Stable space splittings and fusion frames

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ABSTRACT

The concept of stable space splittings has been introduced in the early 1990ies, as a convenient framework for developing a unified theory of iterative methods for variational problems in Hilbert spaces, especially for solving large-scale discretizations of elliptic problems in Sobolev spaces. The more recently introduced notions of frames of subspaces and fusion frames turn out to be concrete instances of stable space splittings. However, driven by applications to robust distributed signal processing, their study has focused so far on different aspects. The paper surveys the existing results on stable space splittings and iterative methods, outlines the connection to fusion frames, and discusses the investigation of quarkonial or multilevel partition-of-unity frames as an example of current interest.

Keywords: Stable space decompositions, fusion frames, iterative methods, preconditioning, atomic decompositions, partition-of-unity methods

1. INTRODUCTION

The decomposition of computational problems into smaller building blocks is a common strategy in large-scale computing, and has been boosted by the increased availability of parallel and distributed computing platforms. Almost any modern algorithm that scales reasonably well with the number of involved problem unknowns and parameters contains instances of this idea. As to simulations based on partial differential equations (PDE), one of the core areas of large-scale computing, the search for and the analysis of efficient adaptive algorithms has benefited from the framework of stable space splittings that emerged in the early 1990ies in various, almost equivalent forms by the efforts of many researchers. At that time, the main impact of this simple yet powerful analysis tool were new optimality proofs for multigrid and domain decomposition methods for elliptic and parabolic problems in Sobolev spaces under minimal assumptions. The early contributions to this area by Nepomnyaschikh, Widlund, Xu, Yserentant, and their collaborators are well documented.[1–5] Below we will follow the formalism developed in our previous work with Griebel[6–11] which was also influenced by earlier developments in the theory of approximation spaces. Roughly speaking, a Hilbert space V with a symmetric V-elliptic bilinear form $a(\cdot,\cdot)$ can be stably split into a collection of Hilbert spaces V_j (equipped with their own symmetric V_j-elliptic bilinear forms $b_j(\cdot,\cdot)$) using the linear restriction operators $R_j : V_j \to V$ iff a norm equivalence of the form

$$a(v,v) \approx \inf_{v_j\in V_j:\, v=\sum_j R_j v_j} \sum_j b_j(v_j,v_j) \qquad (1)$$

holds with constants independent $v \in V$. A particular case of this definition is the case of closed subspaces $V_j \subset V$, and R_j given by the natural imbedding operators. We also realized[8,10] that frames $\Phi = \{\phi_j\}$ in V are a special instance of stable space splittings (they represent decompositions into one-dimensional subspaces $V_j = \{c\phi_j :\ c \in \mathbb{R}\}$ of V if we set $b_j(\cdot,\cdot) = \|\phi_j\|_V^{-2}(\cdot,\cdot)_V$), a fact that did not lead to any further research activity, mainly because at that time the numerical PDE community was very sceptical about the practical impact of the new ideas from wavelet theory, part of which frames were considered. On the other side, researchers on wavelet algorithms[12–14] favored, for various reasons, having a Riesz basis and not a frame, whereas most of the stable space splitting examples involve redundancy. Only very recently, the frame case is also being investigated for PDE applications, see Section 7 of Ref. 14 for further information.

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The recent introduction of frames of subspaces and fusion frames[15–17] was triggered by applications to distributed information processing in large sensor networks. Instead of defining frames of subspaces via a minimization property as was done for stable space splittings in Equ. (1), the following equivalent form was used: A collection of closed subspaces $V_j \subset V$ is called a frame of subspaces with weights $w_j > 0$ iff

$$\|v\|_V^2 \approx \sum_j w_j^2 \|P_j v\|_V^2, \qquad (2)$$

again with constants independent of v, where $P_j:\ V \to V_j$ denotes the ortho-projector onto V_j. This is equivalent to Equ. (1) if we set $a(\cdot,\cdot) = (\cdot,\cdot)_V$, and $b_j(\cdot,\cdot) = w_j^{-2}(\cdot,\cdot)_V$ (with R_j the natural imbeddings). The duality of the two definitions is a well-known fact in frame theory, and there tied to the existence of a dual frame. In the theory of stable space splittings, the counterpart of the scaled ortho-projectors $w_j^2 P_j$ are the solution operators $T_j:\ V \to V_j$ for the auxiliary variational problems of finding $u_j := T_j u \in V_j$ such that

$$b_j(u_j, v_j) = a(u, R_j v_j), \qquad \forall v_j \in V_j, \qquad (3)$$

for given $u \in V$. The operators T_j are the building blocks of iterative schemes associated with a stable space splitting, as will be detailed in the next section.

It is interesting to note that, with the exception of the definitions and basic properties, research on these related topics concentrated on complementary aspects. Stable space splittings were motivated by providing fast solvers for the basic discretization schemes for differential and integral equations. This involves the adaption to realistic physical domains and solution properties, and studying the complexity of the resulting iterative solvers. Also, the practically most useful stable space splittings are not frames of subspaces for L_2 spaces but only for Sobolev spaces H^s for some order $s > 0$. This explains why certain constructive questions (e.g., generalizations of tight frames, and optimized frames of subspaces for L_2 applications) have been treated only recently as a result of the introduction of the new notion of fusion frames. However, there is one common thread important to both application areas - the problem of robustness with respect to deleting parts of the considered space splittings. In applications to sensor networks, information needs to be retrieved even if a certain percentage of the sensors malfunctions[16] while adaptive algorithms for operator equations skip unnecessary parts of the discretization space, and thus of its splittings, during execution. In both cases, the frame property needs to be preserved for the subsystems, including a robust behavior of the equivalence constants in Eqs. (2) and (1). This is a subtle problem since, in contrast to Riesz systems, frame bounds of subsystems of a frame system may deteriorate dramatically.

The purpose of this note is to review the concept of stable space splittings (Sec. 2.1), and the abstract theory of Schwarz iterative methods based on it (Sec. 2.2), together with some typical examples. This material is rather elementary, more details and proofs can be found in Ref. 10. We also describe the connection to fusion frames. In the concluding short Sec. 3 we discuss the so-called quarkonial frames introduced by Triebel[18–20] as a theoretical tool for the study of function spaces, and relate them to interesting unsolved problems in the theory of multilevel partition-of-unity methods.[21–24] The note will hopefully foster the exchange of ideas between the areas of iterative methods for large-scale variational problems, on the one hand, and fusion frames with their applications to information processing, on the other, and lead to new research results and collaborations.

