

Close-to-Optimal Bounds for $SU(N)$ Loop Approximation

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Abstract

In [6], we proved an asymptotic $O(n^{-\alpha/(\alpha+1)})$ bound for the approximation of $SU(N)$ loops ($N \geq 2$) with Lipschitz smoothness $\alpha > 1/2$ by polynomial loops of degree $\leq n$. The proof combined factorizations of $SU(N)$ loops into products of constant $SU(N)$ matrices and loops of the form $e^{A(t)}$ where $A(t)$ are essentially $\mathfrak{su}(2)$ loops preserving the Lipschitz smoothness, and the careful estimation of errors induced by approximating matrix exponentials by first-order splitting methods. In the present note we show that using higher order splitting methods allows us to improve the initial estimates from [6] to close-to-optimal $O(n^{-(\alpha-\epsilon)})$ bounds for $\alpha > 1$, where $\epsilon > 0$ can be chosen arbitrarily small.

1 Introduction

The study of approximation rates for Lie-group-valued loops by polynomial loops is a relatively unexplored topic within the larger area of nonlinearly constrained approximation. Motivation is provided by previous density results [2, 3, 7] for semi-simple Lie groups, and by more practical needs, e.g., for the design of para-unitary FIR filters [2, 5, 10].

In this note, we continue the study of the $SU(N)$ case, $N \geq 2$, and improve upon the following Jackson-type estimate stated in [6]: *For any Lip_α -continuous loop $U : \mathbb{T} \rightarrow SU(N)$ and $\alpha > 1/2$ there exists a sequence of polynomial loops $U_n : \mathbb{T} \rightarrow SU(N)$ of degree $\leq n$ such that the following asymptotic inequality holds*

$$\|U - U_n\|_C := \max_{t \in \mathbb{T}} \|U(t) - U_n(t)\| \leq C_{\alpha, N, U} (n+1)^{-\alpha/(1+\alpha)}, \quad n \geq 0. \quad (1)$$

Even though this estimate is admittedly far from final, to our knowledge it represented the first nontrivial upper estimate for the achievable rate of approximation for Lip_α loops with values in matrix Lie groups. Note that there is a large gap between the exponent $\alpha/(\alpha+1)$ established in (1), and the trivial upper bound α for the maximal order of approximation of Lip_α loops following from the classical

Jackson-Bernstein theorems for univariate trigonometric approximation [1, Theorem 7.3.3].

In the meantime, we realized that some simple modifications in the proof strategy of [6] yield close-to-optimal rates, at least for $\alpha > 1$. The major change is to use higher order splitting methods for exponentials instead of the standard first order approximation $e^{\sum_{j=1}^J X_j} \approx e^{X_1} \dots e^{X_J}$.

Theorem 1 *Let $N \geq 2$, $\alpha > 1$, and $\epsilon > 0$. For any $U(t) \in \text{Lip}_\alpha(\mathbb{T} \rightarrow \text{SU}(N))$, there exists a sequence of $\text{SU}(N)$ -valued polynomial loops $U_n(t)$ of degree $\leq n$ such that*

$$\|U - U_n\|_C \leq C_{\alpha,\epsilon,N,U} (n+1)^{-(\alpha-\epsilon)}, \quad n \geq 0. \quad (2)$$

The proof of this result is given below. The restriction $\alpha > 1$ comes from the fact that we currently miss error formulas for splitting methods of order $k > 2$ in terms of higher order commutators. Another obstacle is the lack of formal proof for factorizations of $\text{SU}(N)$ -valued Lip_α loops into exponentials of $\text{su}(N)$ -valued Lip_α loops if $\alpha \leq 1/2$. The latter problem can possibly be removed by using homotopy arguments as in [3] (the first author acknowledges inspiring discussions with W. M. Lawton on this and related subjects of the present note). See also the remarks at the end of the next section.

The major open question is whether (2) remains true also with $\epsilon = 0$, or if the nonlinear constraints lead to a slight deterioration of approximation results compared to the unconstrained case. Settling this question will probably require a different approach.

2 Proof of Theorem 1

We first recall the facts already proved in [6]. The notation we use is either self-explaining or can be found in [6] (we have opted to keep notation very close to that of [6], to make the comparison easy). The matrix norm of choice is the spectral norm.

