Nonlinear Multi-Scale Transforms: L_p Theory

Peter Oswald, Jacobs University Bremen
email: p.oswald@jacobs-university.de

Abstract

We treat the L_p theory ($1 \leq p < \infty$) for univariate nonlinear subdivision schemes and multi-scale transforms based on the concept of offset invariance and nonlinear derived subdivision operators. The paper covers convergence, smoothness, and stability issues, and complements the recent survey [8], where most of the L_p results have been included without proof.

1 Introduction

Nonlinear subdivision schemes and multi-scale transforms have recently attracted some attention, as means to battle certain deficiencies of linear methods, and to deal with nonlinearly constrained data sets (e.g., manifold-valued functions). Although most applications require multivariate parameterizations, the theory of multi-scale transforms has so far been developed predominantly in the univariate case, and for the uniform metric. The reason is that, on the one hand, most of the univariate results have straightforward extensions to shift-invariant multivariate grid topologies, a case which thus does not represent a serious mathematical challenge, and, on the other, existing nonlinear multi-scale algorithms, e.g., for the extraction of directional features in two- and three-dimensional data sets, require completely new ideas, and a new theoretical framework for their analysis. Concerning the concentration on the uniform metric, it is justified in areas such as geometry processing where local deviations and singularities can easily be spotted in a visualization and thus need to be controlled. Some other applications of multi-scale algorithms, such as modeling of densities with integral balances, image processing using BV type norms, or multi-level schemes for numerically solving elliptic problems in Sobolev spaces, necessitate results for the L_p setting ($1 \leq p < \infty$), especially with $p = 1$ and $p = 2$.

The present paper contributes to the latter topic in the functional setting, i.e., when the multi-scale transform applies to scalar data, and convergence and smoothness are treated using standard parameterizations of discrete real-valued data sequences by piecewise linear functions on uniform grids. It complements Section 2 of the survey [8], where most of the results below are formulated without proof, and a more detailed discussion of other approaches, the case $p = \infty$, and illustrating examples are given. In contrast to previous research [6, 13], where L_p theory is developed using quasi-uniform subdivision operators, we start with the notion of offset invariance for $\mathbb{P}_k$ originally introduced for $k = 1$ in [16], and extended to arbitrary $k \geq 1$ in [11]. This notion allows us to define nonlinear derived subdivision operators $S^{[m]}$ satisfying the commutator property $\Delta^m S = S^{[m]}\Delta^m$, $m = 1, \ldots, k$, and systematically work with them, similarly to the linear case. We establish close-to-final convergence and smoothness results in the Besov scale in terms of spectral radii of the $S^{[m]}$, and also show L_p stability following the recent paper [11] covering the case $p = \infty$. In the concluding section, the results are applied to the family of nonlinear power-q schemes.

2 Offset invariance and derived subdivision operators

Throughout the paper, we consider local, r-shift invariant, stationary multi-scale transforms, recursively acting on data sequences from $\ell_p(\mathbb{Z})$ $(1 \leq p \leq \infty)$ according to

$$v^{j-1} = Rv^j, \qquad d^j = D(v^j - Sv^{j-1}); \qquad v^j = Sv^{j-1} + Pd^j, \qquad j \geq 1, \tag{1}$$

with bounded but generally nonlinear operators $P, D, R, S : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$. For consistency in (1), the relation $(\mathrm{Id} - PD)(\mathrm{Id} - SR) = 0$ needs to hold (Id is the identity operator). Abusing a bit conventions, throughout this paper we call a (not necessarily linear) operator $T : X \to Y$ between two Banach space X and Y *bounded* if there is a constant C_0 such that $\|Tx\|_Y \leq C_0\|x\|_X$ for all $x \in X$, and *Lipschitz continuous* if there exists a constant C_1 such that $\|Tx - Tx'\|_Y \leq C_1\|x - y\|_X$ for all $x, y \in X$. The infimum of all such constants C_0 is denoted by $\|T\|_{X\to Y}$. That the operators in (1) are independent of the scale index $j \geq 1$ makes the scheme *stationary*, while *r-shift invariance* is equivalent to requiring

$$ST_1 = T_r S, \qquad RT_r = T_1 R, \qquad PT_1 = T_1 P, \qquad DT_1 = T_1 D,$$

where T_m, $m \in \mathbb{Z}$, is the shift operator given by $(T_m v)_i = v_{i+m}$, $i \in \mathbb{Z}$. Finally, *locality* of an r-shift invariant transform means that we can write

$$(Sv)_{ri+s} = \phi_s(v_{i-L_1}, \ldots, v_{i+L_2}), \qquad s = 0, \ldots, r-1, \quad i \in \mathbb{Z}, \tag{2}$$

with functions $\phi_s : \mathbb{R}^L \to \mathbb{R}$, where $L = L_1 + L_2 + 1$ is the support length of the subdivision part of the transform, and the integers L_1, L_2 are fixed. Similarly,

$$(Rv)_i = \phi_R(v_{ri-L_3}, \ldots, v_{ri+L_4}), \qquad i \in \mathbb{Z}, \tag{3}$$

and

$$(Dv)_i = \phi_D(v_{i-L_5}, \ldots, v_{i+L_6}), \qquad (Pv)_i = \phi_P(v_{i-L_7}, \ldots, v_{i+L_8}), \qquad i \in \mathbb{Z},$$

for some functions ϕ_R, ϕ_D, ϕ_P, and fixed integers L_s, $s = 3, \ldots, 8$. It is easy to see that due to locality and r-shift invariance, boundedness (Lipschitz continuity, C^1 property, ...) of S on $\ell_p(\mathbb{Z})$ spaces is equivalent to the boundedness (Lipschitz continuity, C^1 property, ...) of the coordinate functions ϕ_s representing S independently of p, similarly for R, D, and P. Moreover, the definition of these local operators automatically extends to the space $\ell(\mathbb{Z})$ of all real-valued sequences. We will always silently assume that $S\mathbf{0} = R\mathbf{0} = D\mathbf{0} = P\mathbf{0} = \mathbf{0}$, where $\mathbf{0}$ is the zero sequence given by $\mathbf{0}_i = 0$, $i \in \mathbb{Z}$.

Sometimes, especially if nonlinear schemes are considered as perturbations of associated linear schemes, the alternative representation

$$(Sv)_{ri+s} = \sum_{l=-L_2}^{L_1} a_{rl+s}(v_{i-L_1}, \ldots, v_{i+L_2})v_{i-l}, \qquad s = 0, \ldots, r-1,$$

or, equivalently,

$$(Sv)_j = \sum_{i \in \mathbf{Z}} a_{j-ri}(v|_{I_{[j/r]}})v_i, \qquad j \in \mathbb{Z}, \tag{4}$$

is chosen. To shorten the notation, by $v|_{I_i}$ we have denoted the restriction of v to the finite index set $I_i := \{i - L_1, \ldots, i + L_2\}$, $i \in \mathbb{Z}$. Coefficient functions with index $s \notin \{-rL_1, \ldots, r(L_2+1)-1\}$ vanish for all arguments: $a_s(\cdot) \equiv 0$. Based on (4) we can now formally write the action of S as an infinite matrix-vector product

$$Sv = S_v v, \tag{5}$$

where S_v is a bi-infinite, data-dependent matrix operator with entries identified from (4):

$$(S_v)_{j,i} := a_{j-ri}(v|_{I_{[j/r]}}), \qquad j, i \in \mathbb{Z}.$$

Note that for a linear S, the matrix operator S_v does not depend on v, and is given by the finitely supported sequence $a := \{a_l\}_{l \in \mathbf{Z}}$ called *mask* of the subdivision operator. The representation (4)-(5) was introduced in [6], and was the departure point for a systematic theory of data-dependent, so-called quasi-linear, subdivision schemes and multi-scale transforms developed in [6, 13]. The transition from (2) to (4)-(5) is not unique.

The concept of polynomial reproduction for subdivision operators is fundamental in the study of multi-scale transforms, therefore we start the exposition with it. For nonlinear S, there are two slightly different extensions of the familiar definition for linear subdivision operators. The first extension proposed in [6] uses the representation (4)-(5), and requires that each S_v reproduces polynomials of order k, we will not pursue it here. Alternatively, following [16, 11] we can introduce offset invariance . Throughout the section, we denote by $\mathbb{P}_k$ the set of algebraic polynomials of degree $< k$ or, equivalently, of order $\leq k$, and by $\mathbf{1}$ the constant sequence given by $\mathbf{1}_i := 1$, $i \in \mathbb{Z}$.