2. STABLE SPACE SPLITTINGS AND ITERATIVE METHODS

2.1 Stable space splittings

Let V be a separable Hilbert space, with scalar product and norm denoted by $(\cdot,\cdot)_V$ and $\|\cdot\|_V$, respectively. Let $a(\cdot,\cdot):\ V \times V \to \mathbb{R}$ be a symmetric bounded V-elliptic bilinear form on V, i.e., there are constants $0 < c \le C < \infty$ such that $|a(u,v)| \le C\|u\|_V\|v\|_V$ and $c\|u\|_V^2 \le a(u,u)$ for all $u, v \in V$. Whenever we write $\{V; a\}$ we have in mind that these assumptions are met. Any $\{V; a\}$ generates an associated linear operator $A:\ V \to V'$ satisfying $\langle Au, v\rangle_{V' \times V} = a(u,v)$ for all $u, v \in V$. The efficient solution of the operator equation $Au = f$ for given $f \in V'$ was the main motivation of developing the concept of stable space splittings (SSS), we always talk about the pair $\{V; a\}$ instead of V alone.

Definition 1. The (at most countable) family of Hilbert spaces $\{V_j; b_j\}$, together with bounded linear restriction operators $R_j : V_j \to V$, is called a *stable space splitting for* $\{V; a\}$ if

$$|||v|||^2 := \inf_{v_j \in V_j:\, v=\sum_j R_j v_j} \sum_j b_j(v_j, v_j) \tag{4}$$

satisfies a two-sided estimate

$$ca(v,v) \le |||v|||^2 \le Ca(v,v), \qquad v \in V, \tag{5}$$

with constants $0 < c \le C < \infty$ independent of $v \in V$. The best possible constants c and C in Equ. (5) are denoted by $\lambda_{\min}$ and $\lambda_{\max}$ and called *lower and upper stability bounds* of the SSS. Their quotient $\kappa := \lambda_{\max}/\lambda_{\min}$ is called the *condition number* of the SSS. The symbolic equation

$$\{V; a\} =_{\lambda_{\min},\lambda_{\max}} \sum_j R_j\{V_j; b_j\} \tag{6}$$

stands for this definition (since often concrete values for the stability bounds are not known, we will not always show them). Part of this definition is the silent assumption that any $v \in V$ possesses at least one V-converging series representation $v = \sum_j R_j v_j$ with certain $v_j \in V_j$ (since the above definition is invariant w.r.t. permutations within the family of $\{V_j; b_j\}$ it turns out that convergence is unconditional). Typically, there are many such representations, i.e., for SSS emphasis is on tolerating some degree of redundancy. We will not dwell on the uniqueness question, i.e., the case of Riesz-type splittings.

Jumping ahead, let us mention that most of previous work on SSS has been devoted to establishing uniform bounds on the condition κ for concrete families of SSS related to numerical discretization processes. For this purpose, a certain toolbox of simple operations on SSS has been introduced in Sections 4.1 and 4.2.2 of Ref. 8. Complemented by the obvious operation of introducing scaling factors into the bilinear forms $b_j(\cdot,\cdot)$, these have been used by many authors to improve and optimize stability bounds.

- **Refinement.** Given an SSS as in Equ. (6), and SSS

$$\{V_j; b_j\} =_{\lambda_{\min,j},\lambda_{\max,j}} \sum_i R_{ji}\{V_{ji}; b_{ji}\} \tag{7}$$

  for each of the V_j, substitution yields the *refined SSS*

$$\{V; a\} =_{\lambda'_{\min},\lambda'_{\max}} \sum_j \sum_i R_j R_{ji}\{V_{ji}; b_{ji}\} \tag{8}$$

  together with the obvious estimates

$$\lambda \cdot \lambda_{\min} \le \lambda'_{\min} \le \lambda'_{\max} \le \Lambda \cdot \lambda_{\max}, \qquad \lambda := \min_j \lambda_{\min,j}, \quad \Lambda := \max_j \lambda_{\max,j}.$$

  Thus, the refined SSS in Equ. (8) has condition number $\kappa' \le \frac{\Lambda}{\lambda}\kappa$. This can be improved by multiplying the forms $b_{ji}(\cdot,\cdot)$ in the SSS for $\{V_j; b_j\}$ by the weight $w_j := \lambda_{\min,j}^{-1}$, yields new values $\lambda = 1$ and $\Lambda = \max_j \kappa_j$, and gives $\kappa' = \kappa \cdot \max_j \kappa_j$. In other words, if combined with scaling, the refinement of a given SSS using uniformly well-conditioned SSS of $\{V_j; b_j\}$ cannot spoil the condition too much!

- **Clustering.** This is the inverse operation to refinement. Suppose that the components of a given SSS have been grouped according to Equ. (8), and that we can find $\{V_j; b_j\}$ such that Equ. (7) holds for all j. Then the SSS of Equ. (6) is the result of *clustering* in the SSS in Equ. (8). Again, similar estimates $\lambda\cdot\lambda'_{\min} \le \lambda_{\min} \le \lambda_{\max} \le \Lambda\cdot\lambda'_{\max}$ and $\kappa \le \frac{\Lambda}{\lambda}\kappa'$ follow easily. If scaling is applied, we arrive at $\kappa \le \kappa' \cdot \max_j \kappa_j$, i.e., clustering within a given SSS, and substituting the subclusters by single $\{V_j; b_j\}$ (with uniform bounds on the condition of the SSS representing the subclusters) does not deteriorate the condition.

- **Selection.** This is a subtle operation which needs most scrutiny in applications. We describe it in its simplest form. Denote the index set for the SSS in Equ. (6) by I. Take any proper subset of it, $\tilde{I} \subset I$, and set $\tilde{V} := \mathrm{clos}(\sum_{j \in I'} R_j V_j)$. We call the collection of all spaces $\{V_j; b_j\}$ with $j \in \tilde{I}$ a *selection* from the SSS in Equ. (6), and may ask the question whether

$$\{\tilde{V}; a\} =_{\lambda_{\min,\tilde{I}},\lambda_{\max,\tilde{I}}} \sum_{j \in \tilde{I}} R_j \{V_j; b_j\} \tag{9}$$

holds, with reasonable stability bounds. The replacement of V by its subspace $\tilde{V}$ is necessary, otherwise the answer is trivially negative. The investigation of which $\tilde{I} \subset I$ do not lead to any serious deterioration of condition is a central question for modern adaptive solvers. By definition Equ. (5), we have

$$\lambda_{\min,\tilde{I}} \geq \lambda_{\min} = \lambda_{\min,I}, \qquad \lambda_{\max,\tilde{I}} \geq \lambda_{\max} = \lambda_{\max,I},$$