Factorization into essentially exponentials of $\text{su}(2)$ loops. Lemma 4 of [6] states that for any $\alpha > 1/2$ and any $U(t) \in \text{Lip}_\alpha(\mathbb{T} \rightarrow \text{SU}(N))$, there exist constant matrices $U_{0,l} \in \text{SU}(N)$ and loops $A_l(t) \in \text{Lip}_\alpha(\mathbb{T} \rightarrow \text{su}(2))$ such that

$$U(t) = \prod_{l=1}^L U_{0,l} e^{\hat{A}_l(t)}, \quad t \in \mathbb{T}, \quad L := N(N-1)/2, \quad (3)$$

where $\hat{A}_l(t) = T_{ij} A_l(t)$ denotes the canonical extension of $A_l(t)$ to a $\text{su}(N)$ loop by the map

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto T_{ij} A = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{21} & 0 \\ 0 & 0 & I_{j-i+1} & 0 & 0 \\ 0 & a_{21} & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 0 & I_{N-j} \end{pmatrix}$$

for some index pair (i, j) with $1 \leq i < j \leq N$ (I_k denotes the $k \times k$ identity matrix). Moreover, smoothness of the factors is controlled by smoothness of $U(t)$:

$$\|A_l\|_{\text{Lip}_\alpha} \leq C_{\alpha, N, U} \|U\|_{\text{Lip}_\alpha}, \quad l = 1, \dots, L. \quad (4)$$

Approximation can be done factor-by-factor. If $P_l(t)$ are polynomial loops in $\text{SU}(2)$ of degree $\leq n$ such that

$$\|e^{A_l(t)} - P_l(t)\|_C \leq \epsilon, \quad l = 1, \dots, L, \quad (5)$$

for the $A_l(t)$ occurring in the factorization (3) then

$$P(t) = \prod_{l=1}^L U_{0,l} \hat{P}_l(t), \quad (\hat{P}_l(t) = T_{ij} P_l(t))$$

is a polynomial loop in $\text{SU}(N)$ of degree $\leq Ln$ and it satisfies the estimate

$$\|U(t) - P(t)\|_C \leq L\epsilon. \quad (6)$$

Use Lemma 5 from [6].

Construction of $P_l(t)$. For any $m > 1$, we can approximate $A_l(t) \in \text{Lip}_\alpha(\mathbb{T} \rightarrow \text{su}(2))$ by a $\text{su}(2)$ -valued polynomial loop

$$R_{l,m}(t) = \sum_{r=1}^6 \sum_{k=0}^m c_{r,k} B_{r,k}(t)$$

of degree $\leq m$ at optimal rate (say, by applying Vallée-Poussin means component-wise), i.e.,

$$\|e^{A_l(t)} - e^{R_{l,m}(t)}\|_C \leq C m^{-\alpha}, \quad m > 0, \quad (7)$$

see Lemma 6 a) in [6]. Here $\{B_{r,k}(t)\}$ is the designated basis (over \mathbb{R}) for the linear space of dimension $6m + 3$ of all $\text{su}(2)$ -valued polynomial loops of degree $\leq m$. To keep notation simple, the dependence of the coefficients $c_{k,r}$ on l and m is not made explicit, moreover, for even r summation should only start from $k = 1$, see [6].

Lemmas 1 and 4 in [6] establish the following facts which are relevant below:

$$\|c_{r,k} B_{r,k}(t)\|_C = |c_{r,k}| \leq C(k+1)^{-\alpha}, \quad k = 0, \dots, m, \quad r = 1, \dots, 6, \quad (8)$$

and if ordered properly (e.g., lexicographically), products of the form

$$\prod_{r=1}^6 \prod_{k=0}^m e^{\lambda c_{r,k} B_{r,k}(t)}$$

represent $\text{SU}(2)$ -valued polynomial loops of degree $\leq 6m$, independently of the choice of $\lambda > 0$ and the set of real coefficients $\{c_{r,k}\}$. For our applications we take $\lambda = 1/M$, for some integer $M > 1$.

With these preparations, we can write down the final formula for the approximation of $e^{A_l(t)}$:

$$P_l(t) := \phi(\{c_{r,k}B_{r,k}(t)/M\}_{r=1,\dots,6;k=0,\dots,m})^M, \quad (9)$$

where $\phi(\{X_j\}_{j=1,\dots,J})$ is a suitable splitting method for approximating $e^{X_1+\dots+X_J}$, see the next paragraph for details. The integers m and M will be fixed later. Note that the point-wise estimate

$$\|e^{A_l(t)} - P_l(t)\| \leq Cm^{-\alpha} + M\|e^{R_{l,m}(t)/M} - \phi(\{c_{r,k}B_{r,k}(t)/M\}_{r=1,\dots,6;k=0,\dots,m})\| \quad (10)$$

follows from applying the triangle inequality to

$$e^{A_l(t)} - P_l(t) = (e^{A_l(t)} - e^{R_{l,m}(t)}) + ((e^{R_{l,m}(t)/M})^M - P_l(t)),$$

then using (7) for the first term, and Lemma 5 in [6] for the second. Thus, the quality of approximation crucially depends on the properties of the chosen splitting method ϕ .

