \le C N^l r_0^N \;, \quad |\alpha|_\infty \le N \;, \tag{65}$$

for some $l \ge 0$. A $q_N(\theta)$ of the form (63) satisfying (64) for any fixed $l > 0$ is obtained by interpolating $q(\theta)$ at the $(2N+1)^d$ points $\theta_{N,\alpha} = 2\pi\alpha/(2N+1)$, $\alpha \in \{0,\ldots,2N\}^d$. In addition, monotonicity properties such as $q_N(\theta) \ge q(\theta)$ or $0 < q_N(\theta) \le q(\theta)$ for all $\theta \in \mathcal{C}$ can be enforced.

We sketch the proof of this lemma. Obviously, it follows from the coefficient bounds in A3 that for large enough N

$$q_N(\theta) = (q_0 + \sum_{|\alpha|_\infty > N} q_\alpha) + \sum_{1 \le |\alpha|_\infty \le N} q_\alpha e^{-i\alpha\theta}$$

is of the form (63) and satisfies (64) and (65) with $l = 0$.

If we define the trigonometric polynomial $q_N(\theta)$ in the form as required in (63) by interpolating $q(\theta)$ at the set $\{\theta_{N,\alpha}\}$, then (64), (65) are satisfied with any fixed $l > 0$ since

$$\|q - q_N\|_{L_\infty(\mathcal{C})} \le C(\log N)^d \inf_{c_\alpha} \|q(\theta) - \sum_{|\alpha|_\infty \le N} c_\alpha e^{-i\alpha\theta}\|_{L_\infty(\mathcal{C})} \le C(\log N)^d r_0^N .$$

Thus, interpolating polynomials, which can efficiently be computed from $q(\theta)$ by FFT, fit into the framework of Lemma 6.

Finally, to achieve monotonicity from above or from below, it suffices to take any q_N that satisfies already (63) and (64) and to add or subtract a suitable multiple $CN^l r_0^N z_1(\theta)$ of the trigonometric polynomial $z_1(\theta) \asymp |\theta|_\infty^2$ mentioned in Theorem 3.

Lemma 7 *Let $r(\theta)$ satisfy assumption A3, and set $r_N(\theta) = p(\theta)q_N(\theta)$, where $q_N(\theta)$ is defined according to Lemma 6, $N \ge N_0$. Then*

$$\mathcal{L}_{r_N} \to \mathcal{L}_r , \quad N \to \infty , \tag{66}$$

in the sense of the trace class norm on E^t, at least for $r_0^{2/d} < t < 1$.

Proof. We give the argument for $d = 2$. Let us denote the Fourier coefficients of $\Delta r_N = r - r_N = p(q - q_N)$ by $(\Delta r_N)_\alpha$. By assumption A3 and (65), one has

$$|(\Delta r_N)_\alpha| \le C \left\{ \begin{array}{lcl} r_0^{|\alpha|_1} & , & |\alpha|_\infty > N + n \\ N^l r_0^N & , & |\alpha|_\infty \le N + n \end{array} \right. ,$$

where n is the degree of the trigonometric polynomial p. Then, using the complete orthonormal system $\{e_{\alpha,t} := t^{-|\alpha|_1} e^{-i\alpha\theta}\}$ in E^t, by a well-known bound for the trace class norm we have

$$\begin{aligned} \|\mathcal{L}_{\Delta r_N}\|_{\sigma_1(E^t)} &\le \sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \in \mathbb{Z}^d} |(\mathcal{L}_{\Delta r_N} e_{\beta,t}, e_{\alpha,t})_{E^t}| \\ &\le 2^d \sum_{\alpha \in \mathbb{Z}^d} t^{-|\alpha|_1} \sum_{\beta \in \mathbb{Z}^d} |(\Delta r_N)_{2\alpha-\beta}| t^{|\beta|_1} \\ &\le C \sum_{\alpha \in \mathbb{Z}^d} t^{-|\alpha|_1} \sum_{\beta \in \mathbb{Z}^d} \left\{ \begin{array}{c} t^{\epsilon|2\alpha-\beta|_1 + |\beta|_1} \\ N^l t^{\epsilon N + |\beta|_1} \end{array} \right\} , \end{aligned}$$

where the first formula applies if $|2\alpha - \beta|_\infty > N + n$, otherwise the second has to be taken. The constant $\epsilon > 0$ is given by $r_0 = t^\epsilon$. Using the definition of the vector norms $|\cdot|_1$ and $|\cdot|_\infty$ involved, the last expression can be bounded from above for $d = 2$ by

$$C\{(\sum_{\alpha_1\in\mathbb{Z}} t^{-|\alpha_1|}A(\alpha_1))(\sum_{\alpha_2\in\mathbb{Z}} t^{-|\alpha_2|}B(N+n,\alpha_2))$$
$$+(\sum_{\alpha_1\in\mathbb{Z}} t^{-|\alpha_1|}B(N+n,\alpha_1))(\sum_{\alpha_2\in\mathbb{Z}} t^{-|\alpha_2|}A(\alpha_2)) + N^l t^{\epsilon N} C(N+n)^2\} \,,$$

where

$$\begin{aligned} A(a) &:= \sum_{b\in\mathbb{Z}} t^{2\epsilon|2a-b|}t^{2|b|} \asymp \begin{cases} t^{4\epsilon|a|}\,, & 0<\epsilon<1 \\ |a|t^{4|a|}\,, & \epsilon = 1 \\ t^{4|a|}\,, & \epsilon > 1 \end{cases} \,, \\ B(N+n,a) &:= \sum_{b\in\mathbb{Z}\,:\,|2a-b|>N+n} t^{\epsilon|2a-b|+|b|}\,, \\ C(N+n) &:= \sum_{a,b\in\mathbb{Z}\,:\,|2a-b|\le N+n} t^{-|a|+|b|}\,. \end{aligned}$$

For the latter sum we have

$$C(N+n) \le C(\sum_{a\in\mathbb{Z}:|2a|\le N+n} t^{-|a|} + \sum_{a\in\mathbb{Z}:|2a|>N+n} t^{N-|a|}) \le Ct^{-N/2}\,.$$

For estimating $B(N+n,a)$ we need to distinguish between the case $|2a| > N+n$, where

$$B(N+n,a) \asymp (t^{2\epsilon|a|} + t^{\epsilon(N+n)+2|a|-(N+n)})\,,$$

(if $\epsilon = 1$ this expression has to be replaced by $(2|a|-(N+n))t^{2|a|}$), and the case $|2a| \le N+n$, where

$$B(N+n,a) \asymp t^{\epsilon(N+n)+(N+n)-2|a|}\,.$$

Consequently, for $\epsilon > 1/2$,

$$\sum_{a\in\mathbb{Z}} t^{-|a|}B(N+n,a) \le C(\sum_{|a|>(N+n)/2} (t^{(2\epsilon-1)|a|} + t^{(\epsilon-1)(N+n)+|a|})$$
$$+ \sum_{|a|\le(N+n)/2} t^{(\epsilon+1)(N+n)-2|a|}) \le Ct^{(\epsilon-1/2)N}\,.$$

Altogether, taking into account also the boundedness of $\sum_{a\in\mathbb{Z}} t^{-|a|}A(a)$ for $\epsilon > 1/2$, we arrive at the final estimate

$$\|\mathcal{L}_r - \mathcal{L}_{r_N}\|_{\sigma_1(E^t)} \le CN^l t^{(\epsilon-1)N} \le CN^l r_0^{(1-1/\epsilon)N}\,, \tag{67}$$

which gives an exponential convergence rate in (66) with respect to the trace class norm in E^t if $\epsilon > 1$. Since for arbitrary d, the largest term will come from estimating $C(N+n)^d$, in general, the restriction $\epsilon > d/2$ or, equivalently, $r_0^{2/d} < t < 1$ is required. Finally, note that, with the same proof, (66) holds for $r(\theta)$ satisfying only (61) and approximations $r_N(\theta)$ with the properties

$$r_N(\theta) = \sum_{|\alpha|_\infty\le N} r_{N,\alpha}e^{-i\alpha\theta}\,, \quad \|r - r_N\|_{L_\infty(\mathcal{C})} \le CN^l r_0^N\,, \qquad N\to\infty\,.$$

2.4 The dual scaling function and its properties

In this subsection, let ϕ, ψ^λ satisfy the assumptions A1 and A2. We will construct $\tilde{\phi}$ and show under which conditions it belongs to $H^s(\mathbb{R}^d)$.

We base our considerations on the definiton of $\hat{\tilde{\phi}}$ by a product formula analogous to (13). The following lemma on its convergence is well known. We reproduce the proof since we need one of the estimates occurring therein.

Lemma 8 *Let $r \in C(\mathbb{T}^d)$ with $r(0) = 1$ be Lipschitz continuous at the origin*

$$|r(\theta) - 1| \leq C|\theta|^\gamma \ , \quad \theta \to 0 \ ,$$

with $\gamma > 0$. Then the infinite product

$$\Pi(r)(\theta) = \prod_{l=1}^{\infty} r(2^{-l}\theta)$$

converges uniformly on compact subsets of $\mathbb{R}^d$ to a continuous function.

Proof. By

$$\Pi(r)_k(\theta) = \prod_{l=1}^{k} r(2^{-l}\theta) \ , \qquad k \geq 1 \ ,$$

with $\Pi(r)_0(\theta) = 1$, we denote the partial products.

Using the Lipschitz continuity, we have

$$\max(0, 1 - C|\theta|^\gamma) \leq |r(\theta)| \leq \min(\|r\|_{L_\infty}, 1 + C|\theta|^\gamma) \ , \ \theta \in \mathcal{C} \ ,$$

from which

$$\begin{aligned} |\Pi(r)_K(\theta)| &\leq |\Pi(r)_k(\theta)| \textstyle\prod_{l=k+1}^{K}(1 + C|2^{-l}\theta|^\gamma) \\ &\leq |\Pi(r)_k(\theta)|(1 + C|\theta|^\gamma) \\ &\leq C|\Pi(r)_k(\theta)| \ , \ |\theta| \leq C_1 2^k \ , \ K \geq k \ . \end{aligned}$$

For large enough $K' > K > k$ and $|\theta| \leq C_1 2^k$ we also have

$$|\log\left(\prod_{l=K+1}^{K'} |r(2^{-l}\theta)|\right)| \leq C \sum_{l=K+1}^{K'} |2^{-l}\theta|^\gamma \leq C 2^{(k-K)\gamma} \ .$$

Now we can put things together: Since

$$|\Pi(r)_{K'}(\theta) - \Pi(r)_K(\theta)| = |\Pi(r)_K(\theta)| \prod_{l=K+1}^{K'} |r(2^{-l}\theta) - 1| \ ,$$

with the first term uniformly bounded for $|\theta| \leq C_1 2^k$ by the above estimates, and the second tending to 0 uniformly on compact subsets if $K, K' \to \infty$, the above convergence statement is verified. The continuity follows from the continuity of the finite products $\Pi(r)_K(\theta)$ which completes the proof.

From the above estimates, letting $K \to \infty$, we finally have

$$|\Pi(r)(\theta)| \leq C|\Pi(r)_k(\theta)| \ , \quad |\theta| \leq C_1 2^k \ , \tag{68}$$

where $C_1 > 0$ is any fixed constant. This inequality will be used below.

Since the dual symbol $\tilde{m}(\theta)$ satisfies the above Lipschitz condition with $\gamma = 1$, see (43), we immediately obtain

Corollary 9 *Let the assumptions A1 and A2 hold and*

$$\hat{\tilde{\phi}}(\theta) := \prod_{l=1}^{\infty} \tilde{m}(2^{-l}\theta) \ , \tag{69}$$

where $\tilde{m}(\theta)$ is the dual symbol (42). Then $\hat{\tilde{\phi}}$ is a continuous function.

In the following, we will extend Theorem 3 to the dual scaling function. Let $\tilde{q}_N(\theta)$ be approximations to $\tilde{q}(\theta) = |\tilde{m}(\theta)|^2$ as in Lemma 6. In particular, for N large enough, we can assume that

$$\|\tilde{q} - \tilde{q}_N\|_{L_\infty(\mathcal{C})} \leq \epsilon_N := CN^l r_0^N \ ,$$

or, similarly,

$$0 < (1 - C\epsilon_N)\tilde{q}_N(\theta) \leq \tilde{q}(\theta) \leq (1 + C\epsilon_N)\tilde{q}_N(\theta) \ , \quad \theta \in \mathcal{C} \ . \tag{70}$$

Lemma 10 *Let $\tilde{q}$ be as defined in (47-50), in particular, A3 is satisfied, and Lemma 6 is applicable. Let $\tilde{q}_N$ be a sequence of approximations of $\tilde{q}$ satisfying (63), (64). Then*

$$\tilde{\rho} = \inf_{L \geq 0} \lim_{N \to \infty} \rho(\mathcal{L}_{\tilde{q}_N}, V_{\tilde{q}_N, z_L}) \ , \tag{71}$$

where the notation is explained in Theorem 3, exists and does not depend on the specific choice of the sequence $\tilde{q}_N$.

Proof. Since $q(\theta)$ is strictly positive and continuous, we conclude from (64) that for sufficiently large N

$$\max\left(\left|1 - \frac{q_N(\theta)}{q(\theta)}\right|, \left|1 - \frac{q(\theta)}{q_N(\theta)}\right|\right) \leq \epsilon_N \tag{72}$$

for some positive $\epsilon_N \leq CN^l r_0^N$, with the last estimate coming from (64) but with another constant depending also on bounds for $q(\theta)$.

The independence follows easily from (72) and the formula

$$\rho(\mathcal{L}_r, V_{r,z}) = \lim_{k \to \infty} \|\mathcal{L}_r^k z\|_{L_\infty(\mathcal{C})}^{1/k} \tag{73}$$

which holds whenever r and z are trigonometric polynomials. For a proof of (73), use that $V_{r,z}$ is finite-dimensional and can be equipped with the L_∞–norm, and that the eigenfunction corresponding to some eigenvalue λ with $|\lambda| = \rho(\mathcal{L}_r, V_{r,z})$ is of the form

$$c = \sum_{l=0}^{n-1} a_l \mathcal{L}_r^l z \ .$$

with some fixed coefficients a_l ($a_n \neq 0$) and an integer n not exceeding the dimension of $V_{r,z}$. Then

$$|\lambda|^k \|c\|_{L_\infty(\mathcal{C})} \le \sum_{l=0}^{n} |a_l| \, \|\mathcal{L}_r^{l+k} z\|_{L_\infty(\mathcal{C})} \le C \|\mathcal{L}_r^k z\|_{L_\infty(\mathcal{C})}$$

for $k \ge 1$, where the boundedness of the operator $\mathcal{L}_r$ in $L_\infty(\mathbb{T}^d)$ has been used. The constant C does not depend on k. On the other hand, by the definition of the spectral radius, we have

$$\|\mathcal{L}_r^k z\|_{L_\infty(\mathcal{C})} \le C_\delta (|\lambda| + \delta)^k$$

for any fixed $\delta > 0$. Thus, raising these inequalities to the power $1/k$, we obtain (73) if limits are taken, first with respect to $k \to \infty$ and then for $\delta \to 0$.