Definition 1 *A local r-shift invariant subdivision operator S is offset invariant for $\mathbb{P}_k$, $k \geq 1$, if for each $v \in \ell(\mathbb{Z})$, and any polynomial p of degree m, $0 \leq m < k$, there exists a polynomial q of degree $< m$ such that*

$$S(v + p|_{\mathbf{Z}}) = Sv + (p+q)|_{r^{-1}\mathbf{Z}}.$$

In particular, S is offset invariant for constants (i.e., the set $\mathbb{P}_1$) if

$$S(v + \alpha \mathbf{1}) = Sv + \alpha \mathbf{1}, \qquad \forall\, \alpha \in \mathbb{R}, \quad \forall\, v \in \ell(\mathbb{Z}).$$

Note that the formulation automatically ensures that offset invariance for $\mathbb{P}_k$, $k > 1$, implies offset invariance for $\mathbb{P}_m$ for all $1 \leq m < k$. The proof of Theorem 2 given below shows that Definition 1 is equivalent to a recursive one: $S = S^{[0]}$ is offset invariant for $k \geq 2$ if it is offset invariant for $\mathbb{P}_{k-1}$, and the scaled version of its associated $(k-1)$-st derived operator $\tilde{S}^{[k-1]} = r^{k-1} S^{[k-1]}$ is offset invariant for constants (which in turn guarantees the existence of the k-th derived subdivision operator $S^{[k]}$). For linear S, Definition 1 coincides with the usual definition of polynomial reproduction of order k, where the above property is required for $v = \mathbf{0}$. As many examples [8] indicate, offset invariance for $\mathbb{P}_k$ of a nonlinear scheme usually holds with $k = 1$ or $k = 2$, but no nonlinear examples of practical relevance are known for larger k. However, it is the right concept for the extension of the notion of *derived subdivision operators* to the nonlinear setting.

Theorem 2 *Let S be a local, r-shift invariant subdivision operator. If S is offset invariant for $\mathbb{P}_k$ for some integer $k \geq 1$ then there exist local, r-shift invariant derived subdivision operators $S^{[m]}$ such that*

$$\Delta^m S v = S^{[m]} \Delta^m v, \qquad \forall\, v \in \ell_p(\mathbb{Z}) \tag{6}$$

for $m = 1, 2, \ldots, k$. Moreover, if S is written in the form (2) then its derived subdivision operators $S^{[m]}$, $m = 1, \ldots, k$, inherit such a representation with the same (or smaller) L_1, L_2, and with functions $\phi_s^{[m]}$ that are obtained from the ϕ_s by superpositions involving only linear transformations. In particular, if S and thus the functions $\phi_s(\cdot)$ are bounded (continuous, Lipschitz continuous, C^1, …) then so are $S^{[m]}$ and the functions $\phi_s^{[m]}$. In particular, if S is bounded then

$$\|S^{[m]} w\|_{\ell_p(\mathbf{Z})} \leq r^{-m+1/p} \|w\|_{\ell_p(\mathbf{Z})} + C \|\Delta w\|_{\ell_p(\mathbf{Z})}, \tag{7}$$

and if S is Lipschitz continuous then

$$\|S^{[m]} w - S^{[m]} w'\|_{\ell_p(\mathbf{Z})} \leq r^{-m+1/p} \|w - w'\|_{\ell_p(\mathbf{Z})} + C \|\Delta(w - w')\|_{\ell_p(\mathbf{Z})}, \tag{8}$$

$m = 0, 1, \ldots, k-1$, with constants C independent of $w, w' \in \ell_p(\mathbb{Z})$.

Proof. The proof extends the standard argument for linear S, see [3, 9]. For $k = 1$ it was first given in [16, Theorem 2.5], see also [11, Lemma 2.1-2]. The case $k > 1$ was suggested in [11, Section 2.1] and can be obtained by induction from $k = 1$. For convenience we give the complete argument.

Let first $k = 1$. For any fixed $i \in \mathbb{Z}$, write

$$v_{i+l} = v_i + w_{i+l}, \qquad w_{i+l} = \begin{cases} \sum_{m=0}^{l-1}(\Delta v)_{i+m}, & l > 0, \\ 0, & l = 0, \\ -\sum_{m=l}^{-1}(\Delta v)_{i+m}, & l < 0. \end{cases}$$

Now apply the definition of off-set invariance for $k = 1$, and (2):

$$(Sv)_{ri+s} = (S(w + v_i\mathbf{1}))_{ri+s} = v_i + \phi_s(w_{i-L_1}, \ldots, w_{i+L_2}), \quad s = 0, \ldots, r-1, \tag{9}$$

note that w depends on the arbitrarily fixed $i \in \mathbb{Z}$. From this, we see that

$$\begin{aligned} (\Delta Sv)_{ri+s} &= \phi_{s+1}(w_{i-L_1}, \ldots, w_{i+L_2}) - \phi_s(w_{i-L_1}, \ldots, w_{i+L_2}) \\ &=: \phi_s^{[1]}((\Delta v)_{i-L_1}, \ldots, (\Delta v)_{i+L_2-1}) \end{aligned}$$

for $s = 0, \ldots, r-2$, and

$$\begin{aligned} (\Delta Sv)_{ri+r-1} &= (\Delta v)_i + \phi_0(-\sum_{m=-L_1}^{-1}(\Delta v)_{i+1+m}, \ldots, \sum_{m=0}^{L_2-1}(\Delta v)_{i+1+m}) \\ &\quad -\phi_{r-1}(w_{i-L_1}, \ldots, w_{i+L_2}) \\ &=: \phi_{r-1}^{[1]}((\Delta v)_{i-L_1}, \ldots, (\Delta v)_{i+L_2}), \end{aligned}$$

which yields the existence of $S^{[1]}$ (uniqueness is obvious). The argument also shows that boundedness and differentiability properties of the functions ϕ_s defining S via (2) automatically carry over to the functions $\phi_s^{[1]}$ defining $S^{[1]}$. Moreover, by $S = S^{[0]}$ and (9) the inequalities (7) and (8) automatically hold for $m = 0$ if S resp. the functions ϕ_s are bounded resp. Lipschitz continuous. This establishes Theorem 2 for $k = 1$.

Now, suppose that S is offset invariant for $\mathbb{P}_k$ for some $k > 1$ (thus also offset invariant for $\mathbb{P}_{k-1}$), and that the existence of $S^{[k-1]}$ is already established. Observe that $\tilde{S}^{[k-1]} := r^{k-1}S^{[k-1]}$ must be offset invariant for constants. Indeed, fix $i \in \mathbb{Z}$. For given w, find v such that $\Delta^{k-1}v_{i+l} = w_{i+l}$, $l = -L_1, \ldots, L_2$, where L_1, L_2 are as in the representation (2) of S resp. $S^{[k-1]}$. Also, set $p(x) = x^{k-1}/(k-1)! \in \mathbb{P}_k$, and note that $\Delta^{k-1}(p|_{\mathbf{Z}}) = r^{(k-1)}\Delta^{k-1}(p|_{r^{-1}\mathbf{Z}}) = \mathbf{1}$. Then

$$\begin{aligned} (\tilde{S}^{[k-1]}(w + \alpha\mathbf{1}))_{ri+s} &= r^{k-1}(S^{[k-1]}\Delta^{k-1}(v + \alpha p|_{\mathbf{Z}}))_{ri+s} \\ &= r^{k-1}(\Delta^{k-1}S(v + \alpha p|_{\mathbf{Z}}))_{ri+s} \\ &= r^{k-1}(\Delta^{k-1}(Sv + (\alpha p + q)|_{r^{-1}\mathbf{Z}}))_{ri+s} \\ &= r^{k-1}(\Delta^{k-1}Sv)_{ri+s} + \alpha r^{k-1}\Delta^{k-1}(p|_{r^{-1}\mathbf{Z}})_{ri+s} \\ &= (\tilde{S}^{[k-1]}w + \alpha\mathbf{1})|_{ri+s} \end{aligned}$$

holds for all $s = 0, \ldots, k-1$ (note that $\Delta^{k-1}(q|_{r^{-1}\mathbf{Z}}) = 0$ since $q \in \mathbb{P}_{k-1}$). This proves the offset invariance of $\tilde{S}^{[k-1]}$ for $\mathbb{P}_1$. Applying the already proved result for $m = 0$ to $\tilde{S}^{[k-1]}$ yields the desired $S^{[k]} := r^{-(k-1)}(\tilde{S}^{[k-1]})^{[1]}$ satisfying (6):

$$\begin{aligned} S^{[k]}\Delta^k &= r^{-(k-1)}((\tilde{S}^{[k-1]})^{[1]}\Delta)\Delta^{k-1} = r^{k-1}\Delta(\tilde{S}^{[k-1]}\Delta^{k-1}) \\ &= \Delta(S^{[k-1]}\Delta^{k-1}) = \Delta^k S. \end{aligned}$$