for any $\tilde{I} \subset I$. However, nothing can be said about controlling the behavior of $\lambda_{\max,\tilde{I}}$ from above. Counterexamples are known from the frame case. E.g., take $V = \mathbb{R}^2$, all bilinear forms coincide with the scalar product of $\mathbb{R}^2$, and consider the SSS of V given by the one-dimensional V_j, $j = 1, 2, 3$, spanned by the vectors $e_1 = (1, 0)$, $e_2 = (1, \epsilon)$, and $e_3 = (0, 1)$, respectively. Then $\kappa \to 2$ as $\epsilon \to 0$, while for $\tilde{I} = \{1, 2\}$ we obviously get $\kappa_{\tilde{I}} \to \infty$. Finally, note that instead of dropping complete spaces $\{V_j; b_j\}$, one could more generally replace each of them by any of its closed subspaces $\{\tilde{V}_j; b_j\}$ (including the previous case where $\tilde{V}_j = \{0\}$ for $j \notin \tilde{I}$) as suggested in Sec. 4.2.2 of Ref. 8.

A couple of remarks are in order. The above mentioned operations of scaling, refinement, clustering, and selection are usually applied in some combination (and the result might be better than the worst-case prediction from the individual steps of their application). Also, one could think of expanding the list of generic operations for SSS. E.g., an inverse operation to selection called *enlargement* would be meaningful when an appropriate SSS for $\tilde{V} \subset V$ needs to be complemented so that a well-conditioned SSS for the whole V results. Constructions that are motivated by exploring tensor-product techniques for high-dimensional problems have been proposed by several authors.[8, 25, 26] The question of how much improvement of desirable properties of an SSS can be gained by adjusting the auxiliary forms $b_j(\cdot, \cdot)$ (e.g., by scaling) for a fixed system $\{V_j\}$ is also of some interest.

The operator view on a SSS is given in the following theorem[8, 10] (compare Ref. 15 for similar statements for fusion frames). To prepare for it, recall the definition of the operators $T_j : V \to V_j$ given by Equ. (3), and set $P = \sum_j R_j T_j$. We call P the *additive Schwarz operator* of the SSS in Equ. (6).

Theorem 1. The following statements are equivalent:

(i) Equation (6) represents a stable space splitting for $\{V; a\}$.

(ii) There are constants $0 < A \leq B < \infty$ such that

$$Aa(v, v) \leq \sum_j b_j(T_j v, T_j v) \leq Ba(v, v), \qquad v \in V. \tag{10}$$

(iii) The additive Schwarz operator P is well-defined as a bounded linear operator on V, is symmetric w.r.t. $a(\cdot, \cdot)$ (i.e., $a(Pu, v) = a(u, Pv)$ for all $u, v \in V$), and possesses a bounded inverse P^{-1} on V.

The connection between the stability bounds $\lambda_{\min}$, $\lambda_{\max}$ of the SSS, the best possible constants $A_{\max}$, $B_{\min}$ in Equ. (10), and the spectral bounds $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ is given by the equations

$$\lambda_{\min}(P) = A_{\max} = \lambda_{\max}^{-1}, \qquad \lambda_{\max}(P) = B_{\min} = \lambda_{\min}^{-1}.$$

As a consequence, $\kappa(P) = \kappa = B_{\min}/A_{\max}$.

The proof[10] (for an earlier sketch of this proof, see Sec. 4.1 of Ref. 8) is based on deriving the identities $a(Pv, v) = \sum_j b_j(T_j v, T_j v)$ and $a(P^{-1}v, v) = |||v|||^2$ valid for arbitrary $v \in V$. The proper framework for studying SSS is to view $\{V; a\}$ as the Hilbert space with scalar product $a(\cdot, \cdot)$, similarly for $\{V_j; b_j\}$, and to introduce the

Hilbert sum $\{\tilde{V},\tilde{b}\}$ consisting of all sequences $\tilde{v}:=(v_j)$ with entries $v_j\in V_j$ such that $\tilde{b}(\tilde{v},\tilde{v}):=\sum_j b_j(v_j,v_j)<\infty$. If $R:\tilde{V}\to V$ is given by $R\tilde{v}=\sum_j R_j v_j$ then its adjoint $R^*:V\to\tilde{V}$ with respect to the chosen scalar products $a(\cdot,\cdot)$ in V and $\tilde{b}(\cdot,\cdot)$ in $\tilde{V}$ (i.e., $a(R\tilde{u},v)=\tilde{b}(\tilde{u},R^*v)$) is given componentwise by $(R^*v)_j=T_jv$. Thus, $P=RR^*$ is the additive Schwarz operator acting on V. Its counterpart in $\tilde{V}$ given by $\tilde{P}:=R^*R$ called *Gramian* in analogy with the frame case plays an important role in the analysis of iterative solvers based on a SSS.

There are only a few general ideas for proving the SSS property (5). Obviously, establishing the upper bound amounts to finding a good decomposition with respect to $\{V_j\}$ which is a matter of approximation theory, and depends very much on the nature of the splitting. The lower estimate is often tied to estimations of the relative angles between the spaces V_j, e.g., by assuming so-called strengthened Cauchy-Schwarz inequalities.[1,2,5]

As was mentioned in the Introduction, frames of subspaces are a special instance of SSS, namely

$$\{V;(\cdot,\cdot)_V\}=_{\lambda_{\min},\lambda_{\max}}\sum_j\{V_j;w_j^{-2}(\cdot,\cdot)_V\},$$

where V is a generic separable Hilbert space (usually, $V=L_2(\Omega)$ serves as the concrete model), all V_j are closed subspaces of V, the restriction operators R_j refer to natural imbeddings and are therefore omitted, and the only modification are the individual scaling factors w_j^{-2} for the scalar product in each V_j. In other words, $V_j\subset V$, $a(u,v)=(u,v)_V$, and $b_j(u_j,v_j)=w_j^{-2}(u_j,v_j)_V$. Indeed, if $P_j:V\to V_j$ is the ortho-projector onto V_j then $T_j=w_j^2P_j$, and (10) reads

$$A\|v\|_V^2\le\sum_j w_j^{-2}(w_j^2P_jv,w_j^2P_jv)_V=\sum_j w_j^2\|P_jv\|_V^2\le B\|v\|_V^2,\qquad\forall\, v\in V,$$

where $A=\lambda_{\max}^{-1}\le B=\lambda_{\min}^{-1}$ are finite constants. This is the definition of a frame of subspaces.[15] Moreover, what we call additive Schwarz operator becomes the frame operator $P=RR^*=\sum_j R_jT_j=\sum w_j^2P_j$ (denoted by S in Ref. 15), with analysis operator R^* defined componentwise by $(R^*v)_j=T_jv=w_j^2P_jv$, and synthesis operator R given by $R\tilde{v}=\sum_j R_jv_j=\sum_j v_j$ (denoted by T in Ref. 15). To our knowledge, the Gramian $\tilde{P}=R^*R$ does not appear explicitly in the initial papers on fusion frames. On the other hand, the construction of dual or tight fusion frames has not been touched for general SSS, see the Remarks in Ref. 10.