With (73) at hand for all choices $r = \tilde{q}_N$ and $z = z_L$, we take two sequences q_N and q'_N from Lemma 6 for which we can assume (72) with the same $\epsilon_N \to 0$. Then we easily get

$$\frac{1-\epsilon_N}{1+\epsilon_N} \tilde{q}_N(\theta) \le \tilde{q}'_N(\theta) \le \frac{1+\epsilon_N}{1-\epsilon_N} \tilde{q}_N(\theta)$$

and, consequently, by definition of $\mathcal{L}_r$ and the non-negativity of all trigonometric polynomials involved,

$$\left(\frac{1-\epsilon_N}{1+\epsilon_N}\right)^k \|\mathcal{L}_{\tilde{q}_N}^k z_L\|_{L_\infty(\mathcal{C})} \le \|\mathcal{L}_{\tilde{q}'_N}^k z_L\|_{L_\infty(\mathcal{C})} \le \left(\frac{1+\epsilon_N}{1-\epsilon_N}\right)^k \|\mathcal{L}_{\tilde{q}_N}^k z_L\|_{L_\infty(\mathcal{C})} \,,$$

for $k \ge 1$. In view of (73) and $\epsilon_N \to 0$, this inequality implies that

$$\tilde{\rho}_L := \lim_{N\to\infty} \rho(\mathcal{L}_{\tilde{q}_N}, V_{\tilde{q}_N, z_L}) = \lim_{k\to\infty} \|\mathcal{L}_{\tilde{q}}^k z_L\|_{L_\infty(\mathcal{C})}^{1/k} \tag{74}$$

exists and is independent of the particular choice of $\{\tilde{q}_N\}$, and that the same holds for $\tilde{\rho} = \inf_L \tilde{\rho}_L$. Since $0 \le z_{L+1}(\theta) \le C z_L(\theta)$, (74) yields

$$\tilde{\rho} = \lim_{L\to\infty} \tilde{\rho}_L \le \tilde{\rho}_{L+1} \le \tilde{\rho}_L \le \ldots \le \tilde{\rho}_0 \ . \tag{75}$$

Lemma 10 is proved.

As will become clear in a moment, the next lemma extends the connection between the Strang-Fix order and regularity exponent to $\tilde{\phi}$.

Lemma 11 *Under the assumptions A1 and A2, we have*

$$\tilde{s} := -\frac{1}{2}\log_2 \tilde{\rho} \le L_{\tilde{m}}. \tag{76}$$

Proof. We can assume that $\tilde{s} > 0$ (there is nothing to prove if $\tilde{s} \le 0$). Considering the definition $q(\theta) = 1/|c_0 D(\theta)|^2$, it is possible to construct a special sequence q_N (satisfying the estimate (64) of Lemma 6) by approximating the function $Q(\theta) := (c_0 D(\theta))^{-1}$ rather than $q(\theta) = |Q(\theta)|^2$. Indeed, by definition of c_0 we have $Q(0) = 1$. Moreover, the Fourier coefficients Q_α of $Q(\theta)$ satisfy the same decay estimates as stated in A3 for (q_α) (however, $Q_{-\alpha} = Q_\alpha^*$ does not hold anymore). Consider the trigonometric polynomial

$$Q_N(\theta) = (Q_0 + \sum_{|\alpha|_\infty > N} Q_\alpha) + \sum_{0<|\alpha|_\infty \le N} Q_\alpha e^{-i\alpha\theta} .$$

Obviously, we have $Q_N(0) = 1$ and

$$\|Q_N - Q\|_{L_\infty(\mathcal{C})} \le C N^l r_0^N , \quad N \to \infty ,$$

for some l and $r_0 < 1$. Then $q'_N(\theta) := |Q_N(\theta)|^2$ satisfies (64) for large enough N, with r_0 replaced by $\sqrt{r_0}$ and the same l as before. The quantity $\tilde{\rho}$ in (71) can be computed on the basis of the specific sequence $\tilde{q}'_N(\theta) = p(\theta) q'_N(\theta)$ where $p(\theta) = |c_0 N(\theta)|^2$.

The advantage of this choice is that we can introduce trigonometric polynomials $\tilde{m}'_N(\theta) := (c_0 N(\theta) Q_N(\theta))^*$ which play the role of symbols associated with $\tilde{q}'_N(\theta) = |\tilde{m}'_N(\theta)|^2$, and generate a sequence of refinement equations with finitely supported masks. It is easy to see that under the assumption $\tilde{s} > 0$, these refinement equations have compactly supported solutions $\tilde{\phi}'_N \in L_2(\mathbb{R}^d)$ if N is large enough, with the corresponding systems $\{\tilde{\phi}'_N(\cdot - \alpha) \,:\, \alpha \in \mathbb{Z}^d\}$ forming Riesz bases in their respective L_2–closed linear spans. Then, by results on scaling functions with compact support and by the same comparison methods as used before, we have $0 < s_{\tilde{\phi}'_N} < L_{\tilde{m}'_N} = L_p = L_{\tilde{m}}$ and

$$s_{\tilde{\phi}'_N} = -\frac{1}{2}\log_2(\rho(\mathcal{L}_{\tilde{q}'_N}, V_{\tilde{q}'_N, z_{L_p/2}})) \;\to\; \tilde{s} , \quad N \to \infty ,$$

see the definition of $\tilde{\rho}$ and Theorem 3. Thus, $\tilde{s} \le L_{\tilde{m}}$ which proves (76) (it is very probable that $\le$ can be replaced by strict inequality, as in the case of trigonometric symbols).

Lemma 12 *Assume that A1 and A2 are satisfied. Let $\tilde{q}$ be as defined in Lemma 10, the dual scaling function $\tilde{\phi}$ be implicitly defined by (69), and $\tilde{\rho}, \tilde{s}$ be given by the formulas in (71), (76), respectively. Then, for $s < \tilde{s}$, $g_{\tilde{\phi},s}$ is a continuous function on $\mathbb{T}^d$. Consequently, $\tilde{\phi} \in H^s(\mathbb{R}^d)$ for all $s < \tilde{s}$. On the other hand, $\tilde{\phi} \notin H^s(\mathbb{R}^d)$ for all $s > \tilde{s}$. Thus,*

$$\tilde{s} = s_{\tilde{\phi}} . \tag{77}$$

Proof. We first show that

$$g_{\tilde{\phi},s}(\theta) = G_{\tilde{q},s}(\theta) := \sum_{\alpha\in\mathbb{Z}^d} (1+|\theta+2\pi\alpha|^2)^s \Pi(\tilde{q})(\theta+2\pi\alpha) \ , \tag{78}$$

belongs to $L_\infty(\mathcal{C})$. We again use the notation of partial products of Lemma 8 and its proof. Let

$$\Omega_k = \{-2^{k-1}+1,\dots,2^{k-1}\}^d \subset \mathbb{Z}^d \ , k\geq 1 \ , \quad \Omega_0 = \emptyset \ ,$$

and introduce for $k \geq 1$ the notation

$$H_{\tilde{q},k}(\theta) = \sum_{\alpha\in\Omega_k} \Pi(\tilde{q})_k(\theta+2\pi\alpha) \ , \ \Delta H_{\tilde{q},k}(\theta) = \sum_{\alpha\in\Omega_k\setminus\Omega_{k-1}} \Pi(\tilde{q})_k(\theta+2\pi\alpha) \ .$$

By definition of Ω_k and $\mathcal{C}$, we have

$$\pi(2^{k-1}-1) \leq |\theta+2\pi\alpha|_\infty \leq \pi(2^k+1) \ , \quad x\in\mathcal{C} \, , \ \alpha\in\Omega_k\setminus\Omega_{k-1} \, . \tag{79}$$

Thus, from (78) and (68) we obtain

$$G_{\tilde{q},s}(\theta) \leq C\sum_{k=1}^{\infty} 2^{2ks}\Delta H_{\tilde{q},k}(\theta) \ , \quad \theta\in\mathcal{C} \ . \tag{80}$$

For estimating the quantities $\Delta H_{\tilde{q},k}(\theta)$ (which represent, in some sense, the Littlewood-Paley blocks of $g_{\tilde{\phi},0}$), we start with recalling the algebraic identity

$$\mathcal{L}_r^k c(\theta) = \sum_{\alpha\in\Omega^k} \Pi(r)_k(\theta+2\pi\alpha)c(2^{-k}(\theta+2\pi\alpha)) \tag{81}$$

which is valid for all reasonable periodic functions r and c (see [7] for a proof). We will apply (81) with $c(\theta) = z_L(\theta)$ (see (54)) and $r(\theta) = \tilde{q}_N(\theta) = p(\theta)q_N(\theta)$, where $q_N(\theta)$ is an approximation to $q(\theta)$ obtained via Lemma 6. Using (72), we have

$$q(\theta) \leq (1+\epsilon_N)q_N(\theta) := 2^{2\delta_N}q_N(\theta) \qquad (\delta_N \leq C\epsilon_N) \ ,$$

which gives

$$\Delta H_{\tilde{q},k}(\theta) \leq 2^{2\delta_N k}\Delta H_{\tilde{q}_N,k}(\theta) \ .$$

By (79), the fact that the trigonometric polynomials $z_L(\theta)$ are strictly positive, except at the points $2\pi\alpha$, $\alpha\in\mathbb{Z}^d$, and by (81) we can continue as follows:

$$\begin{aligned}
\Delta H_{\tilde{q}_N,k}(\theta) &= \sum_{\alpha\in\Omega_k\setminus\Omega_{k-1}} \Pi(\tilde{q}_N)_k(\theta+2\pi\alpha) \\
&\leq C \sum_{\alpha\in\Omega_k\setminus\Omega_{k-1}} \Pi(\tilde{q}_N)_k(\theta+2\pi\alpha)z_L(2^{-k}(\theta+2\pi\alpha)) \\
&\leq C \sum_{\alpha\in\Omega_k} \Pi(\tilde{q}_N)_k(\theta+2\pi\alpha)z_L(2^{-k}(\theta+2\pi\alpha)) \\
&= C(\mathcal{L}_{\tilde{q}_N}^k z_L)(\theta) \ , \quad \theta\in\mathcal{C} \ , \ \ k\geq 2 \ .
\end{aligned}$$

The constant depends on z_L, any integer $L \geq 0$ is allowed. Substituting these inequalities into (80) (for $k = 1$, we can use the boundedness of the expression $\Delta H_{\tilde{q}_N,0}(\theta)$), we get

$$G_{\tilde{q},s}(\theta) \leq C \left(1 + \sum_{k=1}^{\infty} 2^{2(s+\delta_N)k} (\mathcal{L}_{\tilde{q}_N}^k z_L)(\theta) \right) . \tag{82}$$

Now, consider any $s < \tilde{s}$, set $\delta = (\tilde{s} - s)/4$, and choose first L and then N such that both

$$\frac{\rho(\mathcal{L}_{\tilde{q}_N}, V_{\tilde{q}_N,z_L})}{\tilde{\rho}} < 2^{2\delta}$$

and $\delta_N < \delta$ are satisfied. Since $V_{\tilde{q}_N,z_L} \subset L_\infty(\mathbb{T}^d)$, by definition of the spectral radius we have

$$\|\mathcal{L}_{\tilde{q}_N}^k z_L\|_{L_\infty(\mathcal{C})} \leq C_{\rho'} {\rho'}^k \ , \ k \geq 1 \ ,$$

for any $\rho' > \rho(\mathcal{L}_{\tilde{q}_N}, V_{\tilde{q}_N,z_L})$, e.g., we can choose ρ' such that

$$1 < \frac{\rho'}{\tilde{\rho}} < 2^{4\delta} \ .$$

With these choices, it remains to substitute into (82):

$$\begin{aligned} \|G_{\tilde{q},s}\|_{L_\infty(\mathcal{C})} &\leq C(1 + \sum_{k\geq 1} 2^{2(s+\delta_N)k} \|\mathcal{L}_{\tilde{q}_N}^k z_L\|_{L_\infty}(\mathcal{C})) \leq C \sum_{k=0}^{\infty} 2^{2(s+\delta)k} {\rho'}^k \\ &\leq C \sum_{k=0}^{\infty} 2^{2(s+3\delta)k} \rho^k = C \sum_{k=0}^{\infty} 2^{2(s+3\delta-\tilde{s})k} < \infty \ . \end{aligned}$$

It follows that $g_{\tilde{\phi},s} = G_{\tilde{q},s} \in L_\infty(\mathcal{C}) \subset L_1(\mathcal{C})$ and, by Corollary 9, that $\tilde{\phi} \in H^s(\mathbb{R}^d)$.

Moreover, we have implicitly shown that the series

$$g_{\tilde{\phi},s}(\theta) = \sum_{k=1}^{\infty} \sum_{\alpha \in \Omega_k \setminus \Omega_{k-1}} (1 + |\theta + 2\pi\alpha|^2)^s |\hat{\tilde{\phi}}(\theta + 2\pi\alpha)|^2 \ ,$$

converges uniformly on $\mathcal{C}$. Since $\hat{\tilde{\phi}}$ is continuous on $\mathbb{R}^d$ (Corollary 9), this shows the continuity of $g_{\tilde{\phi},s}(\theta)$ for $s < \tilde{s}$.

Assume that $\tilde{\phi} \in H^s(\mathbb{R}^d)$ for some $s > \tilde{s}$. This means that the numbers

$$\Delta h_{\tilde{q},k} = \int_{\mathcal{C}} \Delta H_{\tilde{q},k}(\theta)\, d\theta \, , \quad k \geq 1 \ ,$$

satisfy $\Delta h_{\tilde{q},k} \geq 0$ and

$$\begin{aligned} \sum_{k=1}^{\infty} 2^{2sk} \Delta h_{\tilde{q},k} &= \int_{\mathcal{C}} \sum_{k=1}^{\infty} 2^{2ks} \Delta H_{\tilde{q},k}(\theta)\, d\theta \\ &\leq C \|G_{\tilde{q},s}\|_{L_1(\mathcal{C})} = C \|\tilde{\phi}\|_{H^s}^2 < \infty \ . \end{aligned}$$

If we again consider the specific sequence $\tilde{q}'_N$, together with the associated functions $\tilde{\phi}'_N$, which we have constructed for the above proof of Lemma 11, then by comparison arguments based on the analog of (72) for our q'_N we have

$$\Delta h_{\tilde{q}'_N,k} \le C(1+\epsilon_N)^k \Delta h_{\tilde{q},k}$$

for a certain sequence $\epsilon_N \to 0$. Therefore, fixing any $s' \in (\tilde{s}, s)$, we see that $G_{\tilde{q}'_N,s'} \in L_1(\mathcal{C})$ respectively $\tilde{\phi}'_N \in H^{s'}(\theta)$ for large enough N. This contradicts $s_{\tilde{\phi}'_N} \to \tilde{s} < s'$, $N \to \infty$, and completes the proof of Lemma 12.

Note that in the proof of Lemma 12, we have implicitly established the useful inequality

$$\|\Delta H_{\tilde{q},k}(\theta)\|_{L_\infty(\mathcal{C})} \le C_\delta 2^{-2(\tilde{s}-\delta)k} \ , \quad k \ge 2 \ , \tag{83}$$

for any fixed $\delta > 0$, which we will use later.

2.5 H^s–Riesz basis property: $\tilde{\phi} \notin L_2(\mathbb{R}^d)$

In this subsection, we prove a theorem about the H^s–Riesz basis property of a hierarchical system $\mathbf{\Psi}$ in the case that the dual scaling function does not necessarily belong to $L_2(\mathbb{R}^d)$. We assume A1 and A2. Let $\tilde{q}(\theta) = |\tilde{m}(\theta)|^2$, where the dual symbol $\tilde{m}(\theta)$ is derived from ϕ, ψ^λ as detailed in subsection 2.2. Especially, (47-50) are assumed, and Theorem 4 is applicable. Let

$$\tilde{q}_N(\theta) = p(\theta) q_N(\theta) \ , \quad N \ge N_0 \ ,$$

be any sequence of trigonometric polynomials constructed via Lemma 6. Provided that N is large enough, we can assume that

$$\|\tilde{q} - \tilde{q}_N\|_{L_\infty(\mathcal{C})} \le \epsilon_N := C N^l r_0^N \ ,$$

or, similarly,

$$0 < (1 - C\epsilon_N)\tilde{q}_N(\theta) \le \tilde{q}(\theta) \le (1 + C\epsilon_N)\tilde{q}_N(\theta) \ , \quad \theta \in \mathcal{C} \ .$$

Obviously, since the Fourier expansion of $\tilde{q}_N$ is finite, $\mathcal{L}_{\tilde{q}_N}$ is a trace class operator and belongs to E^t for any $0 < t < 1$. It is easy to check that the eigenfunctions associated with non-zero eigenvalues of any such transfer operator are actually trigonometric polynomials (of degree depending only on the degree of $\tilde{q}_N$).