Estimate for the second term in (10). Now we depart from [6], where the method ϕ of choice was the first order splitting method

$$\phi_1(\{X_j\}_{j=1,\dots,J}) := e^{X_1}e^{X_2} \dots e^{X_J}.$$

The error estimate is stated in Lemma 6 b) of [6], and leads to an overall estimate $\leq CM^{-1}$ for the second term in the right-hand side of (10) if $\alpha > 1/2$ (and to the suboptimal asymptotic approximation rate of that paper). We now show that using higher order symmetric splitting methods leads to significant improvements. The standard 2nd order symmetric splitting method is given by

$$\phi_2(\{X_j\}_{j=1,\dots,J}) := e^{X_1/2} \dots e^{X_{J-1}/2} e^{X_J} e^{X_{J-1}/2} \dots e^{X_1/2}.$$

Following Yoshida (see [4]), symmetric splitting methods of order $2(s+1)$ can be constructed from a given symmetric splitting method of order $2s$ via the formula

$$\phi_{2(s+1)}(\{X_j\}_{j=1,\dots,J}) := \phi_{2s}(\{a_s X_j\}_{j=1,\dots,J}) \phi_{2s}(\{b_s X_j\}_{j=1,\dots,J}) \phi_{2s}(\{a_s X_j\}_{j=1,\dots,J}),$$

if one chooses the constants as follows:

$$a_s = (2 - 2^{1/(2s+1)})^{-1}, \quad b_s = -2^{1/(2s+1)}(2 - 2^{1/(2s+1)})^{-1}.$$

The order condition for these ϕ_{2s} can be stated as follows: For $\lambda \rightarrow 0$, we have

$$\|e^{\lambda(X_1+\dots+X_J)} - \phi_{2s}(\{X_j\}_{j=1,\dots,J})\| = O(\lambda^{2s+1}).$$

Using Taylor expansion and rough estimates, the order requirement translates into the error bound

$$\|e^{\lambda(X_1+\dots+X_J)} - \phi_{2s}(\{X_j\}_{j=1,\dots,J})\| \leq C\lambda^{2s+1} \left(\sum_{j=1}^J \|X_j\| \right)^{2s+1}, \quad (11)$$

valid for $|\lambda| \leq \lambda_{\max}$, with a constant depending on λ_{\max} and $\sum_j \|X_j\|$. More precise error bounds are available for ϕ_2 , see Remark 2 at the end.

We are now ready to apply this to the family $\{c_{r,k}B_{r,k}(t)\}_{r=1,\dots,6; k=0,\dots,m}$ (pointwise with respect to t) with $\lambda = M^{-1}$. By (8), if $\alpha > 1$ we have

$$\sum_{r=1}^6 \sum_{k=0}^m \|c_{r,k}B_{r,k}(t)\| \leq C, \quad (12)$$

where C depends on the Lip_α -norms of the the $\text{su}(2)$ -loops $A_l(t)$, but not on m . Substituting into (11), we obtain for each $l = 1, \dots, L$

$$\|e^{R_{l,m}(t)/M} - \phi_{2s}(\{c_{r,k}B_{r,k}(t)/M\}_{r=1,\dots,6; k=0,\dots,m})\|_C \leq CM^{-(2s+1)}. \quad (13)$$

From now on, set $\phi = \phi_{2s}$ in the formula for $P_l(t)$ (and consequently $P(t)$). Substituting into (10) and taking into account (5), (6) we finally arrive at

$$\|U(t) - P(t)\|_C \leq L \max_l \|e^{A_l(t)} - P_l(t)\| \leq C(m^{-\alpha} + M^{-2s}), \quad (14)$$

where C depends on s, α, N , and on $U(t)$.

Estimating the degree of $P(t)$. Consider a large enough integer $n \geq n_0$ (for $n < n_0$, just use constant $P(t) = I$ to get the complementing trivial bound $\|U(t) - I\| \leq 2$). We will now fix m and M such that the degree of the above constructed $P(t)$ is $\leq n$ and the right-hand side in (14) is asymptotically as small as possible. Due to the recursive definition of ϕ_{2s} , the degree of the polynomial loops $\phi_{2s}(\{c_{r,k}B_{r,k}(t)/M\})$ is bounded by 3^{s-1} times the degree of the polynomial loops $\phi_2(\{c_{r,k}B_{r,k}(t)/M\})$ generated by the 2nd order method (for simplicity, we do not indicate the index set $r = 1, \dots, 6; k = 0, \dots, m$ in the notation). The latter, however, have degree $\leq 12m$. This can be proved as in Lemma 1 of [6]. Indeed, write

$$\phi_2(\{c_{r,k}B_{r,k}(t)/M\}) = \phi_1(\{c_{r,k}B_{r,k}(t)/2M\})\phi_1(\{c_{r,k}B_{r,k}(t)^*/2M\})^*.$$

We already know that the first factor $\phi_1(\{c_{r,k}B_{r,k}(t)/2M\})$ has degree $\leq 6m$. The second factor is the Hermitian transpose of $\phi_1(\{c_{r,k}B_{r,k}(t)/2M\})$, and it remains to check that $\{B_{r,k}^*(t)\}$ is such a permutation of the original basis $\{B_{r,k}(t)\}$ to which Lemma 1 of [6] can be applied, leading to the same degree bound.