We conclude this section with the two main-stream examples that have motivated the development of the SSS framework (for more details and other worked examples, we refer to the literature[1,2,5,8,10–12,27]). Both address the solution of generic elliptic boundary value problems. For simplicity, consider the Poisson equation with homogeneous Dirichlet boundary conditions,

$$-\Delta u=f\qquad\text{in }\ \Omega,\qquad\qquad u=0\qquad\text{on }\ \partial\Omega,$$

in its variational form, i.e., find $u\in V:=H_0^1(\Omega)$ such that

$$a_\Omega(u,v):=\int_\Omega\nabla u\cdot\nabla v=\int_\Omega fv\qquad\quad\forall\, v\in V\tag{11}$$

(we will assume that the reader is familiar with this kind of PDE problem and the generic conditions on the involved domains Ω, and can fill in the necessary details).

Domain decomposition splittings. The idea is to cover Ω by several subdomains Ω_j, and to reduce the solution of Equ. (11) to the solution of similar Dirichlet problems w.r.t. the subdomains Ω_j. For simplicity, consider two overlapping subdomains Ω_1 and Ω_2. For $j=1,2$, set $V_j:=H_0^1(\Omega_j)$, $b_j(\cdot,\cdot):=a_{\Omega_j}(\cdot,\cdot)$, and define R_j by extending functions from V_j by zero to $\Omega\backslash\Omega_j$. The resulting decomposition

$$\{H_0^1(\Omega);a_\Omega\}=R_1\{H_0^1(\Omega_1);a_{\Omega_1}\}+R_2\{H_0^1(\Omega_2);a_{\Omega_2}\}$$

is indeed stable, with the upper stability constant and thus the condition number deteriorating as the distance of the inner boundary pieces $\Gamma_j:=\Omega\cap\partial\Omega_j$ of the subdomains approaches zero (overlap means that Γ_1 belongs to

Ω_2, and Γ_2 to Ω_1). This SSS is of historical interest: It provides the framework for the convergence proof of the classical Schwarz alternating method providing one of the existence proofs for the Dirichlet problem. If we cover Ω with many overlapping subdomains then the condition number of the corresponding SSS deteriorates with the subdomain diameter which necessitates the introduction of further auxiliary spaces. In the non-overlapping case ($\Gamma = \Gamma_1 = \Gamma_2$), the above splitting needs to be augmented by the trace space $V_3 = H_{00}^{1/2}(\Gamma)$ of $H_0^1(\Omega)$ on Γ, and an appropriate extension operator R_3 to retain the stability property. These SSS become very attractive if they are considered on the discretization level. While the discretized problem for Ω may be too large to fit into memory, the smaller subdomain problems can be attacked on a single-processor unit. Domain decomposition methods whose theoretical analysis is predominantly based on the SSS concept[5] provide robust solution algorithms on parallel and distributed computing systems, and are nowadays a widely accepted approach in large-scale scientific computing.

Multiscale splittings. Consider a multiresolution analysis $V_0 \subset V_1 \subset V_2 \subset \ldots L_2(\Omega)$ with dilation factor $r > 1$, consisting of subspaces V_j with dense union in $L_2(\Omega)$. Under the assumption that appropriate Jackson-Bernstein estimates in $L_2(\Omega)$ hold, one can prove[8, 12] that

$$\{H^s(\Omega); a\} = \sum_{j=0}^{\infty} \{V_j; r^{2js}(\cdot,\cdot)_{L_2(\Omega)}\}$$

is a SSS for a certain range of $0 < s < s_{\max}$, and any symmetric $H^s(\Omega)$-elliptic bilinear form $a(\cdot,\cdot)$. Note that due to the nestedness of $\{V_j\}$ the result cannot hold for $L_2(\Omega)$ ($s = 0$) since the lower bound is violated (just write $v \in V_0$ in the form $v = \sum_{j=0}^{N-1} v_j$ with $v_j = N^{-1}v \in V_0 \subset V_j$, $j = 0, \ldots, N-1$, and let $N \to \infty$). Such results have along history in the theory of function spaces, and extend to the non-Hilbert setting of $L_p(\Omega)$, $p \neq 2$.

To give an example of practical interest, for traditional C^0 finite element spaces $V_j = S_k(\mathcal{T}_j)$ of piecewise polynomials of degree $k \geq 1$ on a sequence of nested dyadically refined quasi-uniform partitions $\mathcal{T}_j$ of Ω this holds with $s_{\max} = 3/2$ and $r = 2$.[8, 12] In this case, also boundary conditions can be treated which results in the following infinite SSS for $H_0^1(\Omega)$ (to avoid any technical discussions, let Ω be a polyhedral domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $\mathcal{T}_j$ dyadically refined triangular or tetrahedral partitions of Ω):

$$\{H_0^1(\Omega); a_\Omega\} = \sum_{j=0}^{\infty} \{S_{k,0}(\mathcal{T}_j); 2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\},$$

where the elements of $S_{k,0}(\mathcal{T}_j) = S_k(\mathcal{T}_j) \cap H_0^1(\Omega)$ satisfy homogeneous boundary conditions. Denoting the standard nodal basis functions in $S_{k,0}(\mathcal{T}_j)$ by $\phi_{k,ji}$, and the one-dimensional spaces spanned by them by $V_{k,ji}$, we can use the refinement operation for SSS to conclude that also

$$\{H_0^1(\Omega); a_\Omega\} = \sum_{j=0}^{\infty}\sum_i \{V_{k,ji}; 2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\} = \sum_{j=0}^{\infty}\sum_i \{V_{k,ji}; a_\Omega\} \tag{12}$$

represent SSS. This follows since the nodal basis of $S_{k,0}(\mathcal{T}_j)$ is L_2 stable, with constants independent of j due to the quasi-uniformity of the partitions. The last step follows by the same reasoning since $a_\Omega(\phi_{k,ji}, \phi_{k,ji}) \approx 2^{2j}\|\phi_{k,ji}\|^2_{L_2(\Omega)}$, again with constants independent of j and i. In other words, (12) expresses the fact that for any fixed k the system $\Phi_k := \{\phi_{k,ji}\}$ of all nodal basis functions from all levels associated with interior nodal points, if appropriately scaled, is a frame for $\{H_0^1(\Omega); a_\Omega\}$. More generally, it is a frame for $H_0^s(\Omega)$ for $1/2 < s < 3/2$, and, if extended by all $\phi_{k,ji}$ associated with boundary nodes, for $H^s(\Omega)$ and $0 < s < 3/2$. The stability bounds and the condition number depend on k, s, and the regularity parameter bounds of the involved partitions.