Theorem 13 *Under the assumptions A1 and A2, with approximations $\tilde{q}_N$ to $\tilde{q}$ satisfying the above inequalities, and $\tilde{\rho}_0$ defined by (74), we have*

$$\rho(\mathcal{L}_{\tilde{q}_N}) \to \rho(\mathcal{L}_{\tilde{q}}) = \tilde{\rho}_0 \ge 1 \ , \quad |\rho(\mathcal{L}_{\tilde{q}_N}) - \tilde{\rho}_0| \le C\epsilon_N \ , \ N \to \infty \ . \tag{84}$$

If $\tilde{s}_0 := -\frac{1}{2}\log_2 \tilde{\rho}_0 \le 0$ satisfies $-\tilde{s}_0 < s_\phi$ then the hierarchical system $\mathbf{\Psi}$ associated with the underlying choice for ϕ, ψ^λ is a Riesz basis in $H^s(\mathbb{R}^d)$ for all s in the interval

$$-\tilde{s}_0 < s < s_\phi \ .$$

If $\tilde{s}_0 < 0$, then $\mathbf{\Psi}$ is not a Riesz basis in $H^s(\mathbb{R}^d)$ for any $s < -\tilde{s}_0$.

Proof. We start with a simple observation. Let $r_0^2 < t < 1$ be fixed. By Theorem 5, $\tilde{\rho}_0 = \max_j |\lambda_j| = \tilde{\lambda}$ where $\tilde{\lambda}$ is the largest positive eigenvalue of $\mathcal{L}_{\tilde{q}}$, the corresponding eigenfunction $0 \neq \tilde{c} \in E^t \subset L_\infty(\mathbb{T}^d)$ is non-negative, and can therefore be chosen such that

$$0 \leq \tilde{c}(\theta) \leq 1 \;, \quad \theta \in \mathcal{C} \;.$$

By the monotonicity of $\mathcal{L}_{\tilde{q}}$, we obtain

$$\tilde{\lambda}^k \tilde{c}(\theta) = (\mathcal{L}_{\tilde{q}}^k \tilde{c})(\theta) \leq (\mathcal{L}_{\tilde{q}}^k 1)(\theta) \;.$$

On the other hand, by properties of the spectral radius of linear operators, for any $\delta > 0$ there is a constant C_δ such that $\|\mathcal{L}_{\tilde{q}}^k\|_{E^t} \leq C_\delta(\tilde{\rho}_0 + \delta)^k$. This gives

$$\tilde{\rho}_0^k \|\tilde{c}\|_{L_\infty(\mathcal{C})} \leq \|\mathcal{L}_{\tilde{q}}^k 1\|_{L_\infty(\mathcal{C})} \leq C\|\mathcal{L}_{\tilde{q}}^k 1\|_{E^t} \leq CC_\delta(\tilde{\rho}_0 + \delta)^k \;. \tag{85}$$

This formula holds for arbitrary $k \geq 1$, its analog can also be derived for any of the operators $\mathcal{L}_{\tilde{q}_N}$. But by definition of the transfer operators and (70) we also have for an arbitrary non-negative $c(\theta) \geq 0$ that

$$(1 - C\epsilon_N)(\mathcal{L}_{\tilde{q}_N} c)(\theta) \leq (\mathcal{L}_{\tilde{q}} c)(\theta) \leq (1 + C\epsilon_N)(\mathcal{L}_{\tilde{q}_N} c)(\theta) \;, \quad \theta \in \mathcal{C} \;.$$

Recursive use of this relation, together with the monotonicity of transfer operators, leads to

$$(1 - C\epsilon_N)^k \|\mathcal{L}_{\tilde{q}_N}^k 1\|_{L_\infty(\mathcal{C})} \leq \|\mathcal{L}_{\tilde{q}}^k 1\|_{L_\infty(\mathcal{C})} \leq (1 + C\epsilon_N)^k \|\mathcal{L}_{\tilde{q}_N}^k 1\|_{L_\infty(\mathcal{C})} \;.$$

Substituting now the appropriate parts of inequality (85) for $\mathcal{L}_{\tilde{q}}$ respectively $\mathcal{L}_{\tilde{q}_N}$, raising to the power $1/k$ and letting $k \to \infty$, we obtain

$$(1 - C\epsilon_N)\rho(\mathcal{L}_{\tilde{q}_N}) \leq \tilde{\rho}_0 \leq (1 + C\epsilon_N)\rho(\mathcal{L}_{\tilde{q}_N}) \;.$$

This shows the convergence statement (84) of Theorem 13, together with the exponential rate. The relationship $\tilde{\rho}_0 \geq 1$ is implicitly contained in (85) and Theorem 4, since the projectors Q_j^{j+k} necessarily have norms ≥ 1.

We come to the Riesz basis property. For this, we use equivalent norms for $H^s(\mathbb{R}^d)$ formulated in terms of a MRA $\{V_j\}$ with scaling function ϕ satisfying A1. The triple bar norm, $|||\cdot|||$, associated with this MRA is defined by

$$|||u|||^2 := \inf_{v_j \in V_j \,:\, u = \sum_{j=0}^\infty v_j} \sum_{j=0}^\infty 2^{2js} \|v_j\|_{L_2}^2 \;. \tag{86}$$

It is well known, see e.g., [13, Theorem 5.1] or [4], that

$$\|u\|_{H^s}^2 \asymp \begin{cases} |||u|||^2 & , \quad 0 < s < s_\phi \;, \\ \|P_0 u\|_{L_2}^2 + \sum_{j=1}^\infty 2^{2sj} \|P_j u - P_{j-1} u\|_{L_2}^2 & , \quad -s_\phi < s < s_\phi \;, \end{cases} \tag{87}$$

where P_j is the L_2–orthogonal projection onto V_j, $j \geq 0$.

In subsection 2.1 we have shown that the H^s–Riesz basis property is equivalent to the norm equivalence (31), provided the existence and uniform L_2–boundedness of the projections Q_j. However, under the assumptions of Theorem 13, this cannot be justified. Instead of (31), we will show that

$$\|u_J\|_{H^s}^2 \asymp \|u_J\|_{Q,s,J}^2 := \|Q_0^J u_J\|_{L_2}^2 + \sum_{k=1}^{J} 2^{2sk} \|(Q_k^J - Q_{k-1}^J)u_J\|_{L_2}^2 \ , \qquad (88)$$

uniformly in $u_J \in V_J$ and J. This implies the H^s–Riesz basis property of $\mathbf{\Psi}$ by obtaining it first for the finite sections $\mathbf{\Psi}_J$ (with uniformly bounded Riesz constants), and then letting $J \to \infty$. This reduction repeats the corresponding considerations at the end of subsection 2.1, and is left upon the reader. We will concentrate on the proof of (88).

Let $0 \leq -\tilde{s}_0 < s < s_\phi$. By definition of $\tilde{s}_0$, by Theorem 4, and by taking a sufficiently small $\delta > 0$ in (85), for any fixed s' satisfying $-\tilde{s}_0 < s' < s$ we have

$$\|Q_k^j v_j\|_{L_2}^2 \leq C 2^{2s'(j-k)} \|v_j\|_{L_2}^2 \ , \quad v_j \in V_j \ , \ 0 \leq k < j \ .$$

Let $u_J \in V_J$, and consider the L_2-orthogonal decomposition

$$u_J = \sum_{j=0}^{J} v_j \ , \qquad v_j = P_j u_J - P_{j-1} u_J \in V_j \ , \ j = 1, \ldots, J \ , \ \ v_0 = P_0 u_J \in V_0 \ .$$

Its decomposition with respect to $\mathbf{\Psi}_J$ can be written by means of the projectors Q_k^j as follows:

$$\begin{aligned} u_J &= Q_0^J u_J + \sum_{k=1}^{J} (Q_k^J - Q_{k-1}^J) u_J \\ &= (v_0 + \sum_{j=1}^{J} Q_0^j v_j) + \sum_{k=1}^{J} (v_k + \sum_{j=k+1}^{J} Q_k^j v_j - \sum_{j=k}^{J} Q_{k-1}^j v_j) \ . \end{aligned}$$

Thus, fixing a small $\epsilon \in (0, s - s')$, we have

$$\begin{aligned} \|(Q_k^J - Q_{k-1}^J)u_J\|_{L_2}^2 &\leq C(\|v_k\|_{L_2}^2 + \sum_{j=k+1}^{J} 2^{2\epsilon(j-k)} \|Q_k^j v_j\|_{L_2}^2 \\ &\qquad + \sum_{j=k}^{J} 2^{2\epsilon(j-k)} \|Q_{k-1}^j v_j\|_{L_2}^2) \\ &\leq C \sum_{j=k}^{J} 2^{2(s'+\epsilon)(j-k)} \|v_j\|_{L_2}^2 \ , \quad k \geq 1 \ , \end{aligned}$$

analogously for $\|Q_0^J u_J\|_{L_2}^2$, $k = 0$. Substitution gives

$$\|u_J\|_{Q,s,J}^2 \leq C \sum_{k=0}^{J} 2^{2sk} \sum_{j=k}^{J} 2^{2(s'+\epsilon)(j-k)} \|v_j\|_{L_2}^2 \leq C \sum_{j=0}^{J} 2^{2sj} \|v_j\|_{L_2}^2 \ ,$$

where the constant does not depend on J. Now recall the definition of v_j, and use the second norm equivalence in (87). This yields

$$\|u_J\|^2_{Q,s,J} \le C\|u_J\|^2_{H^s} , \qquad u_J \in V_J .$$

The opposite inequality follows from the infimum definition of the triple bar norm $|||\cdot|||$ and the first norm equivalence of (87) (here, the relation $0 \le -\tilde{s}_0 < s$ is taken into account):

$$\|u_J\|^2_{Q,s,J} \ge |||u_J|||^2 \ge c\|u_J\|^2_{H^s} , \qquad u_J \in V_J .$$

Thus, (88) follows, with constants independent of J.

Finally, suppose $\tilde{s}_0 < 0$ and consider any $0 < s < -\tilde{s}_0$. Fix some $s' \in (s, -\tilde{s}_0)$. By definition of $\tilde{s}_0$, (85), and Theorem 4, we see that we can find a sequence $u_J \in V_J$, $J \to \infty$, such that

$$\|Q_0^J u_J\|^2_{L_2} \ge C 2^{2s'J} \|u_J\|^2_{L_2} .$$

Using the equivalence of the H^s- and the triple bar norm, we obtain

$$\begin{aligned} \|u_J\|^2_{H^s} &\le C|||u_J|||^2 \le C2^{2sJ}\|u_J\|^2_{L_2} \le C2^{2(s-s')J}\|Q_0^J u_J\|^2_{L_2} \\ &\le C2^{2(s-s')J}(\|Q_J^0 u_J\|^2_{L_2} + \sum_{k=1}^{J} 2^{2sk}\|(Q_J^k - Q_J^{k-1})u_J\|^2_{L_2}) . \end{aligned}$$

Note that the factor $2^{2(s-s')J}$ deteriorates exponentially fast if $J \to \infty$. This contradicts the Riesz basis property of $\mathbf{\Psi}$ in $H^s(\mathbb{R}^d)$ (according to the derivation at the end of subsection 2.1, we see that there is no chance to find a finite Riesz constant $B < \infty$ in the upper estimate for the Riesz property in this case). Thus, Theorem 13 is established.

Except for the limiting case $s = -\tilde{s}_0$, Theorem 13 gives the complete answer concerning the Riesz basis property for the systems under consideration in Sobolev spaces $H^s(\mathbb{R}^d)$ with $s > 0$. The case $s \le 0$ will be discussed in the next subsection.

2.6 H^s–Riesz basis property: $\tilde{\phi} \in L_2(\mathbb{R}^d)$

In this subsection, we assume that $\tilde{s}$ defined in (76) is strictly positive. By Lemma 12, $\tilde{\phi}$ belongs to $L_2(\mathbb{R}^d)$, and its regularity exponent satisfies $s_{\tilde{\phi}} = \tilde{s} > 0$. Under this assumption, we show that the conditions of Theorem 2 hold and, consequently, obtain the interval $-s_{\tilde{\phi}} < s < s_\phi$ for which the system $\mathbf{\Psi}$ of (20) is an H^s–Riesz basis. According to the negative results of Theorem 13, we see that $\tilde{s} > 0$ can only happen if $\tilde{s}_0 = 0$ (it can be shown that in the case $\tilde{s}_0 < 0$ we have $\tilde{\rho} = \rho_0 > 1$ and, thus, $\tilde{s}_0 = \tilde{s} < 0$).

If $\phi, \tilde{\phi} \in L_2(\mathbb{R}^d)$ then the validity of Jackson-Bernstein inequalities is connected with studying the functions $g_{\phi,s}, g_{\tilde{\phi},s}$ defined in subsection 2.1. Indeed, we have

Lemma 14 *Let $\phi, \tilde{\phi} \in L_2(\mathbb{R}^d)$ satisfy the refinement equations (10), (44), and the biorthogonality relation expressed by (29), and assume that V_j, $\tilde{V}_j$ are defined as $L_2(\mathbb{R}^d)$–closures of the linear spans of the systems Φ_j, $\tilde{\Phi}_j$, respectively. Let $g_{\phi,s}, g_{\tilde{\phi},s}$ be defined according to (4).*
(i) *ϕ belongs to $H^s(\mathbb{R}^d)$ if and only if $g_{\phi,s} \in L_1(\mathcal{C})$ (analogously for $\tilde{\phi}$).*
(ii) *Φ_j is an L_2–Riesz basis in V_j if and only if $0 < c < g_{\phi,0}(\theta) < C < \infty$, $\theta \in \mathcal{C}$, for some constants c and C (analogously for $\tilde{\Phi}_j$).*
(iii) *Let* (ii) *be satisfied for ϕ and $\tilde{\phi}$. Then the projectors $Q_j, \tilde{Q}_j$ of (30) are uniformly L_2-bounded.*
(iv) *Let* (ii) *be satisfied for ϕ, and let $s > 0$. Then*

$$g_{\phi,s} - |\hat{\phi}(\theta)|^2 = \mathrm{O}(|\theta|^{2s}) , \quad \theta \to 0 \tag{89}$$

implies the Jackson inequality (32) of order s for $\{V_j\}$ (analogously for $\tilde{\phi}$ and $\{\tilde{V}_j\}$).
(v) *Let* (ii) *be satisfied for ϕ, and let $s > 0$. Then $g_{\phi,s} \in L_\infty(\mathcal{C})$ implies the Bernstein inequality (33) of order s for $\{V_j\}$ (analogously for $\tilde{\phi}$ and $\{\tilde{V}_j\}$).*

Proof. (i) is obvious from the definition of Sobolev spaces on $\mathbb{R}^d$ via Fourier transforms. (ii) is well-known, see [23, Theorem 3.3], [2], or subsection 2.1. Formally, (iii) follows from

$$\begin{aligned}
\|Q_0 f\|_{L_2}^2 &= C \int_{\mathbb{R}^d} |\sum_{\alpha\in\mathbb{Z}^d} (f, \tilde{\phi}(\cdot - \alpha))_{L_2} e^{-i\alpha\theta}|^2 |\hat{\phi}(\theta)|^2 \, d\theta \\
&\le C \|g_{\phi,0}\|_{L_\infty(\mathcal{C})} \sum_{\alpha\in\mathbb{Z}^d} |(f, \tilde{\phi}(\cdot - \alpha))_{L_2}|^2 \le C \sum_{\alpha\in\mathbb{Z}^d} |\int_{\mathbb{R}^d} \hat{f}(\theta) \hat{\tilde{\phi}}(\theta)^* e^{i\alpha\theta} \, d\theta|^2 \\
&= C \sum_{\alpha\in\mathbb{Z}^d} |\int_{\mathcal{C}} \sum_{\beta\in\mathbb{Z}^d} \hat{f}(\theta + 2\pi\beta) \hat{\tilde{\phi}}(\theta + 2\pi\beta)^* e^{i\alpha\theta} \, d\theta|^2 \\
&= C \int_{\mathcal{C}} |\sum_{\beta\in\mathbb{Z}^d} \hat{f}(\theta + 2\pi\beta) \hat{\tilde{\phi}}(\theta + 2\pi\beta)^*|^2 \, d\theta \\
&\le C \|g_{\tilde{\phi},0}\|_{L_\infty(\mathcal{C})} \int_{\mathcal{C}} \sum_{\beta\in\mathbb{Z}^d} |\hat{f}(\theta + 2\pi\beta)|^2 \, d\theta \le C \|f\|_{L_2}^2 .
\end{aligned}$$

The changes of summation can easily be justified under the assumptions made. For Q_j, $j > 0$, use a dilation argument. For the proof of the L_2-boundedness of $\{\tilde{Q}_j\}$, interchange the roles of ϕ and $\tilde{\phi}$.