By repeatedly using the corresponding result for $k = 1$, properties of S such as boundedness, Lipschitz continuity, etc., are inherited by $S^{[k]}$. Similarly, the inequalities (7) and (8) also follow from the case $k = 1$. Indeed, by the above observation for any fixed $m = 0, \ldots, k-1$ the subdivision operator $\tilde{S}^{[m]} = r^m S^{[m]}$ is offset invariant for constants. Thus, by applying the result for $k = 1$ to it, we get from

$$\|S^{[m]}w\|_{\ell_p(\mathbf{Z})} = r^{-m}\|\tilde{S}^{[m]}w\|_{\ell_p(\mathbf{Z})} \leq r^{-m}(r^{1/p}\|w\|_{\ell_p(\mathbf{Z})} + C\|\Delta w\|_{\ell_p(\mathbf{Z})})$$

the desired (7) if S (and thus $S^{[m]}$ and $\tilde{S}^{[m]}$) are bounded while

$$\begin{aligned} \|S^{[m]}w - S^{[m]}w'\|_{\ell_p(\mathbf{Z})} &= r^{-m}\|\tilde{S}^{[m]}w - \tilde{S}^{[m]}w'\|_{\ell_p(\mathbf{Z})} \\ &\leq r^{-m}(r^{1/p}\|w - w'\|_{\ell_p(\mathbf{Z})} + C\|\Delta(w - w')\|_{\ell_p(\mathbf{Z})}) \end{aligned}$$

yields (8) under the assumption of Lipschitz continuity for S.

3 Convergence and Smoothness

In the univariate case, L_p convergence of the reconstruction part

$$v^j = Sv^{j-1} + Pd^j, \qquad j \geq 1, \tag{10}$$

of a multi-scale transform resp. the subdivision scheme

$$v^j = Sv^{j-1}, \qquad j \geq 1, \tag{11}$$

associated with it to a limit function $f^\infty \in L_p(\mathbb{R})$, and the L_p smoothness of the latter, is studied by associating with v^j its linear spline interpolant f^j on the grid $\Gamma^j = r^{-j}\mathbb{Z}$:

$$f^j(x) = (i + 1 - r^j x)v_i^j + (r^j x - i)v_{i+1}^j, \qquad r^j x \in [i, (i+1)), \qquad i \in \mathbb{Z}. \tag{12}$$

Alternatively, we can write $f^j = \sum_i v_i^j B_2(r^j \cdot -i)$ using linear B-splines series with the hat function $B_2(x) = (1 - |x|)_+$, or think of f^j as the limit of a linear interpolating subdivision process governed by the subdivision operator

$$(S_2 v)_{ri+s} = r^{-1}((r-s)v_i + sv_{i+1}), \qquad s = 0, \ldots, r-1, \quad i \in \mathbb{Z}.$$

Definition 3 *The multi-scale reconstruction algorithm (10) is called* L_p *convergent if, for any* $v^0 \in \ell_p(\mathbb{Z})$ *and detail sequences* $d^j \in \ell_p(\mathbb{Z})$ *satisfying*

$$\sum_{j\geq 1} r^{-j/p} \|d^j\|_{\ell_p(\mathbf{Z})} < \infty, \tag{13}$$

the corresponding sequence of linear spline interpolants f^j *converges in* $L^p(\mathbb{R})$ *to a limit function* $f^\infty \in L_p(\mathbb{R})$.
Similarly, the subdivision scheme (11) associated with S *is called* L_p *convergent if* $f^j \to f^\infty \not\equiv 0$ *in* $L_p(\mathbb{R})$ *for any* $\mathbf{0} \neq v^0 \in \ell_p(\mathbb{Z})$.

In applications to multi-scale solvers for operator equations [7, 5] and geometric modeling [9] the smoothness characteristics of the limits f^∞ matter. Smoothness of functions that are limits of approximation processes (in our case the recursively constructed sequences $\{f^j\}$ of linear splines) is conveniently measured in the scale of Besov spaces $B^s_{p,q}(\mathbb{R})$, see [13] for various equivalent definitions including the standard one based on moduli of smoothness: $f \in L_p(\mathbb{R})$ belongs to $B^s_{p,q}(\mathbb{R})$ for some $0 < s < k$, and $1 \leq p, q \leq \infty$ if

$$|f|_{B^s_{p,q}(\mathbf{R})} := \|r^{sj}\omega_k(r^{-j}, f)_{L_p(\mathbf{R})}\|_{\ell_q(\mathbf{Z}_+)} < \infty,$$

where

$$\omega_k(t, f)_{L_p(\mathbf{R})} = \sup_{0<h<t} \|\Delta^k_h f\|_{L_p(\mathbf{R})}, \qquad t > 0.$$

The choice of the integer $k > s$ is not critical, the resulting norms are equivalent for any two such integers.

We will deal only with the case $1 \leq p = q < \infty$, and give an alternative definition using an approximation-theoretic characterization of Besov spaces that is very convenient for our setup: For small values $0 < s \leq 1$ of the smoothness parameter a function $f \in L_p(\mathbb{R})$ belongs to $B^s_p(\mathbb{R}) := B^s_{p,p}(\mathbb{R})$ if and only if there exists at least one L_p convergent series representation

$$f = \sum_{j=0}^\infty h^j$$

where the functions h^j are linear splines on the grid $\Gamma^j = r^{-j}\mathbb{Z}$ (and thus possess a B-spline representation of the form $h^j = \sum_i c^j_i B_2(r^j \cdot -i))$ satisfying the constraint

$$\sum_{j=0}^\infty r^{spj} \|h^j\|^p_{L_p(\mathbf{R})} < \infty.$$

Moreover, we can define an equivalent norm in $B^s_p(\mathbb{R})$ by setting

$$\|f\|_{B^s_p(\mathbf{R})} := \inf \left(\sum_{j=0}^\infty r^{spj} \|h^j\|^p_{L_p(\mathbf{R})} \right)^{1/p}, \tag{14}$$

where the infimum is taken with respect to all such representations. For larger values $m < s \leq m+1$, where $m \geq 1$ is an integer, we have by induction $f \in B^s_p(\mathbb{R})$ iff $f^{(m-1)}$

is absolutely continuous, and $f^{(m)} \in B_p^{s-m}(\mathbb{R})$. In this case, an equivalent norm is given by

$$\|f\|_{B_p^s(\mathbf{R})} := \|f\|_{L_p(\mathbf{R})} + \|f^{(m)}\|_{B_p^{s-m}(\mathbf{R})}. \tag{15}$$

Proofs for these statements based on Jackson-Bernstein inequalities for linear splines and further references can be found in, e.g., [4, 14, 7]. Finally, note that for the important subcase $p = 2$, the scale $B_p^s(\mathbb{R})$, $s > 0$, coincides with the scale of Sobolev spaces $H^s(\mathbb{R}) = W_2^s(\mathbb{R})$.

Definition 4 *The subdivision scheme (11) associated with S possesses L_p smoothness $s > 0$ if it is L_p convergent, and limit functions satisfy $f^\infty \in B_p^s(\mathbb{R})$ for all $v^0 \in \ell_p(\mathbb{Z})$. The supremum of all such $s > 0$ is called L_p smoothness exponent of S, and denoted by $s_p(S)$.*

Theorem 5 *Let S be a local, r-shift invariant, bounded subdivision operator operator. Assume that S is offset invariant for $\mathbb{P}_k$ for some integer $k \geq 1$, and $1 \leq p < \infty$.*
a) If

$$\rho_{p,k}(S) = \rho_p(S^{[k]}) := \limsup_{j\to\infty} \|(S^{[k]})^j\|^{1/j}_{\ell_p(\mathbf{Z})\to\ell_p(\mathbf{Z})} < r^{1/p} \tag{16}$$

then S is L_p convergent, and

$$s_p(S) \geq \min(k, -\log_r(r^{-1/p}\rho_{p,k}(S))) > 0. \tag{17}$$

b) If, in addition, S is Lipschitz continuous, and P bounded, then (16) also implies the L_p convergence of the multi-scale reconstruction (10). Moreover, if s is non-integer, satisfies

$$0 < s < \min(k, -\log_r(r^{-1/p}\rho_{p,k}(S))),$$

and

$$\|\{v^0, d^j\}_{j\geq 1}\|_{p,s;r} := \left(\|v^0\|^p_{\ell_p(\mathbf{Z})} + \sum_{j=1}^{\infty} r^{j(sp-1)}\|d^j\|^p_{\ell_p(\mathbf{Z})}\right)^{1/p} < \infty,$$

then the limit function f^∞ belongs to $B_p^s(\mathbb{R})$, and

$$\|f\|_{B_p^s(\mathbf{R})} \leq C\|\{v^0, d^j\}_{j\geq 1}\|_{p,s;r} \tag{18}$$

We conjecture that the restriction on non-integer s in part b) can be removed (see the remark at the end of this section). For a discussion of the properties of the introduced spectral radii, and their connection to similar quantities defined in [6, 13] and to the smoothness equivalence conjecture [16, 15], we refer to [8]. We just mention some useful properties. Firstly, the characterization

$$\rho_{p,k}(S) = \inf_{j\geq 1} \|(S^{[k]})^j\|_{\ell_p(\mathbf{Z})\to\ell_p(\mathbf{Z})} = \inf\{\rho \,:\, \|\Delta^k S^j v\|_{\ell_p(\mathbf{Z})} \leq C_\rho \rho^j \|\Delta^k v\|_{\ell_p(\mathbf{Z})}\} \tag{19}$$

highlights the associated geometric decay property of the quantities $\|\Delta^k S^j v\|_{\ell_p(\mathbf{Z})}$ that is crucially used both in the proofs, and for obtaining numerical upper bounds for $\rho_{p,k}(S)$. Secondly, we have

$$\rho_{p,m}(S) \le \max(r^{-m+1/p}, \rho_{p,k}(S)), \qquad m = 1, \ldots, k-1. \tag{20}$$