This frame inherits the frame property (with uniform bounds on the condition number) to a certain class of its subsystems including the ones that are relevant for adaptive computations using *nested refinement* in the realm of the *h-version* of the finite element method. Roughly speaking, if we use selection in (12) and select only those $V_{k,ji}$, $j \leq J$, whose associated nodal basis function $\phi_{k,ji}$ have support in a certain region $\Omega_j \subset \Omega$, and if these regions are properly nested, $\Omega = \Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_J \supset \Omega_{J+1} = \emptyset$, then the selected subsplitting of (12) possesses a uniform condition number bound for all reasonable choices of $\{\Omega_j,\ j \leq J\}$. How to choose Ω_j

for a given problem instance is usually facilitated by local a posteriori error indicators as part of the adaptive algorithm. The guaranteed uniform condition number estimate leads to optimal preconditioning of the linear systems to be solved at each stage of the adaptive method. A more recent account on similar SSS and adaptive finite element methods, also for the Hilbert spaces $H(\mathrm{div})$ and $H(\mathrm{curl})$ governing flow and electro-magnetics applications, is given in Ref. 28.

2.2 Iterative methods based on SSS

We will use the notation from the previous sections, and talk now about the iterative solution of the operator equation $Au = f$ for given $f \in V'$ and $A : V \to V'$, or, equivalently, of the variational problem for finding $u \in V$ such that

$$a(u,v) := \langle Au, v\rangle_{V'\times V} = \langle f, v\rangle_{V'\times V}, \qquad \forall\, v \in V. \tag{13}$$

To fix notation, introduce operators $B_j : V_j \to V_j'$ and $R_j' : V' \to V_j'$ by

$$\langle B_j u_j, v_j\rangle_j = b_j(u_j, v_j), \qquad \langle R_j' f, v_j\rangle_j = \langle f, R_j v_j\rangle_{V'\times V}, \qquad \forall\, v_j \in V_j,$$

where $\langle\cdot,\cdot\rangle_j := \langle\cdot,\cdot\rangle_{V_j'\times V_j}$ for short. Then $T_j = B_j^{-1} R_j' A$, and if we introduce $\tilde g = (g_j) \in \tilde V$ by $g_j = B_j^{-1} R_j' f \in V_j$, and $g = \sum_j R_j g_j \in V$ we have the following

Theorem 2. For a given SSS (6), the solution $u \in V$ of the operator equation $Au = f$ resp. of (13) coincides with the solution of the operator equation

$$Pu = g \tag{14}$$

in V or, equivalently, by solving the following problem in $\tilde V$:

$$\tilde P \tilde u = \tilde g, \qquad u = R\tilde u. \tag{15}$$

This theorem, and the fact that the additive Schwarz operator $P = \sum_j R_j T_j = (\sum_j R_j B_j^{-1} R_j')A \equiv CA$ leads to a symmetric positive definite preconditioner $C \approx A^{-1}$ for A, is simple algebra. From now on, assume that the SSS splits $\{V; a\}$ into finitely many spaces $\{V_j; b_j\}$, $j = 1, \ldots, J$. Then $\tilde P$ can be interpreted as $J \times J$ matrix operator acting on $\tilde V$ whose entries are $P_{ij} = T_i R_j = B_i^{-1} R_i' A R_j : V_j \to V_i$.

Additive Schwarz method (AS). Given $u^0 \in V$, for $n = 0, 1, \ldots$ execute

$$u^{n+1} = u^n - \omega \sum_{j=1}^{J} R_j (T_j u^n - g_j),$$

until a convergence criterion is met.

Multiplicative Schwarz method (MS). Given $u^0 \in V$, for $n = 0, 1, \ldots$ execute $u^{n,0} = u^n$,

$$u^{n,j} = u^{n,j-1} - \omega R_j (T_j u^{n,j-1} - g_j), \quad j = 1, \ldots, J,$$

and $u^{n+1} = u^{n,J}$, until a convergence criterion is met.

Symmetric Multiplicative Schwarz method (SMS). Given $u^0 \in V$, for $n = 0, 1, \ldots$ execute $u^{n,0} = u^n$,

$$\begin{aligned} u^{n,j} &= u^{n,j-1} - \omega R_j (T_j u^{n,j-1} - g_j), \quad j = 1, \ldots, J, \\ u^{n,2J+1-j} &= u^{n,2J-j} - \omega R_j (T_j u^{n,2J-j} - g_j), \quad j = J, \ldots, 1, \end{aligned}$$

and $u^{n+1} = u^{n,2J}$, until a convergence criterion is met.

In all three abstract algorithms, $\omega > 0$ is a relaxation parameter (usually fixed independently of j and n). It is worthwhile to write down the error propagation operators for the AS, MS, and SMS methods:

$$\begin{aligned} M_{AS} &= I - \omega \sum_{j=1}^{J} R_j T_j = I - \omega P = I - \omega CA, \qquad M_{MS} = (I - \omega R_J T_J) \ldots (I - \omega R_1 T_1), \\ M_{SMS} &= (I - \omega R_1 T_1) \ldots (I - \omega R_J T_J)(I - \omega R_J T_J) \ldots (I - \omega R_1 T_1) = M_{MS}^* M_{MS} \end{aligned}$$

The SMS method also gives rise to a symmetric preconditioner C_{SMS}, defined by $M_{SMS} = I - \omega C_{SMS} A$. Since implementing the SMS method is essentially the same as applying C_{SMS}, both AS and SMS methods are often combined with the conjugate gradient method which usually results in a more robust and efficient algorithm called **Schwarz-preconditioned conjugate gradient method**. There exist many further extensions of the multiplicative version, especially in a multi-resolution framework.[3]