Assertion (iv) has been proved in [1]. Finally, to establish (v) it suffices again to consider $j = 0$. If

$$v_0 = \sum_{\beta\in\mathbb{Z}^d} c_\beta \phi(\cdot - \beta) \in V_0 ,$$

then we have

$$\|v_0\|_{H^s}^2 = C \int_{\mathbb{R}^d} |\hat{v}_0(\theta)|^2 (1 + |\theta|^2)^s \, d\theta$$

$$= C\int_{\mathbb{R}^d} |\sum_{\beta\in\mathbb{Z}^d} c_\beta e^{-i\beta\theta}|^2 |\hat\phi(\theta)|^2 (1+|\theta|^2)^s \, d\theta$$

$$= C\int_{\mathcal{C}} g_{\phi,s}(\theta) |\sum_{\beta\in\mathbb{Z}^d} c_\beta e^{-i\beta\theta}|^2 \, d\theta \le C\|g_{\phi,s}\|_{L_\infty(\mathcal{C})} \sum_{\beta\in\mathbb{Z}^d} |c_\beta|^2 \, .$$

By dilation, we obtain (33).

We first verify that ϕ fulfills the assumptions of Theorem 2. That ϕ generates a MRA is the assumption A1. By (7), we have $g_{\phi,s} \in L_\infty(\mathcal{C})$ for $s < s_\phi$ (according to the derivation of (7), this follows immediately from the H^s-Riesz property of Φ_0, $s < s_\phi$, which, in turn, is a consequence of the L_2-Riesz basis property of Φ_0 in V_0 and $\phi \in H^s(\mathbb{R}^d)$). Thus, part (v) of Lemma 14 shows that the Bernstein inequality for $\{V_j\}$ is satisfied for $0 < s < s_\phi$. For compactly supported scaling functions, it is well known that $\{V_j\}$ satisfies the Jackson inequality of order L_m (the Strang-Fix order of $m(\theta)$). Since $s_\phi \le L_m$, we conclude that ϕ and $\{V_j\}$ satisfy the conditions of Theorem 2 for any $\gamma < s_\phi$.

In the remainder of this subsection we will discuss the essential steps of checking that $\tilde\phi$ and $\{\tilde V_j\}$ satisfy the conditions of Theorem 2. Note that the $g_{\tilde\phi,s}$ is accessible to estimates from knowledge about $\tilde m(\theta)$ via the formula (69) for $\hat{\tilde\phi}$. We will make repeated use of properties of $\mathcal{L}_r$, with various r.

First note that Lemma 12 shows that $g_{\tilde\phi,s}$ is continuous and thus belongs to $L_\infty(\mathcal{C})$ for all $s < \tilde s$. From (v) of Lemma 14, it follows that the Bernstein inequality for $\{\tilde V_j\}$ is satisfied for $0 < s < s_{\tilde\phi}$.

Lemma 15 *Under the assumptions A1, A2 and $\tilde s > 0$, there are constants c and C such that*

$$0 < c \le g_{\tilde\phi,0}(\theta) \le C < \infty \, , \quad \theta \in \mathcal{C} \, . \tag{90}$$

From (ii) of Lemma 14, it follows that $\tilde\Phi_j$ is a Riesz basis in $\tilde V_j$ and that $\tilde\phi$ generates a MRA.

Proof. The upper inequality of (90) holds since $g_{\tilde\phi,0}$ is continuous by Lemma 12 and $\tilde s > 0$. The following simple observation is sufficient to show the lower bound. Since ϕ generates an MRA, by (7), we have $0 < c \le g_{\phi,0}(\theta) \le C < \infty$. Then, the function

$$g(\theta) = \sum_{\beta\in\mathbb{Z}^d} \hat\phi(\theta+2\pi\beta)\hat{\tilde\phi}(\theta+2\pi\beta)^*$$

belongs also to $L_\infty(\mathbb{T}^d)$ (the sum is absolutely convergent on $\mathbb{T}^d$ to a continuous function, use Hölder inequality). Computing its Fourier coefficients leads to

$$g_\alpha = (2\pi)^{-d}\int_{\mathcal{C}} g(\theta)e^{i\alpha\theta}\, d\theta = (2\pi)^{-d}\int_{\mathbb{R}^d} \hat\phi(\theta)(\hat{\tilde\phi}(\theta)e^{-i\alpha\theta})^* \, d\theta$$

$$= \int_{\mathbb{R}^d} \phi(x)\tilde\phi(x-\alpha)^* \, dx = (\phi(\cdot),\tilde\phi(\cdot-\alpha))_{L_2} = \delta_{0\alpha}$$

for $\alpha \in \mathbb{Z}^d$. Thus, from (29) we see that $g(\theta) = 1$ and by the Hölder inequality

$$1 = |g(\theta)|^2 \leq g_{\phi,0}(\theta) g_{\tilde{\phi},0}(\theta) , \quad \theta \in \mathcal{C} .$$

This implies the lower bound with a constant $c = \|g_{\phi,0}\|_{L_\infty}^{-1}$, and, thus, $\tilde{\Phi}_j$ is a Riesz basis in $\tilde{V}_j$.Since $\tilde{\phi}$ satisfies a refinement equation, we have shown that $\tilde{\phi}$ generates a MRA if $\cup \tilde{V}_j$ is dense in $L_2(\mathbb{R}^d)$. But this follows from the fact that $\hat{\tilde{\phi}}$ is continuous and that $\hat{\tilde{\phi}}(0) \neq 0$, see, e.g., [1]. This finishes the proof of the lemma.

Using Lemma 15 and (iii) in Lemma 14, it follows that the projector sequences $\{Q_j\}$ and $\{\tilde{Q}_j\}$ are uniformly L_2–bounded.

We come to the Jackson inequality for $\{\tilde{V}_j\}$. Recall that $\tilde{q}(\theta) = |\tilde{m}(\theta)|^2 = p(\theta) q(\theta)$. Due to the fact that $\tilde{m}(\theta)$ is a C^∞–function, the Strang-Fix orders $L_{\tilde{m}}$ and $L_{\tilde{q}} = L_p = 2L_{\tilde{m}}$ are well-defined.

Lemma 16 *Under the assumptions A1, A2, and $\tilde{s} > 0$, the Jackson inequality of order $L_{\tilde{m}}$ holds for $\{\tilde{V}_j\}$.*

Proof. According to Lemma 14, (iv), it suffices to verify that

$$w(\theta) := G_{\tilde{q},0}(\theta) - \Pi(\tilde{q})(\theta) = \mathrm{O}(|\theta|^{L_{\tilde{q}}}) , \quad \theta \to 0 . \tag{91}$$

We use the inequalities (83) which are valid under our assumptions and lead to

$$0 \leq G_{\tilde{q},0}(\theta) - H_{\tilde{q},k}(\theta) = \sum_{l=k+1}^{\infty} \Delta H_{\tilde{q},l}(\theta) \leq C \sum_{l=k+1}^{\infty} 2^{-\tilde{s}l} \leq C 2^{-\tilde{s}k}$$

for $\theta \in \mathcal{C}$ if we choose $\delta = \tilde{s}/2$. Thus, $H_{\tilde{q},k}(\theta) \to G_{\tilde{q},0}(\theta)$, $k \to \infty$, uniformly on $\mathcal{C}$. Since $\Pi(\tilde{q})(\theta)$ and, thus, $H_{\tilde{q},k}(\theta)$ are continuous functions on $\mathbb{R}^d$, this yields $G_{\tilde{q},0} \in C(\mathbb{T}^d)$ and the continuity of the function $w(\theta)$ introduced in (91).

As a final preparation, we will check that

$$\mathcal{L}_{\tilde{q}} G_{\tilde{q},0} = G_{\tilde{q},0} . \tag{92}$$

Indeed,

$$\begin{aligned} (\mathcal{L}_{\tilde{q}} G_{\tilde{q},0})(\theta) &= \sum_{\lambda \in \Lambda} \sum_{\alpha \in \mathbb{Z}^d} \tilde{q}(\frac{\theta}{2} + \pi\lambda) \prod_{k=1}^{\infty} \tilde{q}(2^{-k}(\frac{\theta + 2\pi\alpha}{2} + \pi\lambda)) \\ &= \sum_{\lambda \in \Lambda} \sum_{\alpha \in \mathbb{Z}^d} \prod_{k=1}^{\infty} \tilde{q}(2^{-k}(\theta + 2\pi(2\alpha + \lambda)) = G_{\tilde{q},0}(\theta) . \end{aligned}$$

Using the definition of the transfer operator $\mathcal{L}_{\tilde{q}}$ and of $L_{\tilde{q}}$ leads, together with (92), to

$$\begin{aligned} |w(\theta)| &= |(\mathcal{L}_{\tilde{q}} G_{\tilde{q},0})(\theta) - \Pi(\tilde{q})(\theta)| \\ &\leq \tilde{q}(\frac{\theta}{2}) |G_{\tilde{q},0}(\frac{\theta}{2}) - \Pi(\tilde{q})(\frac{\theta}{2})| + \sum_{\lambda \in \Lambda'} \tilde{q}(\frac{\theta}{2} + \pi\lambda) G_{\tilde{q},0}(\frac{\theta}{2} + \pi\lambda) \\ &\leq \tilde{q}(\frac{\theta}{2}) |w(\frac{\theta}{2})| + C|\theta|^{L_{\tilde{q}}} . \end{aligned}$$

The constant C depends on $\tilde{q}$ and on $\|G_{\tilde{q},0}\|_{L_\infty}$ but is independent of θ.

From Lemma 11, it follows that $L_{\tilde{q}} \geq 1$ so that $\Pi(\tilde{q})(2\pi\alpha) = 0$ for all $\alpha \in \mathbb{Z}^d \setminus \{0\}$. Thus iterating the inequality for $|w(\theta)|$ and using $w(\theta) \to 0,\ \theta \to 0$, which follows from the continuity of w and from

$$w(0) = G_{\tilde{q},0}(0) - \Pi(\tilde{q})(0) = \sum_{0\neq\alpha\in\mathbb{Z}^d} \Pi(\tilde{q})(2\pi\alpha) = 0 \ ,$$

we obtain

$$\begin{aligned} |w(\theta)| &\leq \Pi(\tilde{q})_k(\theta)|w(2^{-k}\theta)| + C(\sum_{l=0}^{k-1} \Pi(\tilde{q})_l(\theta)|2^{-l}\theta|^{L_{\tilde{q}}}) \\ &\leq C \sup_l \|\Pi(\tilde{q})_l\|_{L_\infty} \sum_{l=0}^{\infty} |2^{-l}\theta|^{L_{\tilde{q}}} \leq C|\theta|^{L_{\tilde{q}}} \ ,\ \theta \to 0 \ . \end{aligned}$$

Thus, (91) holds, and the proof is complete.

The main result of this subsection is

Theorem 17 *Let the functions ϕ, ψ^λ, $\lambda \in \Lambda'$, which define the hierarchical system $\mathbf{\Psi}$ given by (20), satisfy assumptions A1 and A2. Assume further that $\tilde{s} > 0$ (see (76), (71) for the definition of $\tilde{s}$ and $\tilde{\rho}$). Then the dual scaling function $\tilde{\phi}$ is well-defined in $L_2(\mathbb{R}^d)$, and its Sobolev regularity is given by $s_{\tilde{\phi}} = \tilde{s}$. The system $\mathbf{\Psi}$ is a Riesz basis in $H^s(\mathbb{R}^d)$ if $-s_{\tilde{\phi}} < s < s_\phi$. This interval is sharp, in the sense, that the H^s-Riesz basis property cannot hold in either of the cases $s < -s_{\tilde{\phi}}$ and $s \geq s_\phi$.*

Proof. As we have shown in this section (Lemmas 15, 16 and the remarks preceding them), the assumptions of Theorem 2 hold. Thus $\mathbf{\Psi}$ is a $H^s(\mathbb{R}^d)$–Riesz basis for $-s_{\tilde{\phi}} < s < s_\phi$. It remains to show that this interval is sharp. Clearly, $\mathbf{\Psi}$ is not a H^s–Riesz basis for $s \geq s_\phi$ since for these s, ϕ does not even belong to $H^s(\mathbb{R}^d)$. From Lemma 12, $\tilde{\phi} \notin H^s(\mathbb{R}^d)$ for $s > s_{\tilde{\phi}}$. But $\tilde{\phi} \in H^s(\mathbb{R}^d)$ is a necessary condition for the system $\mathbf{\Psi}$ to be a Riesz basis in $H^{-s}(\mathbb{R}^d)$. Indeed, assuming the H^{-s}–Riesz basis property of $\mathbf{\Psi}$, the norm equivalence (31) should hold:

$$\|u\|_{H^{-s}}^2 \asymp \|Q_0u\|_{L_2}^2 + \sum_{k=1}^{\infty} 2^{-2sk}\|(Q_k - Q_{k-1})u\|_{L_2}^2 \qquad \forall\, u \in L_2(\mathbb{R}^d) \ .$$

Thus, by using the biorthogonality relations (29) and the usual density and duality arguments, we obtain

$$\begin{aligned} \|\tilde{\phi}\|_{H^s} &= \sup_{u\in H^{-s}(\mathbb{R}^d)} \frac{\langle\tilde{\phi}, u\rangle_{H^s\times H^{-s}}}{\|u\|_{H^{-s}}} = \sup_{u\in L_2(\mathbb{R}^d)} \frac{\langle\tilde{\phi}, u\rangle_{H^s\times H^{-s}}}{\|u\|_{H^{-s}}} \\ &\leq C \sup_{u\in L_2(\mathbb{R}^d)} \frac{(\tilde{\phi}, Q_0u)_{L_2}}{(\|Q_0u\|_{L_2}^2 + \sum_{k=1}^{\infty} 2^{-2ks}\|(Q_k - Q_{k-1})u\|_{L_2}^2)^{1/2}} \\ &\leq C \sup_{v_0\in V_0} \frac{(\tilde{\phi}, v_0)_{L_2}}{\|v_0\|_{L_2}} \leq C\|\tilde{\phi}\|_{L_2} < \infty \ . \end{aligned}$$

As in section 2.3, the limiting case $s = -s_{\tilde{\phi}}$ remains open, we conjecture that the Riesz basis property does not hold for this exceptional value. It should be mentioned that there is a case which is still left open by Theorem 13 and 17: it may happen that that the $\tilde{\rho}_0$ in (84) and the $\tilde{\rho}$ in (71) both equal 1. Then, by Theorem 13, $\mathbf{\Psi}$ is a Riesz basis in $H^s(\mathbb{R}^d)$ for any $0 < s < s_\phi$. On the other hand, the same considerations as before show that $\tilde{\phi}$ cannot belong to any Sobolev space of positive order, and that $\mathbf{\Psi}$ cannot be a Riesz basis in $H^s(\mathbb{R}^d)$ for any $s < 0$. This completes the picture.

Reviewing the proof of the main Theorems 13 and 17, it is clear that their statements hold not only when ϕ and ψ^λ are of compact support. The essential technical requirement is that $|m(\theta)|^2$ and $|\tilde{m}(\theta)|^2$ satisfy the assumption A3, i.e., that they are the product of a non-negative trigonometric polynomial and a positive C^∞ function with exponentially decaying Fourier coefficients. This is, for example, the case when ϕ and ψ^λ have rational trigonometric symbols. Of course, some of the remaining assumptions in A1 and A2 must also be satisfied.