Finally, we note that computing exact values for $\rho_{p,k}(S)$ (and similar spectral radii) is a hard problem in general, as it is equivalent to evaluating ℓ_p joint spectral radii of finite families of *nonlinear* maps acting on $\mathbb{R}^L$.

Proof. Theorem 5 is established following the same strategy as for proving similar statements for linear S (in the nonlinear setting see [13] and for $p = \infty$ also [1]). We give the argument for part b), the simple changes for part a) are discussed at the end. Due to (20), the assumption on $\rho_{p,k}(S)$ implies $\rho_{p,1}(S) < r^{1/p}$. Thus, we can assume $k = 1$. By definition, proving L_p convergence of (10) is the same as proving L_p convergence of the series $\sum_{j \ge 0} h^j$ composed of linear splines $h^0 = f^0$, $h^j = f^j - f^{j-1}$, $j \ge 1$, where f^j are the linear interpolants to the sequences v^j produced by (10). Recall that the B-spline subdivision operator S_2 is interpolating, and produces f^j as the limit function if recursively applied to v^j. Moreover, the shifts of the B-spline B_2 are L_p stable. This implies $\|h^0\|_{L_p(\mathbf{R})} \asymp \|v^0\|_{\ell_p(\mathbf{Z})}$, and

$$\|h^j\|_{L_p(\mathbf{R})} \le Cr^{-j/p}\|Sv^{j-1} + Pd^j - S_2 v^{j-1}\|_{\ell_p(\mathbf{Z})} \le Cr^{-j/p}(\|\Delta v\|_{\ell_p(\mathbf{Z})} + \|d^j\|_{\ell_p(\mathbf{Z})}), \tag{21}$$

$j \ge 1$. The first estimation step in (21) follows since the nodal values of the linear spline f^{j-1} on the refined grid $r^{-j}\mathbb{Z}$ are given by the sequence $S_2 v^{j-1}$, the second one from the triangle inequality combined with the boundedness of P and with

$$\|Sv - S_2 v\|_{\ell_p(\mathbf{Z})} \le C\|\Delta v\|_{\ell_p(\mathbf{Z})}, \qquad v \in \ell_p(\mathbb{Z}). \tag{22}$$

The latter inequality in turn follows since both S and S_2 are offset invariant for constants, use the representation (9).

Next we prove geometric decay of the quantities $A_j := r^{-j/p}\|w^j\|_{\ell_p(\mathbf{Z})}$, where $w_j := \Delta v^j$. Note that by definition (10)

$$\begin{aligned} w^j &= \Delta(Sv^{j-1} + Pd^j) = S^{[1]}w^{j-1} + \Delta P d^j = \\ &= (S^{[1]})^2 w^{j-2} + S^{[1]}(S^{[1]}w^{j-2} + \Delta P d^{j-1}) - S^{[1]}(S^{[1]}w^{j-2}) + \Delta P d^j \\ &\ \ \ldots \\ &= (S^{[1]})^n w^{j-n} + \sum_{l=1}^{n-1} (S^{[1]})^l (S^{[1]}w^{j-l-1} + \Delta P d^{j-l}) - (S^{[1]})^l (S^{[1]}w^{j-l-1}) + \Delta P d^j. \end{aligned}$$

Thus, using the boundedness of P and the Lipschitz continuity of $S^{[1]}$,

$$\|S^{[1]}w' - S^{[1]}w\|_{\ell_p(\mathbf{Z})} \le C_1\|w' - w\|_{\ell_p(\mathbf{Z})},$$

we get

$$\begin{aligned}\|w^j\|_{\ell_p(\mathbf{Z})} &\leq \|(S^{[1]})^n w^{j-n}\|_{\ell_p(\mathbf{Z})} + \sum_{l=1}^{n-1} (C_1)^l \|\Delta P d^{j-l}\|_{\ell_p(\mathbf{Z})} + \|\Delta P d^j\|_{\ell_p(\mathbf{Z})} \\ &\leq \|(S^{[1]})^n w^{j-n}\|_{\ell_p(\mathbf{Z})} + C \sum_{l=0}^{n-1} \|d^{j-l}\|_{\ell_p(\mathbf{Z})},\end{aligned}$$

where the constant C depends on n. According to the definition of the spectral radius $\rho_{p,1}(S)$, for any fixed ρ such that $\rho_{p,1}(S) < \rho < r^{1/p}$, there is an $n = n_\rho$ such that

$$\|(S^{[1]})^n w\|_{\ell_p(\mathbf{Z})} \leq \rho^n \|w\|_{\ell_p(\mathbf{Z})}.$$

Substituting into the previous inequality, we get

$$\|w^j\|_{\ell_p(\mathbf{Z})} \leq \rho^n \|w^{j-n}\|_{\ell_p(\mathbf{Z})} + C \sum_{l=0}^{n-1} \rho^l \|d^{j-l}\|_{\ell_p(\mathbf{Z})},$$

and, after multiplying by $r^{-j/p}$ and setting $\delta := \rho r^{-1/p} < 1$,

$$A_j \leq \delta^n A_{j-n} + C \sum_{l=0}^{n-1} \delta^l r^{-(j-l)/p} \|d^{j-l}\|_{\ell_p(\mathbf{Z})},$$

with a constant C depending on ρ. By recursion, if j is given by $j = nm + j'$, where $0 \leq j' < n$, then

$$A_j \leq \delta^{nm} A_{j'} + C \sum_{l=0}^{nm-1} \delta^l r^{-(j-l)/p} \|d^{j-l}\|_{\ell_p(\mathbf{Z})} \leq C(\delta^j \|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{i=1}^{j} \delta^{j-i} r^{-i/p} \|d^i\|_{\ell_p(\mathbf{Z})}), \quad (23)$$

where the last step in the estimate follows from the boundedness of S and P:

$$A_{j'} \leq 2r^{-j'/p} \|v^{j'}\|_{\ell_p(\mathbf{Z})} \leq C(\delta^{j'} \|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{i=1}^{j'} \delta^{j'-i} r^{-i/p} \|d^i\|_{\ell_p(\mathbf{Z})}),$$

for $j' = 0, 1, \ldots, n-1$, with another constant depending on ρ.

Thus, substituting into (21), we arrive at

$$\begin{aligned}\sum_{j=0}^{\infty} \|h^j\|_{L_p(\mathbf{R})} &\leq C(\|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{j=1}^{\infty} r^{-j/p} (\|w^j\|_{\ell_p(\mathbf{Z})} + \|d^j\|_{\ell_p(\mathbf{Z})}) \\ &\leq C \Bigg(\|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{j=1}^{\infty} r^{-j/p} \|d^j\|_{\ell_p(\mathbf{Z})} \\ &\qquad + \sum_{j=1}^{\infty} (\delta^j \|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{i=1}^{j} \delta^{j-i} r^{-i/p} \|d^i\|_{\ell_p(\mathbf{Z})}) \Bigg) \\ &\leq C \left(\|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{j=1}^{\infty} r^{-j/p} \|d^j\|_{\ell_p(\mathbf{Z})} + \sum_{i=1}^{\infty} \delta^{-i} r^{-i/p} \|d^i\|_{\ell_p(\mathbf{Z})} \sum_{j=i}^{\infty} \delta^j \right) \\ &\leq C \left(\|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{j=1}^{\infty} r^{-j/p} \|d^j\|_{\ell_p(\mathbf{Z})}) \right).\end{aligned}$$

This proves L_p convergence of the series $\sum_{j=0}^{\infty} h^j$ and consequently of the multi-scale reconstruction (10) under the stated assumption (13).