The convergence speed will be measured by $\rho := \lim_{n\to\infty} \|M^n\|_a^{1/n}$ which simplifies to $\rho := \|M^n\|_a$ if the error propagation operator M is symmetric in $\{V;a\}$, where $\|\cdot\|_a$ denotes the operator norm w.r.t. $\{V;a\}$. It is very instructive[8–10] to rewrite the above Schwarz methods in terms of solving $\tilde{P}\tilde{u} = \tilde{g}$. Indeed, setting $u = R\tilde{u}$ and writing iterations for Equ. (15) in the generic form $\tilde{u}^{n+1} = \tilde{u}^n - \tilde{N}(\tilde{P}\tilde{u}^n - \tilde{g})$, one arrives at the following compact formulas:

$$\tilde{N}_{AS} = \omega \tilde{I}, \quad \tilde{N}_{MS} = (\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1}, \quad \tilde{N}_{SMS} = (\frac{1}{\omega}\tilde{I} + \tilde{U})^{-1}(\frac{2}{\omega}\tilde{I} - \tilde{D})(\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1},$$

where $\tilde{P} = \tilde{L} + \tilde{D} + \tilde{U}$ is the decomposition of the matrix operator $\tilde{P}$ into strictly lower and upper triangular and diagonal parts. These formulas are exact copies of the corresponding formulas for the classical Richardson, Richardson-SOR, and Richardson-SSOR schemes in numerical linear algebra, see Lemma 6 in Ref. 10. This analogy allows us to give proofs[9,10] of the following

Theorem 3. Assume that the SSS in Equ. (6) splits $\{V;a\}$ into finitely many $\{V_j;b_j\}$, $j = 1,\dots,J$, with stability bounds $\lambda_{\min}$, $\lambda_{\max}$, and condition $\kappa = \lambda_{\max}/\lambda_{\min}$.
(i) The AS method converges for $0 < \omega < 2\lambda_{\min}$, with rate $\rho_{AS,\omega} = \max(|1-\omega\lambda_{\min}^{-1}|, |1-\omega\lambda_{\min}^{-1}|)$. The best possible rate is

$$\rho_{AS}^* := \min_{\omega} \rho_{AS,\omega} = \rho_{AS,\omega^*} = 1 - \frac{2}{1+\kappa}, \qquad \omega^* = \frac{2}{\lambda_{\min}^{-1} + \lambda_{\max}^{-1}}.$$

If executed as a pcg-method, we have an improved estimate for the error decay:

$$\|u - u^n\|_a \le 2\left(1 - \frac{2}{1+\sqrt{\kappa}}\right)^n \|u - u^0\|_a, \qquad n \ge 1.$$

(ii) For the MS and SMS methods, convergence is guaranteed for $0 < \omega < 2\hat{\lambda}$, where

$$\hat{\lambda} := \min_{j=1,\dots,J} \min_{v_j \in V_j} \frac{b_j(v_j, v_j)}{a(R_j u_j, R_j u_j)} \ge \lambda_{\min}.$$

The achievable rate depends on estimates for the triangular operator $\tilde{L}$:

$$\rho_{MS,\omega}^2 \le \rho_{SMS,\omega} \le 1 - \frac{\omega(2-\omega\hat{\lambda}^{-1})}{\lambda_{\max}\|\tilde{I} + \omega\tilde{L}\|_{\tilde{b}}}.$$

With no further assumptions on the SSS, one can conclude that the best possible rates satisfy

$$(\rho_{MS}^*)^2 \le \rho_{SMS}^* \le 1 - \frac{1}{\log_2(4J)\kappa}.$$

Theorem 3 tells us that Schwarz iterative methods should converge reliably whenever the condition of the SSS can be controlled. Although the above worst-case estimate for the multiplicative methods cannot be improved, MS and SMS perform often better than AS, see Ref. 29 for a refined analysis of multiplicative methods. Finding appropriate relaxation parameters ω is an issue, applying the pcg-variant is one way to avoid this problem. In contrast to MS and SMS, the AS method looks very attractive for a parallel implementation since the subproblems can be processed independently. This is true for SSS of domain decomposition type but not so much in the multiscale case, where both AS and MS/SMS methods can be interpreted as V-cycle multigrid methods. Lack of space prevents us from giving details.[1–3,27] In standard applications to PDE simulations, the convergence analysis needs to be complemented by a careful analysis of the computational costs associated with the individual components (i.e., the costs of executing R_j, R_j', B_j^{-1}), and communication needs between them if the computation is organized on a processor network. Also note that the above results address only the solving of symmetric positive definite operator equations. Extensions to non-symmetric and indefinite problems have been attempted, however, very little general theory can be expected from the simple SSS framework.

3. MULTILEVEL-PARTITION-OF-UNITY FRAMES

The partition-of-unity method (PUM) introduced by Babuška and co-workers[21] belongs to the class of meshless, particle-type numerical discretization methods. The starting point is a partition of unity (PU) $\Phi := \{\phi_i : i \in I\}$ of locally supported bump functions ϕ_i satisfying $\sum_{i\in I} \phi_i(x) \equiv 1$ on the domain $\Omega \subset \mathbb{R}^d$ of interest. Then, using a set of generic (e.g., polynomials of degree $\le k_i$) or problem-adapted shape functions $\{\phi_{il} : l \in I_i\}$ for each $i \in I$ one creates locally enriched spaces $U_i = \mathrm{span}\{\psi_{il} := \phi_i\phi_{il} : l \in I_i\}$. We call $U = \mathrm{span}\Psi$, where $\Psi := \{\psi_{il} : l \in I_i,\, i \in I\}$, a PUM space. An element $u \in U$ can be represented $u = \sum_{i\in I} u_i$ (often non-uniquely), the $u_i = \sum_l c_{il}\psi_{il} \in U_i$ can be interpreted as particles with support in $\mathrm{supp}\,\phi_i$, and the presence of additional shape functions ϕ_{il} allows for utmost flexibility in each particle position. If we set $W_i := \mathrm{span}\{\phi_{il} : l \in I_i\}$ and assume that W_i reproduces polynomials of degree $< k$ for all i then the PU property of Φ, together with other reasonable assumptions on Φ, more or less automatically guarantees good local approximation properties of order k for the PUM spaces, one of the desirable features of a discretization scheme. The other feature that appeals to practitioners is that smooth Φ can be constructed based on particle positions (i.e., point clouds rather than space partitions as in finite element methods) and overlapping space covers $\Omega \subset \cup_{i\in I}\Omega_i$ by simple formulas. Covers using balls, rectangles, ellipsoids, and multi-scale versions of the PUM have been proposed.[22,23,30]