Note that the argument used in the proof of property (76) implicitly yields a simplified formula for $\tilde{\rho}$. In (71), the infimum with respect to $L \geq 0$ can be replaced by the minimum for all $L \leq L_p/2$. Actually, we can prove

$$\tilde{\rho} = \tilde{\rho}_{L_{\min}} , \tag{93}$$

where $L_{\min}$ is the minimum of $L_p/2$ and the smallest $L \in \mathbb{Z}_+$ such that

$$\tilde{s}_L = -\frac{1}{2}\log_2 \tilde{\rho}_L < L .$$

By the above comparison techniques and Theorem 3, such an L exists in the range $0 \leq L \leq L_p/2+1$, at least. Note that in all examples below, we observed $L \leq L_p/2$ for such L which supports our conjecture that equality in (76) is impossible.

3 Examples: Box spline systems

3.1 Stevenson's system

In the next two subsections, we deal exclusively with the *linear box spline MRA* in $\mathbb{R}^d$. Wavelets based on the MRA of linear box splines on a three-directional mesh are of importance for several reasons. Above all, linear finite element spaces are a widely used in discretization schemes for second order elliptic boundary value problems and boundary integral equations which leads to the need in constructing preconditioning methods, and, in particular, to the interest in finding hierarchical bases with good properties in Sobolev spaces for this special case (see the references below). Although we are imposing a number of simplifications, some of the results on hierarchical Riesz bases $\mathbf{\Psi}$ associated with the linear box spline MRA carry over (with obvious modifications) to bounded polyhedral domains $\Omega \subset \mathbb{R}^d$ equipped with sequences of simplicial partitions $\mathcal{R}_j$ resulting from standard mesh refinement. Finally, the linear spline examples to be considered are still fairly simple. Thus, they allow for some detailed analysis and provide some additional insight which may be useful for applications to more complicated MRA's.

The linear box spline MRA is generated by its scaling function ϕ satisfying the refinement equation

$$\phi(x) = \frac{1}{2}\sum_{\lambda\in\Lambda}(\phi(2x-\lambda)+\phi(2x+\lambda)) \; . \tag{94}$$

The associated symbol is

$$m(\theta) = 2^{-d}\sum_{\lambda\in\Lambda}\cos\lambda\theta = \prod_{l=1}^{d+1}\cos\frac{\theta_l}{2} \; , \tag{95}$$

where θ_l denotes the coordinates of θ if $l=1,\ldots,d$, and where $\theta_{d+1}=\sum_{l=1}^{d}\theta_l$ for simplicity. This specific form of $m(\theta)$ immediately leads to an explicit formula for $\hat{\phi}$ via (13). General information on box splines, and specifically on the linear box spline ϕ, can be found in [3]. It is known that ϕ coincides with the (appropriately scaled) nodal basis function for linear C^0 finite elements with respect to an infinite (2^d-1)-directional uniform simplicial partition $\mathcal{R}_0$ of $\mathbb{R}^d$ which is invariant under $\mathbb{Z}^d$-shifts. The edges of this partition are induced by the 2^d-1 segments connecting the origin with $\lambda\in\Lambda'$. Thus, $s_\phi=3/2$, and $L_m=2$ as can be checked directly from (95). All conditions of a MRA, as formulated at the beginning of section 2, are valid. The linear box spline MRA is interpolatory, in the sense, that $v_0\in V_0$ is equivalent to

$$v_0(x)=\sum_{\alpha\in\mathbb{Z}^d}v_0(\alpha)\phi(x-\alpha)\;,\quad (v_0(\alpha)\,:\,\alpha\in\mathbb{Z}^d)\in\ell^2(\mathbb{Z}^d)\;.$$

This allows for a simple interpretation of the coefficients of the Riesz basis decomposition of $v_j\in V_j$ with respect to Φ_j as function values of v_j on the corresponding grid $\mathcal{V}_j:=2^{-j}\mathbb{Z}^d$. For convenience, set $\mathcal{V}_j^*=\mathcal{V}_j\backslash\mathcal{V}_{j-1},\; j\ge 1$. The support of ϕ is convex and given by its set of extremal points $\Lambda'\cup(-\Lambda')$. More explicitly, it is the union of all simplices in the above-mentioned partition attached to the origin.

In this section, we consider the hierarchical system $\mathbf{\Psi}_{3HB}$ which was introduced by Stevenson in [29] but appeared also in other papers [15]. It is based on the construction of functions $\psi_{j,P},\; P\in\mathcal{V}_j^*$, with small masks, such that a certain *discrete* $L_2(\Omega)$*-orthogonality* is satisfied. More specifically,

$$(f,g)_{L_2,j}:=\sum_{S\in\mathcal{R}_j}\frac{|S|}{d+1}\sum_{P\in\mathcal{V}_j\cap\bar{S}}f(P)g(P)^*$$

is used. Note that the underlying composite quadrature rule is based on the trapezoidal rule for the simplices S, and is exact on V_j. It turns out that for each $P\in\mathcal{V}_j^*$, there is a unique function of the form

$$\psi_{j,P}=\phi_{j,P}+a^P_{j,P'}\phi_{j,P'}+a^P_{j,P''}\phi_{j,P''}$$

which is orthogonal to all of V_{j-1} in the sense of this j-dependent discrete scalar product. Here P', P'' are the endpoints of the edge in $\mathcal{R}_{j-1}$ containing $P \in \mathcal{V}_j^*$. Thus, any ψ-mask contains ≤ 3 non-zero coefficients which explains the name *3-point hierarchical basis* and the notation $\mathbf{\Psi}_{3HB}$. The properties of this system have been studied, under various assumptions on $\{\mathcal{R}_j\}$, in [29, 30, 31]. In particular, the Riesz basis property in $H^s(\Omega)$ has been established for a relatively large range of the smoothness parameter including also some negative values of s.

We complement these results by considering the $\mathbb{R}^d$-counterpart of this system. The results of section 2 will be used to determine the exact s-range for the Riesz basis property in $H^s(\mathbb{R}^d)$ to hold for $\mathbf{\Psi}_{3HB}$. We use this example also to present some further modifications such as factorization techniques for the symbols, and to give some indications on the numerical performance of the methods for determining $s_{\tilde{\phi}}$.

It is obvious that in the case of infinite (2^d-1)-directional uniform partitions of $\mathbb{R}^d$, the above requirement of discrete L_2-orthogonality leads to the following 2^d-1 functions

$$\psi^\lambda(x) = \phi(2x-\lambda) - \frac{1}{2}(\phi(2x) + \phi(2x-2\lambda)) , \quad \lambda \in \Lambda' , \tag{96}$$

which generate the systems Ψ_j according to (20), and the complement spaces $W_j = \mathrm{clos}_{L_2}(\mathrm{span}\,\Psi_j)$. The stability of the splittings $V_j = V_{j-1} \dot{+} W_j$ is obvious (but could also be checked by (35)). We leave it as an exercise to compute the subsymbols $m_\lambda^{\lambda'}(\theta)$ from (94), (96) and to find expressions for $L(\theta)$, $M(\theta)$, $\tilde{m}(\theta)$, and other quantities of interest that have been introduced in subsection 2.1.

The dual symbol can sometimes be computed more conveniently by directly following the elimination procedure which leads to (41) and (42). Using the same notation as in subsection 2.1 associated with an arbitrary decomposition $v_1 = v_0 + w_1$, we have according to (94), (96)

$$c_{2\alpha+\lambda} = \begin{cases} d_\alpha - \frac{1}{2}\sum_{\lambda'\in\Lambda'}(d_\alpha^{\lambda'} + d_{\alpha-\lambda'}^{\lambda'}) & , \quad \lambda = 0 , \\ \frac{1}{2}(d_\alpha + d_{\alpha+\lambda}) + d_\alpha^\lambda & , \quad \lambda \in \Lambda' , \end{cases}$$

which immediately yields

$$c_{2\alpha} + \frac{1}{2}\sum_{\lambda\in\Lambda'}(c_{2\alpha+\lambda} + c_{2\alpha-\lambda}) = \frac{2^d+1}{2} d_\alpha + \frac{1}{4}\sum_{\lambda\in\Lambda'}(d_{\alpha+\lambda} + d_{\alpha-\lambda}) .$$

Switching to the corresponding functions and subsymbols, we see that

$$\sum_{\lambda\in\Lambda} \cos(\lambda\theta)\, c_\lambda(\theta) = \frac{1}{2}(2^d + \sum_{\lambda\in\Lambda} \cos 2\lambda\theta) d(2\theta) .$$

For the system under consideration, this is the counterpart of (41), and according to (42), (95) we obtain

$$\tilde{m}^*(\theta) = \tilde{m}(\theta) = \frac{2m(\theta)}{1+m(2\theta)} \tag{97}$$

for the symbol associated with the dual refinement equation, and

$$\tilde{q}(\theta) = p(\theta)q(\theta) \ , \quad p(\theta) = m(\theta)^2 \ , \quad q(\theta) = \frac{4}{(1+m(2\theta))^2} \ , \tag{98}$$

for the function (47). Obviously, the dual symbol is not a trigonometric polynomial but $p, \tilde{q}$ satisfy all conditions (47-50). For the Strang-Fix order, we have $L_{\tilde{q}} = L_p = 2L_{\tilde{m}} = 2L_m = 4$. Since the expressions are very simple, one could easily obtain some rough preliminary estimates for the quantity $\tilde{\rho}$ from (71). E.g., by

$$-(\cos\frac{\pi}{d+1})^{d+1} \le m(\theta) \le 1 \ ,$$

we get

$$(\mathcal{L}_{\tilde{q}}^k z_L)(\theta) \le \left(\frac{2}{1-(\cos\frac{\pi}{d+1})^{d+1}}\right)^{2k} (\mathcal{L}_p^k z_L)(\theta) \ .$$

By the above properties of ϕ, Theorem 3, and (73), we have

$$\lim_{k\to\infty} \|\mathcal{L}_p^k z_2\|_{L_\infty}^{1/k} = 2^{-2s_\phi} = 2^{-3} \ .$$

Thus, by the above estimates and (75), (74), we obtain

$$\tilde{\rho} = \tilde{\rho}_2 \le \frac{1}{2(1-(\cos\frac{\pi}{d+1})^{d+1})^2} = \begin{cases} 1/2 & , \quad d=1 \\ 32/49 & , \quad d=2 \\ 8/9 & , \quad d=3 \end{cases} \ . \tag{99}$$

For $d \ge 4$ this estimation yields a number > 1 which is not sufficient for verifying the assumption (71). Theorem 17 implies the Riesz basis property in $H^s(\mathbb{R}^d)$ for the system $\mathbf{\Psi}_{3HB}$ if $-s_{\tilde{\phi}} < s < 3/2$, where the regularity exponent of the dual scaling function $\tilde{\phi}$ satisfies

$$s_{\tilde{\phi}} = \frac{1}{2} + \log_2(1-(\cos\frac{\pi}{d+1})^{d+1}) \ge \begin{cases} 0.5 & , \quad d=1 \\ 0.316338 & , \quad d=2 \\ 0.084962 & , \quad d=3 \end{cases} \ . \tag{100}$$

To obtain the exact value, we will demonstrate several approaches related to the methods of section 2. We concentrate on the case $d \le 3$ and prove

Theorem 18 *For $d \le 3$, the system $\mathbf{\Psi}_{3HB}$ is a Riesz basis in $H^s(\mathbb{R}^d)$ if*

$$-0.990236... < s < 3/2 \ .$$

The lower bound is exact within rounding error and given by $-s_{\tilde{\phi}} = \frac{1}{2}\log_2\tilde{\rho}$, where $\tilde{\rho} = \tilde{\rho}_1 = 0.25340693...$ has been computed on the basis of (74) and (75) from approximations to the function $\tilde{q}$ given in (98).

We give different arguments for proving this result. First of all, in Table 1 and 2 we present numerical values for the leading 7 (for $d=1$) respectively 10 (for $d=2$) eigenvalues of the operator $\mathcal{L}_{\tilde{q}_N}$ for some small N. The approximating polynomials $\tilde{q}_N$ are determined by interpolation along the lines of Lemma 6.

$\lambda_{k,N}$	$N=5$	$N=10$	$N=15$	$N=20$
k=1	1.0	1.0	1.0	1.0
k=2	0.5	0.5	0.5	0.5
k=4	0.25308501	0.25339921	0.25340786	0.25340693
k=3	0.25	0.25	0.25	0.25
k=5	0.125	0.125	0.125	0.125
k=6	0.10616925	0.06922710	0.06881609	0.06873386
k=7	0.07114080	0.06653668	0.06728336	0.06718530

Table 1. Leading eigenvalues $\lambda_{k,N}$ of $\mathcal{L}_{\tilde{q}_N}$, $d=1$.

$\lambda_{k,N}$	$N=5$	$N=10$	$N=15$	$N=20$
k=1	1.0	1.0	1.0	1.0
k=2	0.5	0.5	0.5	0.5
k=3	0.5	0.5	0.5	0.5
k=4	0.25308501	0.25343486	0.25340786	0.25340693
k=5	0.25308501	0.25339921	0.25340786	0.25340693
k=6	0.25150103	0.25339921	0.25340740	0.25340692
k=7	0.25	0.25	0.25	0.25
k=8	0.25	0.25	0.25	0.25
k=9	0.25	0.25	0.25	0.25
k=10	0.18419043	0.18435227	0.18435150	0.18435164

Table 2. Leading eigenvalues $\lambda_{k,N}$ of $\mathcal{L}_{\tilde{q}_N}$, $d=2$.

A further increase of N did not change these values (except for the last 3 digits of $\lambda_{6/7}$ in Table 1). The results demonstrate the stability and superlinear convergence of the approximation method, and show a typical difference between univariate and multivariate calculations. In the latter case, multiple eigenvalues are a major obstacle. Due to our construction which preserves the Strang-Fix conditions the trivial eigenvalues 2^{-l} of transfer operators are reproduced exactly (with the correct multiplicity). Table 3 shows the calculations of the spectral radii $\tilde{\rho}_{N,L} := \rho(\mathcal{L}_{\tilde{q}_N}, V_{\tilde{q}_N,L})$ for $L \le L_p/2 = 2$. According to the rules for choosing an appropriate $L_{\min}$ stated at the end of subsection 2.5, we see that $L_{\min} = 1$ is appropriate. Thus, the results are identical for $L \ge 1$.

All calculations presented so far use simultaneous iteration (in a crude implementation of the accelerated version by Stewart [33]), and are based on the fact that the value $\tilde{\rho}_{N,L}$ is the largest eigenvalue of $\mathcal{L}_{\tilde{q}_N}$ restricted to the finite-dimensional Krylov space $V_{\tilde{q}_N,z_L}$ generated by applying the iterates of this operator to the low-order polynomial z_L. This subspace is, for $L \le L_p/2$, contained in the following space of trigonometric polynomials (compare [22, Section 4] or [6]):

$$V_{\tilde{q}_N,z_L} \subset V_{N,L} := \{ \sum_{|\alpha|_\infty \le N+L} c_\alpha e^{-i\alpha\theta} \ : \ \sum_\alpha c_\alpha \alpha^\beta = 0 \ \text{ if } |\beta|_1 \le 2L-1 \} \, .$$

N	$L=0$	$L=1$	$L=2$
5	1.00000000	0.25308501	0.25308501
10	1.00000000	0.25343486	0.25343486
15	1.00000000	0.25340786	0.25340786
20	1.00000000	0.25340693	0.25340693
25	1.00000000	0.25340693	0.25340693
30	1.00000000	0.25340693	0.25340693
$\tilde{\rho}_L$	1.00000000	0.25340693	0.25340693

Table 3. Computation of $\tilde{\rho}_{N,L}$ by the approximation method, $d=2$.