The remaining statements about the Besov smoothness of the limit function and the estimate (18) are proved by induction in k. Let first $k = 1$, and $0 < s < \min(1, -\log_r(r^{-1/p}\rho_{p,1}(S)))$. By our definition (14) of the norm in a Besov space, the limit function f belongs to $B_p^s(\mathbb{R})$ and (18) holds if we can show that the above decomposition $f^\infty = \sum_{j=0}^{\infty} h^j$ of f^∞ into linear splines satisfies

$$\sum_{j=0}^{\infty} r^{jsp}\|h^j\|_{L_p(\mathbf{R})}^p \le C\|\{v^0, d^j\}_{j\ge 1}\|_{p,s,r}^p \ ,$$

provided that the right-hand side of this inequality is finite. But this follows almost line-by-line from the proof of L_p convergence if we replace the quantities A_j by the quantities $\hat{A}_j := r^{j(s-1/p)}\|\Delta v^j\|_{\ell_p(\mathbf{Z})}$, choose $\epsilon > 0$ such that

$$r^{1/p} > r^{1/p-s} > \hat{\rho} := r^{1/p-s-\epsilon} > \max(\rho_{p,1}(S), r^{1/p-1})$$

(that this is possible follows from the assumption on s), and set $\hat{\delta} := \hat{\rho} r^{s-1/p} = r^{-\epsilon} < 1$. Then, (23) translates into

$$\hat{A}_j \le C(\hat{\delta}^j\|v^0\|_{\ell_p(\mathbf{Z})} + \sum_{i=1}^{j}\hat{\delta}^{j-i}r^{i(s-1/p)}\|d^i\|_{\ell_p(\mathbf{Z})}), \qquad j \ge 0.$$

Applying the inequality

$$(\sum_{i=0}^{j}\hat{\delta}^{j-i}a_i)^p \le C_{\rho,p}\sum_{i=1}^{j}\hat{\delta}^{j-i}a_i^p \qquad (a_i \ge 0),$$

we have

$$\hat{A}_i^p \le C(\hat{\delta}^j\|v^0\|_{\ell_p(\mathbf{Z})}^p + \sum_{i=1}^{j}\hat{\delta}^{j-i}r^{i(sp-1)}\|d^i\|_{\ell_p(\mathbf{Z})}^p), \qquad j \ge 0.$$

Substitution of this bound yields the desired estimate:

$$\begin{aligned}
\sum_{j=0}^{\infty} r^{jsp}\|h^j\|_{L_p(\mathbf{R})}^p &\le C\left(\|v^0\|_{\ell_p(\mathbf{Z})}^p + \sum_{j=1}^{\infty}(\hat{A}_{j-1}^p + r^{j(sp-1)}\|d^j\|_{\ell_p(\mathbf{Z})}^p)\right) \\
&\le C\left(\|v^0\|_{\ell_p(\mathbf{Z})}^p + \sum_{j=1}^{\infty} r^{j(sp-1)}\|d^j\|_{\ell_p(\mathbf{Z})}^p\right. \\
&\qquad \left. + \sum_{j=0}^{\infty}(\hat{\delta}^j\|v^0\|_{\ell_p(\mathbf{Z})}^p + \sum_{i=1}^{j}\hat{\delta}^{j-i}r^{-i/p}\|d^i\|_{\ell_p(\mathbf{Z})}^p)\right) \\
&\le C\left(\|v^0\|_{\ell_p(\mathbf{Z})}^p + \sum_{j=1}^{\infty} r^{j(sp-1)}\|d^j\|_{\ell_p(\mathbf{Z})}^p + \sum_{i=1}^{\infty}\hat{\delta}^{-i}r^{i(sp-1)}\|d^i\|_{\ell_p(\mathbf{Z})}\sum_{j=i}^{\infty}\hat{\delta}^j\right) \\
&\le C\|\{v^0, d^j\}_{j\ge 1}\|_{p,s,r}^p < \infty.
\end{aligned}$$

This proves the result for $k = 1$ for $1 \leq p < \infty$.

The induction step for the case $k > 1$ is shown for $k = 2$, it can be recursively repeated without difficulty. We can concentrate on the case $\rho_{p,2}(S) < r^{-1+1/p}$, and assume $1 \leq s < \min(2, -\log_r(r^{-1/p}\rho_{p,2}(S))$. Indeed, if $\rho_{p,2}(S) \geq r^{-1+1/p}$ then $\rho_{p,1}(S) = \rho_{p,2}(S) \geq r^{-1+1/p}$ by (20), and the already established result for $k = 1$ applies. Similarly for the case $\rho_{p,2}(S) < r^{-1+1/p}$ and $s < 1$, since then $\rho_{p,1}(S) \leq r^{-1+1/p} < r^{-s+1/p}$, and again the result for $k = 1$ is applicable. Consider $\tilde{S}^{[1]} := rS^{[1]}$. Obviously, this operator satisfies the conditions of the theorem with $k = 1$ and s replaced by $\tilde{s} = s - 1$ since $\rho_{p,1}(\tilde{S}^{[1]}) = r\rho_{p,2}(S)$ and thus

$$0 < \tilde{s} < \min(2, -\log_r(r^{-1/p}\rho_{p,2}(S)) - 1 = \min(1, -\log_r(r^{-1/p}\rho_{p,2}(\tilde{S}^{[1]})).$$

Thus, if we consider the reconstruction algorithm

$$w^j = \tilde{S}^{[1]}w^{j-1} + r\Delta P d^j, \qquad , w^0 = \Delta v^0,$$

then its limit function g belongs to $B_p^{s-1}(\mathbb{R})$, and

$$\|g\|_{B_p^{s-1}(\mathbf{R})} \leq C\|\{\Delta v^0, rd^j\}_{j\geq 1}\|_{p,s-1,r}^p \leq C\|\{v^0, d^j\}_{j\geq 1}\|_{p,s,r}^p.$$

Now, from the definition of $\tilde{S}^{[1]}$ and the recursion for w^j it is obvious that $w^j = r^j\Delta v^j$. Thus, the piecewise constant functions $g^j := \sum_i w_i^j B_1(r^j \cdot -i)$, where B_1 is the characteristic function of $[0.1)$, represent the derivatives of the piecewise linear interpolants f^j associated with the sequence v^j. From this and the already established L_p convergence $f^j \to f$, it is not difficult to see that $g = f'$ in $L_p(\mathbb{R})$. The result the follows since for $s > 1$

$$\|f\|_{B_p^s(\mathbf{R})} \leq C(\|f\|_{L_p(\mathbf{R})} + \|f'\|_{B_p^{s-1}(\mathbf{R})}).$$

The changes for part a) are minor. The proof can be repeated by setting $d^j = \mathbf{0}$. This implies $w^j = (S^{[1]})^n w^{j-n}$ without any reference to the Lipschitz stability of S, and convergence and smoothness estimates follow for $k = 1$. The induction argument for $k > 1$ is also the same (note that for the proof of (17) the consideration of integer s is not necessary due to the monotonicity of the Besov scale with respect to s). This finishes the proof of Theorem 5.

We end this section with the following remark. The restriction to non-integer s in part b) of Theorem 5 is due to the method of proof (as a matter of fact, the restriction has not been mentioned in [8], nor does it appear in [13] although the proof given there also does not cover integer values of s). The technical hurdle is the crude estimate (22) which suffices for the case $0 < s < k = 1$ but not for $s = 1$ and $k = 2$. What would be needed is a replacement of $\|\Delta v\|_{\ell_p(\mathbf{Z})}$ by $\|\Delta^2 v\|_{\ell_p(\mathbf{Z})}$ in the right-hand side of the inequality in (22). Such a replacement is obvious if S is interpolatory, or if the construction of f^j by linear B-spline series interpolating v^j on $r^{-j}\mathbb{Z}$, as done in this paper, is modified by introducing an appropriate shift (this trick works for linear S). We believe that this gap in the statement of Theorem 5 can also be closed in the nonlinear case.

4 Stability

Stability of multi-scale transforms, i.e., the robustness with respect to small perturbations, is not a major issue for linear schemes since convergence of a linear subdivision scheme implies stability. However, for nonlinear schemes stability is by no means obvious, and deserves consideration. In this section, we consider only Lipschitz stability in $\ell_p(\mathbb{Z})$ of the simplified version (1) of a nonlinear multi-scale transform.