Note that the *hp*-version of the finite element method[31] can be interpreted as PUM. For simplicity, let $\mathcal{T}$ be a simplicial partition of a polyhedral domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$. The nodal basis for the space $S_1(\mathcal{T})$ of linear C^0 finite elements represents a PU (sufficiently smooth to deal with, e.g., H^1-elliptic problems). For each vertex v_i of $\mathcal{T}$ we have one ϕ_i, with support Ω_i consisting of the union of all simplices attached to v_i. Let W_i be the restriction of $S_{k_i-1}(\mathcal{T})$ to the patch Ω_i if $k_i > 1$ resp. the one-dimensional space of all constant functions if k_i. Then the resulting PUM space represents a *hp*-version finite element space on $\mathcal{T}$ with local degree distribution $\{k_i\}$. Indeed, the local space $U_i := \phi_i W_i \subset U$ contains all Lagrange nodal basis functions from $S_{k_i}(\mathcal{T})$ with support in the interior of Ω_i. Thus, near v_i we have all discretization power of polynomials of degree $\le k_i$, and k_i is variable. This is the typical additional feature of the *hp*-version compared to the *h*-version (for simplicity, we have assumed that the latter, i.e., the local adaption of the size of the simplices, is included into the choice of $\mathcal{T}$). It allows for not only resolving local singularity behavior of solutions of standard operator equations by local mesh refinement but also capturing regions of high solution regularity by increasing k_i in an efficient way as in spectral methods. The adaptive multilevel PUM[23,24] pursues the same idea. There a whole sequence $\Phi_j = \{\phi_{ji}\}$ of PU for each $j \ge 0$ (with support size exponentially decreasing with j) is used, and U_{ji} is obtained by multiplying ϕ_{ji} with a space $W_{ji} = \mathbb{P}_{k_{ji}}$ of polynomials of degree $< k_{ji}$, where $k_{ji} \ge 0$ is a degree distribution with finitely many $k_{ji} > 0$ (if $k_{ji} = 0$ then $W_{ji} = U_{ji} = \{0\}$ is trivial, and can be dropped). The discretization space of interest is $U = \mathrm{span}(\cup_{j,i}U_{ji})$, and has similar features as the above finite element *hp*-spaces.

It is a challenging question to *find SSS for the above PUM spaces with uniform control on their condition*, to guarantee robust iterative solution procedures, the target being again $H^s(\Omega)$-elliptic problems. The main difficulty is to achieve robustness with respect to the degree distribution $\{k_i\}$ resp. $\{k_{ji}\}$. This problem has been addressed and partially solved for the original finite element *hp*-method and $H^1(\Omega)$- resp. $H^1_0(\Omega)$-elliptic operator equations on polyhedral domains in $\mathbb{R}^d$, $d = 2, 3$.[32,33] Simplifying a bit, a potentially good SSS for the above described *hp*-finite element PUM space U and $H^1(\Omega)$-elliptic $a(\cdot,\cdot)$ is given by

$$\{U, a\} = \{S_1(\mathcal{T}); a\} + \sum_i \{U_i, a\}.$$

Computationally, this SSS corresponds to local condensation of variables, i.e., to solving many $H^0_1(\Omega_i)$-elliptic problems w.r.t. U_i on the local patches (since almost all basis functions spanning U_i have overlapping support this leads to dense matrix inversion with matrices of dimension $\approx k_i^d$, a problem that needs extra scrutiny), and one global problem w.r.t. the standard linear finite element space $S_1(\mathcal{T})$ for which many robust solvers exist, among them SSS-based domain decomposition and multilevel algorithms. Numerical experiments for the adaptive multilevel PUM spaces are less conclusive.[34]

A rigorous treatment of the raised question on SSS for PUM spaces is also motivated by theoretical results[18–20] on wavelet frames composed of *quarks*. We consider the simplest case of quarks attached to uniform grids $2^{-j}\mathbb{Z}^d$ for $H^s(\mathbb{R}^d)$, $s > 0$. Let $\phi \in C^\infty(\mathbb{R}^d)$ be non-negative, with compact support in the interior of the positive quadrant $\mathbb{R}^d_+$, such that its $\mathbb{Z}^d$-shifts form a PU: $\sum_{\alpha\in\mathbb{Z}^d} \phi(x-\alpha) \equiv 1$ for $x \in \mathbb{R}^d$. Define the prototypical quarks attached to the point $0 \in \mathbb{Z}^d$ by $\psi^\beta(x) = x^\beta \phi(x) \geq 0$ for all $\beta \in \mathbb{Z}^d_+$, and apply dilation, shifts, and appropriate scaling to define the *quarkonial system*

$$\Psi_{q,s} = \{\psi^\beta_{j,\alpha}(x) := q^{|\beta|} 2^{-j(s-d/2)} \psi^\beta(2^j x - \alpha) : \alpha \in \mathbb{Z}^d,\ \beta \in \mathbb{Z}^d_+,\ j \geq 0\},$$

where $|\beta| = \sum_{i=1}^d \beta_i$ denotes the degree of the monomial x^β. Theorem 1 in Ref. 20 claims that *for any fixed* $0 < q \leq q_0$ *(with* $q_0 < 1$ *depending only on the support size of* ϕ*), and arbitrary* $s > 0$ *the system* $\Psi_{q,s}$ *is a frame for* $H^s(\mathbb{R}^d)$. Dual functionals for these frames that lead to (quasi-)optimal frame decompositions are also specified, similar results are available for the whole scale of Besov- and Triebel-Lizorkin spaces. At first glance, this is a surprising result about the frame property of a highly redundant system which raises hopes for finding suitable SSS of multilevel PUM spaces by selection and clustering operations. Indeed, as any frame, $\Psi_{q,s}$ is a SSS, and its subsystems

$$\Psi^{j,\{k_{j\alpha}\}}_{q,s} = \{\psi^\beta_{j,\alpha} : |\beta| < k_{j,\alpha},\ \alpha \in \mathbb{Z}^d\}$$

span a typical PUM space (for a uniformly spaced point cloud $2^{-j}\mathbb{Z}^d$, and with polynomial spaces $W_{j,\alpha} = \mathbb{P}_{k_{j,\alpha}}$) on level j. Further combining spaces with different $j \geq 0$ yields also multilevel PUM spaces. At the moment, there are no definite results on the behavior of the stability bounds and condition for these quarkonial subsystems of reasonable generality. Even the questions if the subsystems

$$\Psi^{J,k}_{q,s} = \{\psi^\beta_{j,\alpha} : \alpha \in \mathbb{Z}^d,\ |\beta| < k,\ 0 \leq j \leq J\}$$

are frames for their closed span in $H^s(\mathbb{R}^d)$ for all $k \geq k_0(s)$, $J \geq 0$, and how their condition behaves for $J \to \infty$ resp. $k \to \infty$, are open.