Thus, the finite-dimensional eigenvalue problem is of size $\leq CN^d$. Though direct methods could have been used for small N, we have used iterative methods here. These are unavoidable for large N and $d \geq 2$, since the discrete representation of $\mathcal{L}_{\tilde{q}_N}$ on the above subspace $V_{N,L}$ leads to dense non-symmetric matrices. As can be easily derived from the definition of the transfer operator in terms of Fourier coefficients, the corresponding matrix-vector multiplication can be carried out by FFT (or, equivalently, by fast higher-dimensional convolution) and requires $\leq CN^d \log N$ operations. Although only the maximal eigenvalue has to be found, simultaneous iteration rather than the power method needs to be applied for $d \geq 2$, at least, due to multiple eigenvalues resp. clusters of eigenvalues for the infinite-dimensional operator $\mathcal{L}_{\tilde{q}}$ under consideration. As can be seen from Table 2, the triple eigenvalue 0.25340693... which characterizes the smoothness of $\tilde{\phi}$ splits into a cluster of eigenvalues for $\mathcal{L}_{\tilde{q}_N}$ in the approximation process, and leads to very slow convergence of the power iteration. The implementation of the iteration process also takes the summing rules valid for (c_α) according to the definition of $V_{N,L}$, $L>0$, into account. Without this, rounding immediately leads to computing the maximal eigenvalue of the operator with respect to the subspace $V_{N,0}$ which in this case would be 1. As an internal stopping criteria for the iterative eigenvalue solver we required $|\lambda_{k,N}^{(j+1)} - \lambda_{k,N}^{(j)}| < 10^{-10}$ for the largest k_0 eigenvalues in the corresponding computational subspace (the dimension of the latter was typically $\approx 2k_0$). Under these restrictions, the number of iterations does not grow with N, and the overall number of arithmetical operations for computing $\tilde{\rho}_{N,L}$ can be bounded by $\leq CN^d \log N$. This favorably compares with the exponential convergence

$$|\rho_L - \tilde{\rho}_{N,L}| \leq CN^l r_0^N \;, \quad N \to \infty \;,\; 0 < r_0 < 1 \;, \tag{101}$$

which results from the proof of Theorem 17, compare also the statement and proof of Theorem 13. Note that in the above example the choice $L=2>L_{\min}=1$ leads to a reduced number of iterations for all dimensions $d \geq 1$ since the clustering near the eigenvalue of interest is reduced. Since for $d=3$ and moderate $N \leq 30$ the matrix dimension of the eigenvalue problem easily reaches 10^5 and more, other performance improvements (e.g., nested iteration) have been tried. Details will be discussed elsewhere.

The constants C, r_0, and the above complexity estimates themselves depend on d (and the example under consideration) which still leads to significant computational time even for simple examples such as considered in this subsection, especially for $d = 3$. There is no relief in this respect if Fredholm determinant approximations as recommended in [7, 18] are used.

The idea of the Fredholm determinant method is to use the properties of $\mathcal{L}_r$ stated in Theorem 5 in various ways to access those of its spectral properties that we are interested in, compare subsection 2.2. E.g., the trace class property ensures that $\mathcal{L}_r$ is compact and has a well-defined trace

$$\mathrm{Tr}(\mathcal{L}_r) = \sum_j \lambda_j \ , \tag{102}$$

where the summation is with respect to all eigenvalues of $\mathcal{L}_r$ taking into account their algebraic multiplicity (it does not depend on the ordering of the eigenvalues since the sum converges absolutely by another definition of trace class operators). One can obtain the trace value explicitly in terms of $r(\theta)$ by using the formula

$$\mathrm{Tr}(A) = \sum_k (Ae_k, e_k)_{E^t}$$

valid for any trace class operator $A : E^t \to E^t$ and any complete orthonormal system in E^t ([19] or [28]). Indeed, taking the system $\{e_{\alpha,t}\}$ introduced in the proof of lemma 7 as the orthonormal system of our choice, we have

$$\mathrm{Tr}(\mathcal{L}_r) = \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_r e_{\alpha,t}, e_{\alpha,t}) = 2^d \sum_{\alpha\in\mathbb{Z}^d} r_\alpha = 2^d r(0) \ .$$

More generally, in [7, 18], the formula

$$\mathrm{Tr}(\mathcal{L}_r^k) = \frac{2^{kd}}{(2^k-1)^d} \sum_{m_1=0}^{2^k-2} \cdots \sum_{m_d=0}^{2^k-2} \left(\prod_{l=0}^{k-1} r(2^l\theta) \right)\Bigg|_{\theta=\frac{2\pi m}{2^k-1}} \tag{103}$$

has been derived. Thus, for these operators the traces are non-negative real numbers, and do not depend on t ($r_0^2 < t < 1$). The same is true for the spectral radius $\rho(\mathcal{L}_r) = \rho(\mathcal{L}_r, E^t)$ which can be approximately computed from relations such as

$$\rho(\mathcal{L}_r, E^t) := \max_j |\lambda_j| = \lim_{k\to\infty} \frac{\mathrm{Tr}(\mathcal{L}_r^{k+1})}{\mathrm{Tr}(\mathcal{L}_r^k)} = \lim_{k\to\infty} (\mathrm{Tr}(\mathcal{L}_r^k))^{1/k} \ . \tag{104}$$

For a trace class operator A, the Fredholm determinant is given by the entire function

$$D_A(z) := \prod_j (1-\lambda_j z) = \sum_{k=0}^{\infty} \gamma_k z^k \ ,$$

of a complex variable $z \in \mathbb{C}$, see [19]. The product is with respect to all eigenvalues of A (counted with their algebraic multiplicity). The coefficients γ_k in the power series representation of $D_A(z)$ can be computed recursively from the traces (103): we have $\gamma_0 = 1$ and

$$\gamma_{k+1} = -\frac{1}{k+1} \sum_{j=0}^{k} \gamma_j \mathrm{Tr}(A^{k+1-j}) \ , \quad k \geq 0 \ .$$

When applying this to our transfer operator, we see that $D_{\mathcal{L}_r}(z)$ does not depend on t. Since the reciprocals of the non-zero eigenvalues λ_j of $\mathcal{L}_r$ are the zeros of $D_{\mathcal{L}_r}(z)$, the whole spectrum of $\mathcal{L}_r$ does not depend on t either. In principle, using the explicit formula (103), polynomial approximations to $D_{\mathcal{L}_r}(z)$ are accessible and lead to approximations for the eigenvalues. For details, we refer to [7].

Even though the trace formula (103) is extremely simple and represents the only computationally expensive part of this method, it leads to an exponential increase of cpu-time: asymptotically $\asymp k2^{kd}$ operations are needed to compute the k-th degree polynomial approximation to $D_{\mathcal{L}_{\tilde{q}}}(z)$. Special care is required in the summation processes to avoid excessive round-off error. Tables 4 and 5 contain computations of traces $t_k = \mathrm{Tr}(\mathcal{L}_{\tilde{q}}^k)$ and approximations to some leading eigenvalues of $\mathcal{L}_{\tilde{q}}$ obtained as absolute values of the reciprocals of the zeros of the k-th degree Taylor polynomial of $D_{\mathcal{L}_{\tilde{q}}}(z)$ (see [7] and subsection 2.2 for the details), respectively. Trace calculations for $k > 15$ ($d = 2$) respectively $k > 10$ ($d = 3$) are very time-consuming which limits the application of the Fredholm determinant method in higher dimensions. Except for $d = 1$, the method fails to provide reasonable insight into the distribution of the leading eigenvalues. This is obviously due to ill-conditioning of multiple zeros of $D_{\mathcal{L}_{\tilde{q}}}(z)$ which cannot be resolved well by Taylor approximations. As a matter of fact, most of the (real) multiple roots of $D_{\mathcal{L}_{\tilde{q}}}$ split into complex roots for the respective Taylor polynomials. The calculations were based on the Taylor polynomials of maximal degree ($k = 15$ for $d = 1, 2$ and $k = 10$ for $d = 3$) obtainable from the trace calculations in Table 4, only the leading values are shown. Compare the first two columns of Table 5 with the (exact) results documented in Tables 1 and 2.

The a priori knowledge about the eigenvalues 2^{-l} could be used to factorize $D_{\mathcal{L}_{\tilde{q}}}(z)$ and to compute the k-th degree Taylor polynomial of the factored Fredholm determinant, without computing new traces. The advantage is that $\tilde{\rho}$ corresponds now to the reciprocal of the smallest zero of the new sequence of approximating polynomials. However, only for $d = 1, 2$ does this lead to a certain improvement. Rounding errors (cancellation of true digits when manipulating with the trace values t_k respectively the Taylor coefficients γ_k) seem to become an issue. The major problem of the Fredholm determinant method, to provide very poor approximations in the case of multiple eigenvalues (or eigenvalue clusters), is well documented in this real-life application to the case of $\mathbf{\Psi}_{3HB}$. It would be interesting to know whether more advanced methods of polynomial and rational approximation to entire functions (e.g., Pade approximations) could help to overcome this drawback.

k	$d=1$	$d=2$	$d=3$
1	2.000000000000000	4.000000000000000	8.000000000000000
2	1.401600000000000	2.074538275718451	3.227639779298985
3	1.158222751304226	1.364014099155181	1.637823281166944
4	1.070816660823510	1.152170413697154	1.246830440461046
5	1.033299119714271	1.068990013270801	1.107464500450192
6	1.016137948442982	1.032844471520890	1.050179786151266
7	1.007941099634121	1.016020091322666	1.024246705882474
8	1.003938573182569	1.007911062988216	1.011919105557116
9	1.001961255985642	1.003930921618738	1.005909279784349
10	1.000978609014974	1.001959314015891	1.002942164896746
11	1.000488796481029	1.000978117073214	
12	1.000244270360982	1.000488672064724	
13	1.000122102983553	1.000244238930547	
14	1.000061043384321	1.000122095050132	
15	1.000030519650454	1.000061041383024	

Table 4. Traces t_k for $\mathcal{L}_{\tilde{q}}^k$.

k	$d=1$	$d=2$	$d=3$
1	1.00000000000000	1.00000001942418	1.83223071579613
2	0.50000000000129	0.51466946129377	1.83223071579613
3	0.25340674129876	0.51466946129377	1.71200516190882
4	0.25000021925075	0.43472222056263	1.71200516190882
5	0.12499719869243	0.43472222056263	1.49136564252763
6	0.07029908309411	0.36825589196147	1.49136564252763
7	0.06468930188479	0.36825589196147	
8	0.03837940595449	0.29353363955150	
9	0.03427154400901	0.29353363955150	
10	0.03427154400901	0.18679705669337	

Table 5. Approximations to the leading eigenvalues λ_k of $\mathcal{L}_{\tilde{q}}$.

The possibility of factorizing the Fredholm determinant is closely related to the zero-order of $\tilde{q}(\theta)$ at the points $\theta = \pi\lambda$, $\lambda \in \Lambda'$. For $d = 1$, the zero-order $L_{\tilde{q}} = 2L_{\tilde{m}}$ is equivalent to a factorization

$$\tilde{q}(\theta) = \cos^{L_{\tilde{q}}}(\theta/2)\tilde{r}(\theta) \ .$$

and can be used to reduce the study of $\mathcal{L}_{\tilde{q}}$ to the transfer operator $\mathcal{L}_{\tilde{r}}$. For $d \geq 2$, such factorization tricks (although they have been suggested in, e.g., [7, 9]) are usually not possible. One could just divide the zero out. But this does not lead, in general, to a trigonometric polynomial $\tilde{r}$ if $\tilde{q}$ was a trigonometric polynomial itself. In our context, where we start with a non-trigonometric function $\tilde{q}$, this is no longer an argument. Moreover, in the example of $\mathbf{\Psi}_{3HB}$, the associated $\tilde{q}$ respectively p contain factors of the type $\cos^2(\nu\theta/2)$ with integer vectors $\nu \in \mathbb{Z}^d$. We have

Lemma 19 *Let $\{\nu_l : l = 1, \ldots, d\} \subset \mathbb{Z}^d$ be a linearly independent set of integer vectors. Assume that (47-50) hold for $\tilde{q}$, and that we can factorize p as*

$$p(\theta) = 4\cos^2 \frac{\nu_l \theta}{2} p_l(\theta) \ , \quad l = 1, \ldots, d \ ,$$

for some polynomials $p_l(\theta)$. Set $\tilde{r}_l = p_l(\theta) q(\theta)$. Then

$$\rho_{L,\tilde{q}} = \max_{l=1,\ldots,d} \rho_{L-1,\tilde{r}_l} \ , \tag{105}$$

where $\rho_{L,\tilde{q}} = \tilde{\rho}_L$ is given by (74). The quantities $\rho_{L-1,\tilde{r}_l}$ are analogously defined, with $\tilde{q}$ replaced by $\tilde{r}_l$, $l = 1, \ldots, d$, in (74).

For a proof of the upper bound, compare [24]. We will not give the proof of this simple reduction lemma which is essentially based on the corresponding one-dimensional result and uses the identity

$$\mathcal{L}_{\tilde{q}} d(\theta) = \sin^2(\nu_l \theta/2) \mathcal{L}_{\tilde{r}_l} c(\theta)$$

for $d(\theta) = \sin^2(\nu_l \theta/2) c(\theta)$. The proof is straightforward if one has

$$\sum_{l=1}^{d} \sin^2 \frac{\nu_l \theta}{2} \asymp \sum_{l=1}^{d} \sin^2 \frac{\theta_l}{2} = z_1(\theta) \qquad \forall \, \theta \in \mathbb{T}^d \ .$$

All examples considered in this paper satisfy this property (for general choices of linearly independent integer vectors ν_l it holds true only in a neighborhood of the origin).

Applied to our current example, (105) allows us to deduce the formula

$$\tilde{\rho} = \rho_{2,\tilde{q}} = \rho_{0,\tilde{r}} = \rho(\mathcal{L}_{\tilde{r}}, E^t) \ , \tag{106}$$

where

$$\tilde{r}(\theta) = \frac{\prod_{l=1}^{d-1} \cos^2(\theta_l/2)}{4(1 + m(2\theta))^2} \ .$$

To see this, observe that p is the product of $d+1$ factors of the above type, generated by the coordinate vectors e_l, $l = 1, \ldots, d$ and $e_{d+1} = e_1 + \ldots + e_d$. Since any subset of d such vectors is linearly independent, we can apply Lemma 19 twice and express $\tilde{\rho} = \rho_{2,\tilde{q}}$ by the maximum of the quantities $\rho_{0,\tilde{r}_{m,n}}$, $m, n = 1, \ldots, d+1$ ($m < n$), corresponding to

$$\tilde{r}_{m,n}(\theta) = \frac{\prod_{l \neq m,n} \cos^2(\theta_l/2)}{16(1 + m(2\theta))^2} \ .$$

That all these numbers equal $\rho_{0,\tilde{r}} := \rho_{0,\tilde{r}_{d,d+1}}$ follows easily by permutation if $n = d+1$ and by a simple change of variables if $n \leq d$.

Table 6 shows the computations of $\tilde{\rho} = \rho_{0,\tilde{r}}$ based on approximation of $\tilde{r}$ by polynomials $\tilde{r}_N$. It contains the related values $\tilde{\rho}_{N,0}$ for several N obtained by subspace iteration as described before. That the results are identical for $d = 1, 2, 3$ seems to be due to the special structure of $\tilde{r}$ which is preserved by the approximation procedure. More precisely, for a given dimension d, the invariant subspace of the transfer operator $\mathcal{L}_{\tilde{r}}$ under consideration is naturally embedded into the corresponding one for larger $d' > d$, and one can prove that the sets of eigenvalues are monotonically increasing with d. By chance and for $d \leq 3$, the leading eigenvalue is already present for $d = 1$.

N	$d = 1$	$d = 2$	$d = 3$
5	0.25308501	0.25308501	0.25308501
7	0.25393170	0.26016283	0.26016283
9	0.25337414	0.25337414	0.25337414
11	0.25339145	0.25339145	0.25339145
13	0.25340802	0.25340802	0.25340802
15	0.25340786	0.25340786	0.25340786
17	0.25340689	0.25340689	0.25340689
20	0.25340693	0.25340693	0.25340693
25	0.25340693	0.25340693	0.25340693
30	0.25340693	0.25340693	0.25340693

Table 6. Approximations to $\tilde{\rho}$ by the approximation method using the factorized symbol

k	$d = 1$	$d = 2$	$d = 3$
1	0.125	0.25	0.5
2	0.24369395	0.35059729	0.48053642
3	0.25372852	0.29922145	0.36991617
4	0.25340384	0.24148643	0.34827888
5	0.25340694	0.23733585	0.33191016
6	0.25340693	0.24876978	0.32209288
7	0.25340693	0.25328726	0.27334273
8	0.25340693	0.25345458	0.27744939
9	0.25340693	0.25341147	0.24618380
10	0.25340693	0.25340672	0.26506758
11	0.25340693	0.25340689	
12	0.25340693	0.25340693	
13	0.25340693	0.25340693	

Table 7. Approximations to $\tilde{\rho}$ by the Fredholm determinant method using the factorized symbol

Alternatively, we have computed $\tilde{\rho}$ by the Fredholm determinant method based on trace calculations for $\mathcal{L}_{\tilde{r}}^k$, compare (106). Table 7 contains the reciprocal of the smallest zero of the approximating Taylor polynomials for different k. The results should be compared with the values obtained from the unfactorized symbol, see Tables 1-5. We see that for $d = 3$ results of high precision $\leq 10^{-8}$ can be obtained only with the approximation method showing its superiority with respect to stability and computational expenses in the asymptotical range.