Definition 6 *The reconstruction algorithm (10) is called L_p stable if there is a constant C_U such that*

$$r^{-J/p}\|v^J-\tilde{v}^J\|_{\ell_p(\mathbf{Z})} \le C_U(\|v^0-\tilde{v}^0\|_{\ell_p(\mathbf{Z})}+\sum_{j=1}^{J} r^{-j/p}\|d^j-\tilde{d}^j\|_{\ell_p(\mathbf{Z})})$$

holds for all $v^0, \tilde{v}^0 \in \ell_p(\mathbb{Z})$, and $J \ge 1$.
The subdivision algorithm (11) is called L_p stable if there is a constant C_S such that

$$r^{-J/p}\|v^J-\tilde{v}^J\|_{\ell_p(\mathbf{Z})} \le C_S\|v^0-\tilde{v}^0\|_{\ell_p(\mathbf{Z})}$$

holds for all $v^0, \tilde{v}^0 \in \ell_p(\mathbb{Z})$, and $J \ge 1$.
For all these definitions it is assumed that the associations

$$\begin{array}{ccc} v^J & \longleftrightarrow & \{v^0, d^1, \dots, d^J\} \\ \tilde{v}^J & \longleftrightarrow & \{\tilde{v}^0, \tilde{d}^1, \dots, \tilde{d}^J\} \end{array}$$

are given by the corresponding recursions in (1), where in the subdivision case detail sequences are set to $\mathbf{0}$.

For a brief discuss of the decomposition part of the multi-scale transform (1), see [8]. We stick to the above finite-dimensional versions of L_p stability since in this form they are valuable for realistic algorithms, e.g., for compression based on detail thresholding. We also note that the inclusion of the fore-factors $r^{-j/p}$ is dictated by the interpretation of the sequences v^j as coarse-scale representations of an L_p limit function. Indeed, assuming L_p convergence, the stability of (10) implies

$$\|f^\infty-\tilde{f}^\infty\|_{L_p(\mathbf{R})} \le C_U(\|v^0-\tilde{v}^0\|_{\ell_p(\mathbf{Z})}+\sum_{j=1}^{\infty} r^{-j/p}\|d^j-\tilde{d}^j\|_{\ell_p(\mathbf{Z})})$$

for the L_p limits of the associated sequences $\{f^j\}_{j\ge 0}$ and $\{\tilde{f}^j\}_{j\ge 0}$.

General results on the L_p stability of the multi-scale reconstruction (10) and of the subdivision scheme (11) have been developed in [6, 13] for $1 \le p \le \infty$ (note that these papers consider the limit case $J \to \infty$, and also the Besov space setting), and more recently for $p = \infty$ in [11] and [1] (see also [12, 2] for earlier stability results). We extend the result from [11] to the range $1 \le p < \infty$. We state it for $k = 1$.

Theorem 7 *Let S be an r-shift invariant, local, offset invariant for $\mathbb{P}_1$, and Lipschitz continuous subdivision operator, and P be Lipschitz continuous. Then the existence of a $0 < \rho < 1$ and some integer $n \geq 1$ such that*

$$r^{-n/p}\|\Delta(v^n - \tilde{v}^n)\|_{\ell_p(\mathbf{Z})} \leq \rho\|\Delta(v^0 - \tilde{v}^0)\|_{\ell_p(\mathbf{Z})} + C\sum_{l=1}^{n} r^{-l/p}\|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})} \tag{24}$$

for any two sets $\{v^0, d^j\}$, $\{\tilde{v}^0, \tilde{d}^j\}$ of multi-scale data implies that the multi-scale reconstruction (10) is L_p stable.
If (24) holds in the special case when $v^0, \tilde{v}^0 \in \ell_p(\mathbb{Z})$ are arbitrary but $d^j = \tilde{d}^j = \mathbf{0}$, $j = 1, \ldots, n$, then the subdivision scheme (11) is L_p stable.

Proof. From the assumptions and (8) with $m = 0$ we have

$$\begin{aligned}
&r^{-J/p}\|v^J - \tilde{v}^J\|_{\ell_p(\mathbf{Z})} \\
&\quad \leq r^{-J/p}(\|Sv^{J-1} - S\tilde{v}^{J-1}\|_{\ell_p(\mathbf{Z})} + \|Pd^J - P\tilde{d}^J\|_{\ell_p(\mathbf{Z})} \\
&\quad \leq r^{-J-1/p}\|v^{J-1} - \tilde{v}^{J-1}\|_{\ell_p(\mathbf{Z})} + Cr^{-J/p}(\|\Delta(v^{J-1} - \tilde{v}^{J-1})\|_{\ell_p(\mathbf{Z})} + \|d^J - \tilde{d}^J\|_{\ell_p(\mathbf{Z})}) \\
&\qquad \ldots \\
&\quad \leq \|v^0 - \tilde{v}^0\|_{\ell_p(\mathbf{Z})} + C\sum_{j=1}^{J} r^{-j/p}\|d^j - \tilde{d}^j\|_{\ell_p(\mathbf{Z})} + C\sum_{j=0}^{J-1} r^{-l/p}\|\Delta(v^l - \tilde{v}^l)\|_{\ell_p(\mathbf{Z})}.
\end{aligned}$$

To see that the last sum can is majorized by the remaining terms in this upper bound, we now explore (24). The reasoning is analogous to our estimates for the L_p convergence proof for Theorem 5. Write $j = sn + j'$ with $j' = 0, \ldots, r-1$, and denote $\tilde{\rho} := \rho^{1/n}$. Then

$$\begin{aligned}
&r^{-j/p}\|\Delta(v^j - \tilde{v}^j)\|_{\ell_p(\mathbf{Z})} \\
&\qquad \leq \rho r^{-(j-n)/p}(\|\Delta(v^{j-n} - \tilde{v}^{j-n})\|_{\ell_p(\mathbf{Z})} + C\sum_{l=j-n+1}^{j} r^{-l/p}\|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})} \\
&\qquad \leq \tilde{\rho}^n r^{-(j-n)/p}(\|\Delta(v^{j-n} - \tilde{v}^{j-n})\|_{\ell_p(\mathbf{Z})} + C\sum_{l=j-n+1}^{j} \tilde{\rho}^{j-l} r^{-l/p}\|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})} \\
&\qquad\quad \ldots \\
&\qquad \leq \tilde{\rho}^{sn}\|\Delta(v^{j'} - \tilde{v}^{j'})\|_{\ell_p(\mathbf{Z})} + C\sum_{l=j'+1}^{j} \tilde{\rho}^{j-l} r^{-l/p}\|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})} \\
&\qquad \leq C\left(\tilde{\rho}^j\|v^0 - \tilde{v}^0\|_{\ell_p(\mathbf{Z})} + C\sum_{l=1}^{j} \tilde{\rho}^{j-l} r^{-l/p}\|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})}\right).
\end{aligned}$$

Here, the last step comes from recursively estimating finitely many expressions $\|\Delta(v^{j'} - \tilde{v}^{j'})\|_{\ell_p(\mathbf{Z})}$, $j' = 0, \ldots, n-1$, using the Lipschitz continuity of S and P, clearly, the constant C depends on the fixed values of $\rho < 1$ and n. With this at hand, it remains to substitute and change order of summation:

$$\begin{aligned}
&\sum_{j=0}^{J-1} r^{-l/p} \|\Delta(v^l - \tilde{v}^l)\|_{\ell_p(\mathbf{Z})} \\
&\qquad \le C \sum_{j=0}^{J-1} \left(\tilde{\rho}^j \|v^0 - \tilde{v}^0\|_{\ell_p(\mathbf{Z})} + C \sum_{l=1}^{j} \tilde{\rho}^{j-l} r^{-l/p} \|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})} \right) \\
&\qquad \le C \left(\|v^0 - \tilde{v}^0\|_{\ell_p(\mathbf{Z})} \sum_{j=0}^{J-1} \tilde{\rho}^j + \sum_{l=1}^{J-1} \tilde{\rho}^{-l} r^{-l/p} \|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})} \sum_{j=l}^{J-1} \tilde{\rho}^j \right) \\
&\qquad \le C \left(\|v^0 - \tilde{v}^0\|_{\ell_p(\mathbf{Z})} + \sum_{l=1}^{J-1} r^{-l/p} \|d^l - \tilde{d}^l\|_{\ell_p(\mathbf{Z})} \right).
\end{aligned}$$

This shows the desired estimate, and the proof of the L_p stability of (10) is complete. The L_p stability of (11) follows under the stated conditions if one repeats the above argument with $d^j = \tilde{d}^j = \mathbf{0}$.