We finish with the following elementary observation that cautions our enthusiasm a bit. Assume that V is fixed, and that the systems $\Psi_k = \{\psi_{ki} : i \in I_k\}$, $k \geq 0$, are such that $\Psi^K = \cup_{k=0}^K \Psi_k$ is a frame for the whole V (and not only a frame system for a subspace of V), with optimal frame bounds $0 < A_K \leq B_K < \infty$, and condition $\kappa^K = B_K/A_K$, for each $K \geq 0$. We call $\{\Psi^K : K \geq 0\}$ a *hierarchical frame family* for V.

Lemma 4. Let $\{\Psi^K; K \geq 0\}$ be a hierarchical frame family in V. Then, for any $\hat\kappa > \inf_{K\geq 0} \kappa_K$, there exist a sequence of scaling factors $w = \{w_k > 0 : k \geq 0\}$ such that $\Psi^\infty_w := \cup_{k\geq 0} w_k \Psi_k$ is a frame in V with frame bounds satisfying $1 \leq A_w \leq B_w \leq \hat\kappa$.

Proof. Intuitively, this is obvious (just choose K such that $\hat\kappa > \kappa_K$, and set $w_k = A_K^{-1/2}$ for $k \leq K$, and choose sufficiently small w_k for $k > K$). We give the main steps using the original frame definition. I.e., we have

$$A_K \|v\|_V^2 \leq \inf_{v=\sum_{k=0}^K \sum_i c_{ki}\phi_{ki}} \sum_{k=0}^K \sum_i c_{ki}^2 \leq B_K \|v\|_V^2, \qquad v \in V, \quad K \geq 0.$$

We can assume that $\hat\kappa > \kappa_0$ (the general case $\hat\kappa > \kappa_K$ for some $K \geq 0$ can be reduced to this one by considering the new family $\{\tilde\Psi^{K'} = \Psi^{K+K'} : K' \geq 0\}$). Let w be arbitrary. Using the upper frame bound for $\Psi^0 = \Psi_0$, for any v we find a decomposition $v = \sum_i c_{0i} \cdot (w_0 \phi_{0i})$ w.r.t. $w_0 \Psi_0 \subset \Psi^\infty_w$ such that $\sum_i w_0^2 c_{0i}^2 = w_0^2 \sum_i c_{0i}^2 \leq B_0 \|v\|_V^2$. Thus, $B_w \leq w_0^{-2} B_0$ holds for the upper frame bound of Ψ^∞_w.

To estimate A_w, consider an arbitrary decomposition w.r.t. Ψ^∞_w and rewrite it it w.r.t. Ψ^K:

$$v = \sum_{k=0}^\infty \sum_i c_{ki} \cdot (w_k \phi_{ki}) = \sum_{K=0}^\infty \left(\sum_{k=0}^K \sum_i a^K_{ki} \phi_{ki}\right) \equiv \sum_{K=0}^\infty v_K,$$

where $\sum_{K=k}^\infty a^K_{ki} = w_k c_{ki}$ for all i and k. Take any sequence $\alpha_K > 0$ such that $\alpha_0 = 1$, and $C_\alpha := \sum_{K\geq 0} \alpha_K \leq \hat\kappa/\kappa_0$. Then, using the lower frame bounds for the Ψ^K, we get

$$\|v\|_V^2 = \|\sum_{K=0}^\infty v_K\|_V^2 \leq \left(\sum_{K=0}^\infty \alpha_K\right) \sum_{K=0}^\infty \alpha_K^{-1} \|v_K\|_V^2 \leq C_\alpha \sum_{K=0}^\infty (A_K \alpha_K)^{-1} \sum_{k=0}^K \sum_i (a^K_{ki})^2 = C_\alpha \sum_{k=0}^K \sum_i \left(\sum_{K=k}^\infty \frac{(a^K_{ki})^2}{A_K \alpha_K}\right).$$

Now we choose the a_{ki}^K, $K \geq k$, to minimize this bound. This gives

$$\min_{\sum_{K=k}^\infty a_{ki}^K = w_k c_{ki}} \sum_{K=k}^\infty \frac{(a_{ki}^K)^2}{A_K \alpha_K} \leq (\sum_{K=k}^\infty \alpha_K A_k)^{-1} w_k^2 c_{ki}^2.$$

Since $A_0 \geq A_1 \geq \ldots$, the involved series converges, and we set $w_k^2 := C_\alpha^{-1} \sum_{K=k}^\infty \alpha_K A_k$ for the weights. Taking into account the previous estimates, this gives the desired result

$$1 \leq A_w \leq B_w \leq C_\alpha B_0 (\sum_{K=0}^\infty \alpha_K A_k)^{-1} \leq C_\alpha B_0 A_0^{-1} = C_\alpha \kappa_0 < \hat{\kappa}.$$

.

In other words, independently of how badly the condition numbers κ_J of the Ψ^K behave, the union Ψ_w^∞ of all Ψ_k (if properly scaled) can be made as good as possible. E.g., the Lemma can be applied to the multilevel finite element frames $\Psi^K := \Phi_K$ discussed at the end of Sec. 2.1 if instead of the Lagrange nodal bases hierarchical finite element bases w.r.t. $S_{1,0}(\mathcal{T}) \subset S_{2,0}(\mathcal{T}) \subset \ldots$ are used for each $\mathcal{T} = \mathcal{T}_j$. Subsystems of Ψ_w^∞ would then span all possible *hp*-finite-element PUM spaces. To treat quarkonial frames, we would set

$$\Psi^K = \{2^{-j(s-d/2)} \psi^\beta (2^j x - \alpha) : \; |\beta| \leq k_0 + K, \, \alpha \in \mathbb{Z}^d, \, j \geq 0\},$$

with $k_0 = k_0(s)$ such that all Ψ^K are frames in $H^s(\mathbb{R}^d)$ (a fact that can be proved, under relatively weak assumptions on ϕ,[30] using the machinery of Jackson-Bernstein inequalities). The above shown scaling factors depend on j and s, and assure that the functions in Ψ^K have norms ≈ 1 in $H^s(\mathbb{R}^d)$. Using Lemma 4, the frame property of the described quarkonial system is not so surprising anymore since the exponentially fast decaying additional scaling factors $q^{|\beta|}$ take care of any (currently not well investigated) growth of condition numbers κ_K with increasing polynomial degree $k_0 + K$ of the involved PUM functions. This makes the posed problem, i.e., the study of good SSS for PUM spaces, even more interesting.

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