3.2 More linear finite element systems

As before, let ϕ be the linear box spline given by the refinement equation (94) with symbol (95). We consider other choices of the ψ^λ.

First, let us discuss the influence of *moment conditions*. We say that a compactly supported, integrable function ψ satisfies moment conditions of order M if

$$\int_{\mathbb{R}^d} \psi(x) x^\beta \, dx = 0 \qquad \forall\, \beta \in \mathbb{Z}_+^d \, : \, |\beta|_1 \leq M - 1 \, . \tag{107}$$

In short, we will also say that a hierarchical system $\mathbf{\Psi}$ satisfies moment conditions of order M if all functions ψ^λ, $\lambda \in \Lambda'$, do so. Clearly, the notion can be generalized to domains Ω and more general ψ. E.g., the hierarchical basis $\mathbf{\Psi}_{HB}$ defined by $\psi^\lambda(x) = \phi(2x - \lambda)$, $\lambda \in \Lambda'$, satisfies no moment conditions ($M = 0$) while for $\mathbf{\Psi}_{3HB}$ we have $M = 2$. A system with $M = 1$ which is intermediate to $\mathbf{\Psi}_{HB}$ and $\mathbf{\Psi}_{3HB}$ is defined by

$$\psi^\lambda(x) = \phi(2x - \lambda) - \phi(2x) \, , \quad \lambda \in \Lambda' \, . \tag{108}$$

Since 2 non-zero coefficients determine the ψ-masks, we will denote it by $\mathbf{\Psi}_{2HB}$. From our calculations below it will be clear that $\mathbf{\Psi}_{2HB}$ satisfies the conditions of subsection 2.1. We sketch the main steps in calculating the dual symbol $\tilde{m}(\theta)$. In analogy to the approach used in subsection 3.1, we have now

$$c_{2\alpha+\lambda} = \begin{cases} d_\alpha - \sum_{\lambda' \in \Lambda'} d_\alpha^{\lambda'} & , \quad \lambda = 0 \, , \\ \frac{1}{2}(d_\alpha + d_{\alpha+\lambda}) + d_\alpha^\lambda & , \quad \lambda \in \Lambda' \, , \end{cases}$$

and, thus,

$$\sum_{\lambda \in \Lambda} c_{2\alpha+\lambda} = \frac{1}{2} \sum_{\lambda \in \Lambda} (d_\alpha + d_{\alpha+\lambda}) \, .$$

After turning to symbols and subsymbols, we have

$$\sum_{\lambda \in \Lambda} e^{i\lambda\theta} c_\lambda(\theta) = \frac{1}{2} d(2\theta)(2^d + \sum_{\lambda \in \Lambda} e^{2i\lambda\theta}) \, ,$$

from which, according to the definition of $\tilde{m}$ given by (41), (42), we derive

$$\tilde{m}(\theta) = \frac{2 \sum_{\lambda \in \Lambda} e^{-i\lambda\theta}}{2^d + \sum_{\lambda \in \Lambda} e^{-2i\lambda\theta}} \, .$$

Using the identities in (95) and

$$\sum_{\lambda \in \Lambda} e^{i\lambda\theta} = \prod_{l=1}^{d} (1 + e^{i\theta_l}) \, ,$$

we finally get

$$\tilde{q}(\theta) = \frac{4\prod_{l=1}^{d}\cos^2\frac{\theta_l}{2}}{1+2\prod_{l=1}^{d+1}\cos\theta_l + \prod_{l=1}^{d}\cos^2\theta_l} \ . \tag{109}$$

We detect the trigonometric polynomial

$$p(\theta) = \prod_{l=1}^{d}\cos^2\frac{\theta_l}{2}$$

(which is known from the MRA given by the characteristic function of the unit cube). We have $L_{\tilde{q}} = L_p = 2L_m = 2$, and the reduction argument of Lemma 19 is applicable once. We formulate the result as

Theorem 20 *For $d \le 3$, the system $\mathbf{\Psi}_{2HB}$ is a Riesz basis in $H^s(\mathbb{R}^d)$ if*

$$-0.044117... < s < 3/2 \ .$$

The exact lower bound is given by $-s_{\tilde{\phi}} = \frac{1}{2}\log_2(\rho_{1,\tilde{q}})$ where $\rho_{1,\tilde{q}} = 0.94067266...$ can be computed, e.g., from the relationship

$$\rho_{1,\tilde{q}} = \rho_{0,\tilde{r}} \ , \quad \tilde{r}(\theta) = \frac{\prod_{l=1}^{d-1}\cos^2\frac{\theta_l}{2}}{1+2\prod_{l=1}^{d+1}\cos\theta_l + \prod_{l=1}^{d}\cos^2\theta_l} \ .$$

That we can again restrict ourselves to one function $\tilde{r}(\theta)$ follows by symmetry arguments. The computations for obtaining the numerical value of $\rho_{\tilde{r},0}$ which use the approximation method are documented in Table 8. What concerns the identical numbers for $d \le 3$, see the comments to Table 6. For $d = 2$, some computations with the Fredholm determinant method can be found in [24]. Note that $\mathbf{\Psi}_{2HB}$ was mainly considered because of its simplicity, its adaption to polyhedral domains is not that easy, see the preliminary numerical experiments in [26]. The case $d = 1$ has already been considered before [7, 18, Section 6]. Even though $\mathbf{\Psi}_{2HB}$ turns out to be a Riesz basis in $L_2(\mathbb{R}^d)$, $d \le 3$, its L_2-condition is expected to be quite poor.

N	$d=1$	$d=2$	$d=3$
5	0.94080413	0.94080413	0.94080413
10	0.94050316	0.94050316	0.94050316
15	0.94074410	0.94074410	0.94074410
20	0.94067265	0.94067265	0.94067265
25	0.94067268	0.94067268	0.94067268
30	0.94067267	0.94067267	0.94067267

Table 8. Approximations to $\tilde{\rho}$ for Φ_{2HB}.

Note that there is a whole family of *edge-oriented* $(M+1)$*-point hierarchical systems* $\mathbf{\Psi}_{(M+1)HB}$ which satisfy moment conditions of order $M = 0, 1, 2, \ldots$

and generalize the systems $\mathbf{\Psi}_{2HB}$ and $\mathbf{\Psi}_{3HB}$ considered so far. For $d=1$, they will be described in the next subsection. We did not yet investigate their higher-dimensional counterparts in detail. The system $\mathbf{\Psi}_{1HB}$ $(M=0)$ has been considered by many authors (it is closely related to the *hierarchical basis preconditioner* useful in two-dimensional finite element applications), As is well-known (see, e.g., [27]), this system is a Riesz basis in $H^s(\mathbb{R}^d)$ only in the cases $d=1$, $1/2<s<3/2$, and $d=2$, $1<s<3/2$.

The next case of interest is $M=3$, where

$$\psi^\lambda(x)=\frac{1}{3}\phi(2x+\lambda)-\phi(2x)+\phi(2x-\lambda)-\frac{1}{3}\phi(2x-2\lambda)\ ,$$

from which one derives

$$\tilde{m}(\theta)=\frac{\sum_{\lambda\in\Lambda}d(\lambda\theta)}{\sum_{\lambda\in\Lambda}d(\lambda\theta)\cos\lambda\theta}\ ,\quad d(t)=\frac{1+3e^{-i2t}}{e^{-i3t}+3e^{-it}}\ .$$

Computations for $d=2$ show that the smoothness of the corresponding $\tilde{\phi}$ decreases to $s_{\tilde{\phi}}=0.340064...$ resulting in the interval $-0.340063...<s<3/2$ for which Φ_{4HB} is a Riesz basis in $H^s(\mathbb{R}^2)$. Unfortunately, $s=-1/2$ is not covered anymore (compare also subsection 3.3 for an analogous system based on piecewise constant functions).

There is another goal often achieved by biorthogonal wavelet constructions: to guarantee *compact support properties for the dual MRA*, i.e., for $\tilde{\phi},\tilde{\psi}^\lambda$. How to do this in a systematic way, is discussed, e.g., in [5, 35]. The requirement is typical for many applications to image processing where one needs fast decomposition and reconstruction algorithms. Even though this is not a central issue in our study (information on the dual scaling function respectively refinement equation does not enter the resulting preconditioners as briefly mentioned in section 1), for comparison we wish to state some results for a specific one-parameter family of such systems in $\mathbb{R}^2$, partial cases of which have appeared in the literature [10, 5, 36, 37]. As is easy to see, any system with

$$\psi^\lambda(x)=\phi(2x-\lambda)-\sum_{\beta\in\mathbb{Z}^d}b_\beta^\lambda\phi(x-\beta)\ ,\quad \lambda\in\Lambda'\ , \tag{110}$$

produces a trigonometric $\tilde{m}(\theta)$ if all sequences (b_β^λ) are finite. Indeed, let us again use the notation for the coefficients associated with a decomposition $v_1=v_0+w_1$, as introduced in subsection 2.1. Then, by linearity of functions from V_0 along edges, we have for arbitrary $\lambda\in\Lambda'$ and $\alpha\in\mathbb{Z}^d$

$$\begin{aligned}c_{2\alpha+\lambda}-(c_{2\alpha}+c_{2\alpha+2\lambda})/2 &= v_1(\alpha+\lambda/2)-(v_1(\alpha)+v_1(\alpha+\lambda))/2\\ &= w_1(\alpha+\lambda/2)-(w_1(\alpha)+w_1(\alpha+\lambda))/2)=d_\alpha^\lambda\ .\end{aligned}$$

Thus,

$$d_\alpha=v_0(\alpha)=v_1(\alpha)-w_1(\alpha)=c_{2\alpha}+\sum_{\lambda\in\Lambda'}\sum_{\gamma\in\mathbb{Z}^d}b_{\alpha-\gamma}^\lambda d_\gamma^\lambda$$

$$= c_{2\alpha} + \sum_{\lambda\in\Lambda'}\sum_{\gamma\in\mathbb{Z}^d} b^\lambda_{\alpha-\gamma}(c_{2\gamma+\lambda} - \frac{1}{2}(c_{2\gamma} + c_{2\gamma+2\lambda})) \ .$$

This gives an explicit expression for $d(2\theta)$ of the form (41), with polynomial $\tilde{m}^\lambda$. After some simple calculations, the formula

$$\tilde{m}(\theta)^* = 1 + \sum_{\lambda\in\Lambda'}(e^{i\lambda\theta} - \frac{1}{2}(1 + e^{i2\lambda\theta}))b^\lambda(2\theta) \qquad b^\lambda(\theta) = \sum_{\beta\in\mathbb{Z}^d} b^\lambda_\beta e^{-i\beta\theta} \ , \qquad (111)$$

can be established from (42) which shows the polynomiality of the dual symbol.

Now we turn to the case $d = 2$ and choose the b^λ_β in (110) according to the following rules: Let $a, b \in \mathbb{R}$ such that $a + b = 1/8$, and set

$$\begin{array}{lll} b^\lambda_{(0,0)} = b^\lambda_{(1,0)} = a \ , & b^\lambda_{(0,-1)} = b^\lambda_{(1,1)} = b \ , & \lambda = (1,0) \ , \\ b^\lambda_{(0,0)} = b^\lambda_{(0,1)} = a \ , & b^\lambda_{(-1,0)} = b^\lambda_{(1,1)} = b \ , & \lambda = (0,1) \ , \\ b^\lambda_{(0,0)} = b^\lambda_{(1,1)} = a \ , & b^\lambda_{(1,0)} = b^\lambda_{(0,1)} = b \ , & \lambda = (1,1) \ . \end{array}$$

All other b^λ_β are set to 0. We denote the resulting hierarchical system by $\mathbf{\Psi}_{5HB,a}$ indicating that it depends on a parameter a ($b = 1/8 - a$), and that the complexity of intergrid transfer operations is comparable with a 5-point hierarchical basis (even though the ψ-mask when expressed in the standard form (17) would contain significantly more non-zero coefficients, the basis transformation from $\mathbf{\Psi}_J$ to Φ_J can be implemented with less operations). With the above choice, the functions ψ^λ satisfy moment conditions of order $M = 2$, as in the case of $\mathbf{\Psi}_{3HB}$. The dual symbol can be computed from (111) as

$$\begin{aligned} \tilde{m}_a(\theta) \ &= \ 1 + 2(1 - \cos(\theta_1 + \theta_2))(a\cos(\theta_1 + \theta_2) + b\cos(\theta_1 - \theta_2)) \\ &\quad +2(1 - \cos\theta_1)(a\cos\theta_1 + b\cos(\theta_1 + 2\theta_2)) \\ &\quad +2(1 - \cos\theta_2)(a\cos\theta_2 + b\cos(2\theta_1 + \theta_2)) \ . \end{aligned}$$

This symbol possesses no obvious factorizations (of the form observed in the previous examples) except for the parameter choice $a = 3/16$:

$$\tilde{m}_{3/16}(\theta) = m(\theta)(3 - 2m(\theta)) \ .$$

Details of computations are left to the reader.

The special case $a = 3/16$ was proposed in [10], in a slightly different fashion. As a by-product of our calculations below, we will see that this is the choice which maximizes the s-interval for which $\mathbf{\Psi}_{5HB,a}$ is a Riesz basis in $H^s(\mathbb{R}^d)$. Some other prominent special cases are $a = 1/8$ (since $b = 0$, this is the counterpart of the system $\mathbf{\Psi}_{3HB}$), $a = 1/6$ which is implicit in [5], and $a = 5/48$ which results from analyzing the construction in [36, 37]. Actually, in these papers constructions for bounded polyhedral domains are considered. The general idea of such constructions is to choose an appropriate linear operator $\tilde{P}_{j-1} : V_j \to V_{j-1}$ which is local in the sense that $\tilde{P}_{j-1}\phi_{j,P}$ is a linear combination of a few $\phi_{j-1,Q}$ with support near $P \in \mathcal{V}_j$ and to set

$$\psi_{j,P} = \phi_{j,P} - \tilde{P}_{j-1}\phi_{j,P} \ , \quad P \in \mathcal{V}_j^* \ .$$

In [5] the use of a standard quasi-interpolant projector for the case of linear finite elements is advised while [36, 37] use specific approximations to the L_2-orthoprojector P_{j-1}. To obtain the above value of $a = 5/48$ for the case of a uniform 3-directional partition of $\mathbb{R}^2$ one has to set $m = 1$ and $\beta = 1$ in [37]. Other values lead to more non-zero coefficients in (110) or to the violation of the moment conditions.

a	$\tilde{\rho}_a$	$\tilde{s}_a$	a	$\tilde{\rho}_a$	$\tilde{s}_a$
0.00	5.53394750	-1.234154	0.17	0.59176261	0.378455
0.05	3.06445061	-0.807814	0.18	0.55357852	0.426049
0.10	1.50400061	-0.294403	**3/16**	0.54279115	0.440765
5/48	1.41224257	-0.248994	0.19	0.54428829	0.438159
0.11	1.29286617	-0.185286	0.20	0.62042168	0.344339
0.12	1.11190616	-0.076518	0.21	0.78164559	0.177707
1/8	1.03213754	-0.022818	0.22	0.98550714	0.010531
0.13	0.95917364	0.030068	0.23	1.22640881	-0.147220
0.14	0.83264571	0.132113	0.25	1.81693079	-0.430752
0.15	0.73035012	0.226670	0.27	2.55199935	-0.675814
0.16	0.65050835	0.310180	0.29	3.43177692	-0.889478
1/6	0.60905307	0.357680	0.31	4.45655944	-1.077965

Table 9. Values for $\tilde{\rho}_a$ and $\tilde{s}_a$.