The statement of this theorem carries over to $k > 1$ if an estimate of the form

$$\|Sv - Sw\|_{\ell_p(\mathbf{Z})} \le r^{1/p} \|v - w\|_{\ell_p(\mathbf{Z})} + C \|\Delta^k (v - w)\|_{\ell_p(\mathbf{Z})} \tag{25}$$

can be established, and (24) holds with Δ replaced by Δ^k, see [11]. Moreover, in [11] the condition (24) has been replaced by a spectral radius estimate on the derivatives of derived subdivision operators. For the sake of simplicity we assume that S and thus all $S^{[m]}$, $m \le k$, are C^1, i.e., all functions ϕ_s in the definition (2) of S possess uniformly bounded, continuous partial derivatives, and refer to [11] for the exact conditions of piecewise continuous differentiability under which the statement can be proved. Denote by $D_v S^{[k]}$ the Frechet derivative of $S^{[k]}$ at $v \in \ell_p(\mathbb{Z})$. Due to our simplifying assumptions, the linear operator family $D_v S^{[k]} : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ depends continuously on v. Now define the spectral radii $\rho_{p,k}^{stab}(S) = \rho_p^{stab}(S^{[k]})$ and $\tilde{\rho}_{p,k}^{stab}(S) = \tilde{\rho}_p^{stab}(S^{[k]})$ as follows:

$$\rho_p^{stab}(S^{[k]}) := \limsup_{j \to \infty} \sup_{w \in \ell_p(\mathbf{Z})} \|DS^{[k]}_{(S^{[k]})^{j-1}w} DS^{[k]}_{(S^{[k]})^{j-2}w} \cdots DS^{[k]}_{w}\|^{1/j}_{\ell_p(\mathbf{Z}) \to \ell_p(\mathbf{Z})}, \tag{26}$$

and

$$\tilde{\rho}_p^{stab}(S^{[k]}) := \limsup_{j \to \infty} \sup_{w^0, w^1, \ldots, w^{j-1} \in \ell_p(\mathbf{Z})} \|DS^{[k]}_{w^{j-1}} DS^{[k]}_{w^{j-2}} \cdots DS^{[k]}_{w^0}\|^{1/j}_{\ell_p(\mathbf{Z}) \to \ell_p(\mathbf{Z})}. \tag{27}$$

Note that in (27) the supremum is taken with respect to an arbitrary collection of w^l, $l = 0, \ldots, j-1$, while in (26) it is taken with respect to a single w. Set $w^l = (S^{[k]})^l w$, $l = 0, \ldots, j-1$, to see that

$$\rho_{p,k}^{stab}(S) \le \tilde{\rho}_{p,k}^{stab}(S).$$

As demonstrated in [11] for the dyadic median interpolating scheme, this inequality can be strict. On the other hand, in [10, Lemma 4.2] it was observed that

$$\lim_{n \to \infty} \tilde{\rho}_p^{stab}((S^{[k]})^n)^{1/n} = \rho_{p,k}^{stab}(S),$$

a property that is useful for establishing approximation results, see [10, 8]. The proof of the following theorem, and its generalization to certain classes of piecewise-differentiable S, is given in [11, Section 2.3] in the case $p = \infty$, it straightforwardly carries over to $1 \le p < \infty$.

Theorem 8 *Let S be an r-shift invariant, local, C^1 continuous subdivision operator, and P be bounded and Lipschitz continuous. In addition, assume that S is offset invariant for $\mathbb{P}_k$, and that (25) holds. The multi-scale reconstruction (10) is L_p stable if $\tilde{\rho}^{stab}_{p,k}(S) < r^{1/p}$, and the subdivision algorithm (11) is L_p stable if $\rho^{stab}_{p,k}(S) < r^{1/p}$.*

We note that under mild conditions L_p stability fails to hold when the corresponding inequalities are reversed. The C^1 assumption on S can be replaced by the weaker k-differentiability, as done in [11] for $p = \infty$. This extension is necessary to deal with the example considered in the next section.

5 Example

For future reference, we want to summarize the currently known results on L_p convergence, Besov smoothness, and stability for the family of nonlinear interpolating power-q subdivision operators. They represent a simple enough test case for the application of general theories where we despite its simplicity do not yet have final answers to all questions.

The power-q subdivision operator is defined by

$$(Sv)_{2i} = v_i, \qquad (Sv)_{2i+1} = \frac{v_i + v_{i+1}}{2} - \frac{1}{8} H_q(\Delta^2 v_{i-1}, \Delta^2 v_i), \qquad i \in \mathbb{Z}, \tag{28}$$

where the so-called limiter H_q is defined by

$$H_q(x,y) = \begin{cases} \frac{x+y}{2}\left(1 - \left|\frac{x-y}{x+y}\right|^q\right), & xy > 0, \\ 0, & xy \le 0. \end{cases} \tag{29}$$

The parameter $q \in [1, +\infty)$ is fixed. References and basic facts can be found in [8]. The most studied case is $q = 2$. Obviously, S realizes a nonlinear, data-dependent interpolation between the subdivision rule S_2 which results if $(\Delta^2 v_{i-1})(\Delta^2 v_i) \le 0$, and the standard Deslauriers-Dubuc 4-point scheme the definition of which coincides with the case $\Delta^2 v_{i-1} = \Delta^2 v_i$ in (28). The S of the power-q family are Lipschitz continuous for all $q \in [1, \infty)$, are 2-shift invariant, and offset invariant for $\mathbb{P}_2$.

Thus, both $S^{[1]}$ and $S^{[2]}$ are well-defined, and our theorems can be used with either $k = 1$ or $k = 2$:

$$\begin{array}{rl} (S^{[1]}w)_{2i} & = \frac{w_i}{2} - \frac{1}{8} H_q(\Delta w_{i-1}, \Delta w_i), \\ (S^{[1]}w)_{2i+1} & = \frac{w_i}{2} + \frac{1}{8} H_q(\Delta w_{i-1}, \Delta w_i), \\ (S^{[2]}w)_{2i} & = \frac{1}{4} H_q(w_{i-1}, w_i), \\ (S^{[2]}w)_{2i+1} & = \frac{w_i}{2} - \frac{1}{8}(H_q(w_{i-1}, w_i) + H_q(w_i, w_{i+1})), \end{array} \qquad i \in \mathbb{Z}. \tag{30}$$

By working with $S^{[1]}$, the L_p convergence of the power-q multi-scale reconstruction algorithm with $P = \mathrm{Id}$ (and thus also the L_p convergence of the associated subdivision scheme) can easily be established. By Theorem 5 it is sufficient to show that $\rho_{p,1}(S) < 2^{1/p}$ which evidently follows from the stronger statement

$$\|S^{[1]}\|_{\ell_p(\mathbf{Z})\to\ell_p(\mathbf{Z})} < 2^{1/p}, \qquad 1 \le p < \infty.$$

Indeed, since

$$\frac{1}{8}|H_q(\Delta w_{i-1}, \Delta w_i)| \le \frac{1}{16}|\Delta w_{i-1} + \Delta w_i| \le \frac{1}{16}(|w_{i-1}| + |w_{i+1}|),$$

we have

$$\|S^{[1]}w\|^p_{\ell_p(\mathbf{Z})} \le 2\|\frac{1}{2}|w_i| + \frac{1}{16}(|w_{i-1}| + |w_{i+1}|)\|^p_{\ell_p(\mathbf{Z})} \le 2(\frac{5}{8}\|w\|_{\ell_p(\mathbf{Z})})^p,$$

independently of $q \in [1,\infty)$.

It was conjectured in [8] that a much sharper result holds, namely, that for all $q \in [1,\infty)$

$$\rho_{p,2}(S) = \frac{1}{2} \quad\Longrightarrow\quad s_p(S) = 1 + \frac{1}{p}, \qquad 1 \le p \le \infty. \tag{31}$$

The upper bound $s_p(S) \le 1 + \frac{1}{p}$ is obvious since the limit function $S^\infty\delta$ for the δ-sequence is the linear B-spline B_2, independently of q, and $\omega_2(t, B_2)_{L_p(\mathbf{R})} \asymp t^{1+1/p}$. The lower bound in (31) follows from the stronger inequality

$$\|S^{[2]}\|_{\ell_p(\mathbf{Z})\to\ell_p(\mathbf{Z})} = \frac{1}{2}, \qquad 1 \le p < \infty, \tag{32}$$

for the range $1 \le q \le 2$ (for $p = \infty$, this equality holds for all q, see [8]).