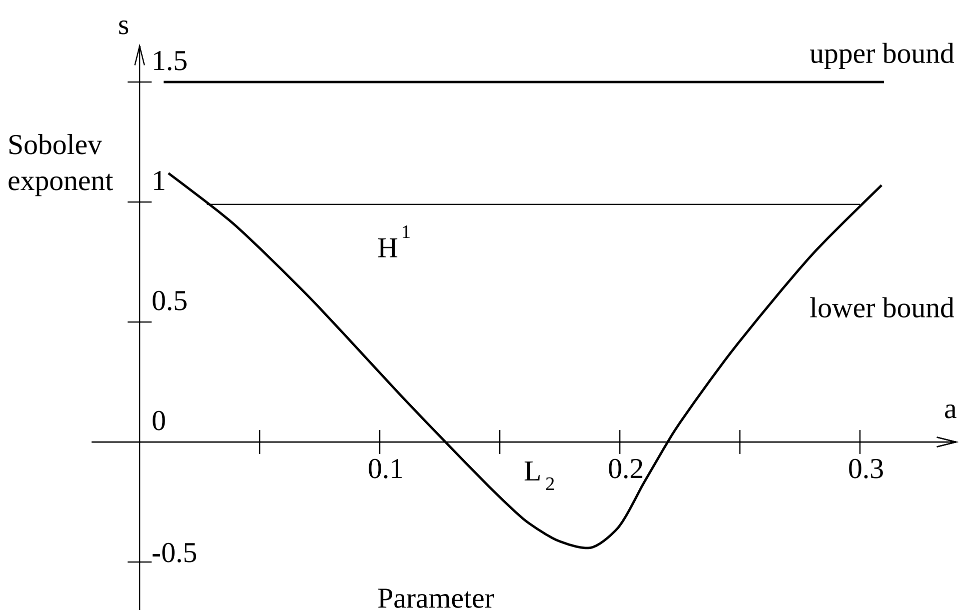

Figure 1.

Theorem 21 *For $d = 2$, the system $\mathbf{\Psi}_{5HB,a}$ is a Riesz basis in $H^s(\mathbb{R}^2)$ if*

$$-\tilde{s}_a < s < 3/2 \ ,$$

where values of $\tilde{s}_a$ for various a are given in Table 9 and visualized in Figure 1. If $\tilde{s}_a > 0$ then $\tilde{\phi}_a$ is well-defined in $L_2(\mathbb{R}^2)$ and $\tilde{s}_{\tilde{\phi}_a} = \tilde{s}_a$. The calculations show that, within an error tolerance of $\leq 10^{-6}$, that $\mathbf{\Psi}_{5HB,a}$ is a Riesz basis in $L_2(\mathbb{R}^2)$ if and only if

$$0.1271462 \leq a \leq 0.2206475 \ ,$$

and a Riesz basis in $H^1(\mathbb{R}^2)$ if and only if

$$0.0287592 \leq a \leq 0.3014364 \ .$$

The smoothest dual scaling function is obtained for $a = 3/16$: $\tilde{s}_a \leq \tilde{s}_{3/16} =$ 0.440765... .

The calculations for Table 9 are based (due to the finite mask corresponding to $\tilde{m}_a$) on Theorem 4.4 of [22], and were performed by standard eigenvalue algorithms (the size of the matrices involved does not exceed 61). The approximation methods designed for the case of non-trigonometric $\tilde{q}$ gave the same results. The Fredholm determinant method had problems with this example, compare Table 9 of [24] due to eigenvalue clustering. It turned out that for $a = 3/16$, the eigenvalue of interest has multiplicity 3 (as can be expected from the above factorization in this case) which splits into a pair of close eigenvalues (of multiplicity 1 and 2, respectively) for a close to this value.

3.3 Other box spline examples

We conclude this paper by presenting some more examples, using other low order box splines as the basic ϕ in dimension $d = 1, 2$. They are partly motivated by applications to boundary integral equations. For applications involving the single layer potential operator, the Riesz basis property for $s = -1/2$ and $d = 2$ is necessary, and systems with small ϕ- and ψ-masks and enough vanishing moments for the ψ^λ are desired.

We start with $d = 1$ and a family $\mathbf{\Psi}_{r,M}$ of hierarchical systems which is characterized by choosing the one-dimensional B-spline of order r (degree $r - 1$) as the ϕ. The ψ-masks will be analogous to those used for constructing the ψ^λ in the $(M + 1)$-point hierarchical basis $\mathbf{\Psi}_{(M+1)HB}$ and yield moment conditions of order M for $\psi := \psi^1$. We will make these choices precise by fixing the symbols corresponding to the ϕ- and ψ-masks (the resulting functions differ from those appearing for $d = 1$ in the special cases considered in subsections 3.1 and 3.2 only by integer shifts and scaling factors):

$$m_{r,M}(\theta) := m_{\phi_{r,M}}(\theta) = \left(\frac{1 + e^{-i\theta}}{2}\right)^r \ , \quad m_{\psi_{r,M}}(\theta) = e^{-ia\theta}\left(\frac{1 - e^{-i\theta}}{2}\right)^M \ .$$

The parameter a is set to 1 if $M+r$ divides by 4, otherwise it is 0. As is well-known (and can be immediately derived from the material of subsection 2.1), we have

$$\begin{aligned}\tilde{m}_{r,M}(\theta)^* &= \frac{m_{\psi_{r,M}}(\theta+\pi)}{m_{r,M}(\theta)m_{\psi_{r,M}}(\theta+\pi)-m_{\psi_{r,M}}(\theta)m_{r,M}(\theta+\pi)} \\ &= 2^r(1+e^{-i\theta})^M\left((1+e^{-i\theta})^{M+r}-e^{ia\pi}(1-e^{-i\theta})^{M+r}\right)^{-1} \\ &= \frac{\cos^M(\theta/2)}{\cos^{M+r}(\theta/2)-i^{M+r+2a}\sin^{M+r}(\theta/2)}\end{aligned}$$

for the dual symbol, from which

$$\tilde{q}_{r,M}(\theta)=\begin{cases}\frac{\cos^{2M}(\theta/2)}{\cos^{2(M+r)}(\theta/2)+\sin^{2(M+r)}(\theta/2)} & , \quad M+r \text{ odd}, \\ \\ \frac{\cos^{2M}(\theta/2)}{(\cos^{(M+r)}(\theta/2)+\sin^{(M+r)}(\theta/2))^2} & , \quad M+r \text{ even}.\end{cases} \tag{112}$$

The case $r=0$ and odd M is considered in [7], [18, Section 6.2] and corresponds to the Butterworth scheme known in signal processing. See Table 2 in [18] for the regularity exponents $\tilde{s}_{0,M}$ associated with (112). There are discrepancies with the values shown below of order $\approx 10^{-4}$ which are, in our opinion, another indication of limitations of the Fredholm determinant method if the value of r_0 becomes close to 1. Since

$$\tilde{s}_{r,M}=\tilde{s}_{M+r,0}+r\ ,$$

(this is immediate from the factorization technique, compare [18, Section 4] or Lemma 19), the case of odd $M+r$ is essentially covered. Some computations for the case of even $M+r$ are carried out in [24, Table 5]. Unfortunately, some entries of this table are incorrect due to errors in the formulae for $p_{n,0}$ for $n=6,8$ there.

$r\setminus M$	0	1	2	3	4
1	-0.50000000	0.50000000	1.04411766	1.99023605	2.64881733
2	-0.50000000	0.04411766	0.99023605	1.64881733	2.53944059
3	-0.95588234	-0.00976395	0.64881733	1.53944059	2.26458677
4	-1.00976395	-0.35118267	0.53944059	1.26458677	2.11977128
5	-1.35118267	-0.46055941	0.26458677	1.11977128	1.88117327
6	-1.46055941	-0.73541323	0.11977128	0.88117327	1.71493986
7	-1.73541323	-0.88022872	-0.11882673	0.71493986	1.49573101
8	-1.88022872	-1.11882673	-0.28506014	0.49573101	1.31652581
9	-2.11882673	-1.28506014	-0.50426899	0.31652581	1.10753951

Table 10. Values for $\tilde{s}_{r,M}$.

Table 10 gives an indication about the potential of the systems $\mathbf{\Psi}_{r,M}$ as decomposition systems in Sobolev spaces. Note that the s-range for which the Riesz basis property holds is given by $-\tilde{s}_{r,M} < s < r - 1/2$. The entries have been computed for $M = L = 0$ by the approximation method using subspace iteration as outlined before (here, a subspace dimension of 3 turned out to be most efficient). Generally speaking, the s-interval for which the H^s–Riesz basis property holds increases with the number of moment conditions. For fixed M, it shifts to the right with increasing B-spline order r. For the examples of edge-oriented systems $\mathbf{\Psi}_{(M+1)HB}$ with small M and $r = 2$ considered in the previous subsections, we have observed that the regularity exponent associated with the dual MRA does not change with d. An exception was the case $M = 0$.

The next example shows that, in general, the qualitative behavior detected in Table 10 does not carry over to higher dimensions. This can be also seen from the results for the system $\mathbf{\Psi}_{4HB}$ mentioned in subsection 3.2. We will consider the piecewise constant MRA ($r = 1$) for $d = 2$ and $M \le 3$. Thus, the scaling function ϕ is now given by the characteristic function of the unit cube in $\mathbb{R}^d$, with the symbol

$$m(\theta) = 2^{-d} \sum_{\lambda \in \Lambda} e^{-i\lambda\theta} = 2^{-d} \prod_{l=1}^{d} (1 + e^{-i\theta_l}) \ .$$

The functions ψ^λ are constructed by using the same formulas as in the linear case (see subsection 3.1 for $M = 2$, and subsection 3.2 for $M = 1, 3$). The cases $M = 0$ (consisting of characteristic functions of dyadic cubes) and $M = 1$ (which is equivalent to a two-dimensional Haar basis) are straightforward: For $M = 0$ the Riesz basis property with respect to $H^s(\mathbb{R}^d)$ does not hold for any s, for $M = 1$ it holds for $-1/2 < s < 1/2$.

We give the necessary formulas and show some computations for $M = 2, 3$. The above elimination procedure gives for $M = 2$

$$\tilde{m}(\theta) = \frac{\sum_{\lambda \in \Lambda} \cos(\lambda\theta)}{\sum_{\lambda \in \Lambda} e^{i\lambda\theta} \cos(\lambda\theta)}$$

and

$$\tilde{q}(\theta) = \frac{4 \prod_{l=1}^{d+1} \cos^2(\theta_l/2)}{1 + 2 \prod_{l=1}^{d+1} \cos\theta_l + \prod_{l=1}^{d} \cos^2\theta_l} = p(\theta) q(\theta) \ .$$

Since the polynomial $p(\theta) = \prod_{l=1}^{d+1} \cos^2(\theta_l/2)$ is the same as for the 3-point hierarchical basis $\mathbf{\Psi}_{3HB}$ studied in subsection 3.1, we can apply Lemma 19 twice. Looking at all occurring factorized symbols $\tilde{r}$, symmetry arguments show that

$$\tilde{\rho} = \rho_{2,\tilde{q}} = \max_{k=1,2} \rho_{0,\tilde{r}_k} \ ,$$

where

$$\tilde{r}_1(\theta) = \frac{1}{16} q(\theta) \prod_{l=1}^{d-1} \cos^2(\theta_l/2) \ , \qquad \tilde{r}_2(\theta) = \frac{1}{16} q(\theta) \prod_{l=3}^{d+1} \cos^2(\theta_l/2) \ .$$

Recall that $\theta_{d+1} = \sum_{l=1}^{d} \theta_l$. The calculations in the first two columns of Table 11 contain approximations to $\rho_{0,\tilde{r}_k}$ for $d = 2$ obtained by the approximation method (subspace dimension 3 was chosen in the simultaneous iteration, only the largest eigenvalue is shown). The maximum is achieved for $k = 2$, and is smaller than for $d = 1$, see Table 10 for $r = 1$, $M = 2$.

In the third column of Table 11, approximations to $\tilde{\rho}$ for the case $M = 3$ are obtained. Since in this case

$$\tilde{m}(\theta) = \frac{\sum_{\lambda\in\Lambda} d(\lambda\theta)}{\sum_{\lambda\in\Lambda} d(\lambda\theta)e^{i\lambda\theta}}\ , \ d(t) = \frac{1+3e^{-i2t}}{e^{-i3t}+3e^{-it}}\ ,$$

and an explicit factorization of $\tilde{q}(\theta) = |\tilde{m}(\theta)|^2$ is not straightforward, we have decided to compute approximations to $\rho_{L,\tilde{q}}$ for various L by the approximation method (to accelerate the convergence a subspace dimension of 4 has been tried). As turns out, $L = 1$ is already enough since $\rho_{1,\tilde{q}} > 1/4$.

N	$\rho_{0,(\tilde{r}_1)_N}$	$\rho_{0,(\tilde{r}_2)_N}$	$\rho_{1,\tilde{q}_N}$
5	0.23520103	0.24909491	0.30690784
10	0.23512579	0.25353403	0.30888135
15	0.23518602	0.25340684	0.30883113
20	0.23516816	0.25340692	0.30882578
25	0.23516817	0.25340689	0.30882563
30	0.23516817	0.25340693	0.30882563
35	0.23516817	0.25340693	0.30882563

Table 11. Approximations to $\tilde{\rho}$ by the approximation method for P0-element systems ($M = 2, 3$).

To summarize, for the MRA consisting of P0 elements (with respect to squares) in $\mathbb{R}^2$, the $\mathbf{\Psi}_{(M+1)HB}$ systems form Riesz bases in $H^s(\mathbb{R}^d)$ if

$$-\tilde{s} < s < 1/2\ , \quad \tilde{s} = \begin{cases} 0.5 & , \quad M = 1 \\ 0.990236... & , \quad M = 2 \\ 0.847568... & , \quad M = 3 \end{cases} .$$

We conclude with a few remarks. With the exception of the systems $\mathbf{\Psi}_{5HB,a}$ considered in subsection 3.2, all our systems are *edge-oriented*, in the sense that the subsymbols $m^{\lambda}_{\lambda'}$ associated with the wavelets ψ^{λ}, $\lambda \in \Lambda'$, are non-vanishing only for $\lambda' = \lambda$ or $\lambda' = 0$. In this case, explicit expressions for $M^{-1}(\theta)$ and for the dual symbol $\tilde{m}(\theta)$ can be computed. Without going to the details, we quote the result which has implicitly been used before:

$$\tilde{m}(\theta)^* = 2^{-d}\frac{1-\sum_{\lambda\in\Lambda'} d_\lambda(\theta)}{m_0(\theta)-\sum_{\lambda\in\Lambda'} d_\lambda(\theta)m_\lambda(\theta)}\ , \quad d_\lambda(\theta) = \frac{m^\lambda_0(\theta)}{m^\lambda_\lambda(\theta)}\ . \tag{113}$$

This formula simplifies further if all ψ^λ-masks are obtained by rotation in which case $d_\lambda(\theta) = d(\lambda\theta)$ for a univariate rational trigonometric function $d(t)$. If in addition $m^\lambda(0) = 0$ (i.e., the ψ^λ satisfy moment conditions of order $M \geq 1$) then

$d(0) = -1$ which leads to a further simplification of (113). This observation helps to explicitly compute certain symbols in higher dimensions.

From the definition (42) of $\tilde{m}(\theta)$ via the determinants $N(\theta)$ and $D(\theta)$ it is evident that the polynomial $p(\theta)$ is determined by the ψ^λ-masks only, and does not depend on the ϕ–mask. Thus, $p(\theta)$ does not change if the same ψ^λ-masks are explored for different MRA's. In [24], we have experimented with the analogs of $\mathbf{\Psi}_{2HB}$ and $\mathbf{\Psi}_{3HB}$ for the simplest C^1 box spline MRA for $d = 2$ which might be of some interest for boundary element respectively domain decomposition codes for the biharmonic problem. Details can be found in that paper.

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