Indeed, for this range of values the q-power scheme is convexity preserving: If a finite segment of v is convex, i.e., for $w = \Delta^2 v$ we have $w_{j-1}, w_{j'+1} \le 0 < w_i$, $i = j, \ldots, j'$, for some $j \le j'$, then Sv remains convex there, more precisely

$$(S^{[2]}w)_{2j} = (S^{[2]}w)_{2j'+2} = 0, \qquad (S^{[2]}w)_i > 0, \qquad i = 2j+1, \ldots, 2j'+1.$$

Similarly for concave segments, with all inequality signs reversed. This follows from the elementary inequality $0 < H_q(x,y) = -H_q(-x,-y) < q\min(x,y)$, $x, y > 0$. Since any finitely supported v decomposes into finitely many convex/concave segments, in order to prove (32) it is enough to prove the corresponding ℓ_p inequality for a finite convex segment, i.e., to show that

$$\sum_{i=j}^{j'}(\frac{w_i}{2} - x_{i-1} - x_i)^p + \sum_{i=j}^{j'-1}(2x_i)^p \le \sum_{i=j}^{j'}(\frac{w_i}{2})^p,$$

where the notation

$$0 < x_i := \frac{1}{8}H_q(w_i, w_{i+1}) < \frac{1}{4}\min(w_i, w_{i+1}), \qquad i = j, \ldots, j'-1,$$

is used ($x_{j-1} = x_{j'} = 0$). But this follows by induction in the length $j'-j$ of the underlying index segment $[j,j']$. The case of a length $j'-j=0$ segment is trivial. Suppose $j'-j>0$. Since

$$x_j < 2x_j, \frac{1}{2}w_j - x_j < \frac{1}{2}w_j, \qquad 0 < x_j, \frac{1}{2}w_{j+1} - x_{j=1} - x_j < \frac{1}{2}w_{j+1} - x_{j+1},$$

the convexity of $\phi(t) = |t|^p$ implies

$$(\frac{1}{2}w_j - x_j)^p + (2x_j)^p \le (\frac{1}{2}w_j)^p + x_j^p$$

and

$$x_j^p + (\frac{1}{2}w_{j+1} - x_{j+1} - x_j)^p \le (\frac{1}{2}w_{j+1} - x_{j+1})^p.$$

Substitution yields

$$(\frac{1}{2}w_j - x_j)^p + (2x_j)^p + (\frac{1}{2}w_{j+1} - x_{j+1} - x_j)^p \le (\frac{1}{2}w_j)^p + (\frac{1}{2}w_{j+1} - x_{j+1})^p.$$

This reduces the proof of the above ℓ_p estimate for the index segment $[j,j']$ to the case of the shorter index segment $[j+1,j']$, and the induction is complete.

For $q>2$, it is easy to see that (32) is not necessarily true anymore (e.g., consider $p=1$, the finitely supported sequence v given by $v_0 = -(1+\epsilon)$, $v_1 = v_{-1} = -1$, $v_i = 0$, $|i| \ge 2$, and choose $\epsilon > 0$ small enough. Thus, (31) is formally established for $1 \le q \le 2$, and still open for $2 < q < \infty$.

Concerning stability, since we know that S is 2-differentiable (see [11, Section 3.2]), and satisfies (25), Theorem 8 is applicable with $k=2$. By crude estimates, we can show that

$$\|D_v S^{[2]}\|_{\ell_p(\mathbf{Z})\to\ell_p(\mathbf{Z})} < 2^{1/p}, \qquad 1 \le p < \infty, \tag{33}$$

holds, independently of v, for a certain range $q \in [1, q_p)$, where $2 < q_p < 4$. this obviously implies $\tilde{\rho}_{p,2}^{stab}(S) < 2^{1/p}$. For $p=\infty$, the latter inequality was established for $q \in [1, 8/3)$ in [11], there it was also shown that $\tilde{\rho}_{\infty,2}^{stab}(S) > 1$ if $q > 4$.

To prove (33), we assume $q \in [1,4]$, and set $a := q/4 \in [1/4, 1]$. From [11, Section 3.2] we quote the following properties of the matrix representation of $D_v S^{[2]}$:

$$\begin{aligned} |(D_v S^{[2]}u)_{2i}| &\le 2\alpha_i |u_{i-1}| + 2\alpha_i' |u_i|, \\ |(D_v S^{[2]}u)_{2i+1}| &\le \alpha_i |u_{i-1}| + (1/2 - \alpha_i' - \alpha_{i+1})|u_i| + \alpha_{i+1}' |u_{i+1}|, \end{aligned}$$

where the data-dependent coefficients α_i, α_i' satisfy $0 < \alpha_i < \alpha_i + \alpha_i' < a/2$. For even indices, we get

$$\sum_{i\in\mathbf{Z}} |(D_v S^{[2]}u)_{2i}|^p \le a^{p-1} \sum_{i\in\mathbf{Z}} 2\alpha_i |u_{i-1}|^p + 2\alpha_i' |u_i|^p \le 2a^p \|u\|_{\ell_p(\mathbf{Z})}^p,$$

and for odd indices

$$\begin{aligned}\sum_{i\in\mathbf{Z}}|(D_vS^{[2]}u)_{2i+1}|^p &\le (\frac{1}{2}+a)^{p-1}\sum_{i\in\mathbf{Z}} a_i|u_{i-1}|^p+(\frac{1}{2}-a'_i-a_{i+1})|u_i|^p+a'_{i+1}|u_{i+1}|^p \\ &\le \frac{1}{2}(\frac{1}{2}+a)^{p-1}\|u\|^p_{\ell_p(\mathbf{Z})}.\end{aligned}$$

Together this gives

$$\|D_vS^{[2]}u\|^p_{\ell_p(\mathbf{Z})}\le (2a^p+\frac{1}{2}(\frac{1}{2}+a)^{p-1})\|u\|^p_{\ell_p(\mathbf{Z})},$$

and (33) follows as long as $2a^p+\frac{1}{2}(\frac{1}{2}+a)^{p-1}<2$. If $a\le 1/2$ it is trivially satisfied for all p. Since the left-hand side as a function of $a\in(0,1]$ is monotone and continuous for each fixed p, and exceeds 2 if $a=1$, there is an $1/2<a_p<1$ such that (33) holds iff $0<a<a_p$ which is equivalent to $q\in[1,q_p)$ with some $2<q_p<4$. In particular, $q_1=3$, $q_2=(\sqrt{57}-1)/2\approx 3.275$, and $q_p\to 2$ if $p\to\infty$.

To summarize, for the subfamily of convexity-preserving power-q schemes ($1\le q\le 2$) L_p convergence, smoothness exponents, and stability are fully established while for $q>2$ only partial results are known.

References

[1] Amat, S., Dadourian, K., Liandrat, J.: Analysis of a class of subdivision schemes and associated non-linear multiresolution transforms. (submitted 2008), arXiv:0810.1146[math.NA]

[2] Amat, S., Liandrat, J.: On the stability of the PPH nonlinear multiresolution. Appl. Comput. Harmon. Anal. **18**, 198–206 (2005)

[3] Cavaretta, A.S., Dahmen, W., Micchelli, C.A.: Stationary subdivision. Memoirs AMS **93**, AMS, Providence (1991)

[4] Ciesielski, Z.: Constructive function theory and spline systems. Studia Math. **58**, 277–302 (1975)

[5] Cohen, A.: Numerical Analysis of Wavelet Methods. Elsevier (2003)

[6] Cohen, A., Dyn, N,, Matei, B.: Quasilinear subdivision schemes with application to ENO interpolation. Appl. Comput. Harmon. Anal. **15**, 89–116 (2003)

[7] Dahmen, W.: Wavelet and multiscale methods for operator equations. Acta Numerica **6**, 55–228 (1997)

[8] Dyn, N., Oswald, P.: Univariate subdivision and multiscale transforms: The nonlinear case. In: Multiscale, Nonlinear, and Adaptive Approximation (R.A. DeVore, A. Kunoth eds.), pp. 203–247, Springer, Berlin (2009)

[9] Dyn, N., Levin, D.: Subdivision schemes in geometric modelling. Acta Numerica **11**, 73–144 (2002)

[10] Grohs, P.: Approximation order from stability of nonlinear subdivision schemes. J. Approx. Th. (submitted), also a Geometry preprint 2008/06, TU Graz (2008)

[11] Harizanov, S., Oswald, P.: Stability of nonlinear subdivision and multiscale transforms. Constr. Approx. (2009)

[12] Kuijt, F., van Damme, R.: Stability of subdivision schemes. TW Memorandum 1469, Fac. Appl. Math., Univ. Twente (1998)

[13] Matei, B.: Smoothness characterization and stability in nonlinear multiscale framework: theoretical results. Asymptotic Analysis **41**, 277–309 (2005)

[14] Oswald, P.: Multilevel Finite Element Approximation: Theory & Applications. Teubner, Leipzig (1994).

[15] Xie, G., Yu, T.P.-Y.: On a linearization principle for nonlinear p-mean subdivision schemes. In: Neamtu,M. Saff, E.B. (eds.) Advances in Constructive Approximation, pp. 519–533, Nashboro Press (2004)

[16] Xie, G., Yu, T.P.-Y.: Smoothness analysis of nonlinear subdivision schemes of homogeneous and affine invariant type. Constr. Approx. **22**, 219–254 (2005)