Putting things together, we see that the degree of $P(t)$ is bounded by $12L3^{s-1}Mm$. Thus, choosing the integers M, m according to

$$M = [(12L3^{s-1})^{-1}n^{\alpha/(\alpha+2s)}], \quad m = [n^{2s/(\alpha+2s)}],$$

we guarantee that the degree of $P(t)$ does not exceed n . On the other hand, substitution into (14) yields

$$\|U(t) - P(t)\|_C \leq Cn^{-2s\alpha/(\alpha+2s)} = Cn^{-\alpha+\alpha^2/(\alpha+2s)}.$$

This establishes the claim of our theorem, if, for given $\alpha > 1$ and $\epsilon > 0$, we choose the order $2s$ of the splitting method large enough.

Remark 1. There are at least three shortcomings of the asymptotic estimate (2). First, the constant $C_{\alpha,\epsilon,N,U}$ depends on $U(t)$ in an unspecified way. Secondly, the restriction $\alpha > 1$ is mainly due to the use of the crude error estimate (11) for higher order splitting methods (see the comments in Remark 2). In addition, for $\alpha \leq 1/2$ the factorization technique of Lemma 4 from [6] breaks down (an alternative is addressed in Remark 3). Thirdly, it is not clear at the moment if one can set $\epsilon = 0$ in (2).

Remark 2. For low-order splitting methods such as ϕ_1 and ϕ_2 , the error bound can be made more precise in terms of commutator expressions which paves the way for proving (13) (and thus also (14)) for some $\alpha \leq 1$. For ϕ_1 this was demonstrated in [6] (see Lemma 6 for the more precise error bounds). An improved error bound for the symmetric 2nd order method ϕ_2 has been established in [9]:

$$\|e^{\lambda(X_1+\dots+X_J)} - \phi_2(\{\lambda X_j\}_{j=1,\dots,J})\| \leq \lambda^3 \Delta(X_1, \dots, X_J), \quad (15)$$

where $\Delta(X_1, \dots, X_J) = \sum_{k=1}^{J-1} \Delta_2(X_k, X_{k+1} + \dots + X_J)$, and

$$\Delta_2(A, B) = \frac{1}{12} \{ \|[[A, B], B]\| + \frac{1}{2} \|[[A, B], A]\| \}.$$

The advantage is that in our application, where we would apply (15) to matrix sets of the form $\{c_k B_{r,k}\}_{k=0,\dots,m}$ related to the terms of the Fourier series of a $\mathfrak{su}(2)$ -valued Lip_α -loop, the sum of the norms of the appearing triple commutators can be estimated by a sum of the form $\sum_{k=1}^m (\log k)^2 k^{-3\alpha}$ which remains uniformly bounded for $\alpha > 1/3$ (the details are worked out in [6] for the first-order case). In contrast, using (11) with $s = 1$ leads to a constant factor of the form $(\sum_{k=1}^m k^{-\alpha})^3$.

Unfortunately, generalizations of (15) to Yoshida-type or any other higher order splitting methods seem to be unknown.

Remark 3. W. M. Lawton drew our attention to a possible alternative to loop factorizations of the form (3) proposed in [6]. It is well known that $\text{SU}(N)$ is a simply connected compact C^∞ -manifold. Thus, any $\text{SU}(N)$ -valued continuous loop $U(t)$ can be contracted to a point by a homotopy map $\psi : [0, 1] \rightarrow C(\mathbb{T} \rightarrow \text{SU}(N))$ (i.e., ψ is continuous, $\psi(1) = U(t)$, and (without loss of generality) $\psi(0) = I$). Let us assume that for $U(t) \in \text{Lip}_\alpha(\mathbb{T} \rightarrow \text{SU}(N))$, the homotopy map ψ can be found in such a way that $\psi(\xi) \in \text{Lip}_\alpha(\mathbb{T} \rightarrow \text{SU}(N))$ for all $\xi \in [0, 1]$ (i.e., preserves Lipschitz smoothness along the homotopy path). We do not have a reference for this assumption but strongly believe that it holds for all $\alpha > 0$.

Now, take a fine enough partition $\xi_0 = 0 < \xi_1 < \dots < \xi_{K-1} < \xi_K = 1$ of $[0, 1]$ such that $\|\psi(\xi_{k-1}) - \psi(\xi_k)\|_C \leq r_N$, where r_N is the injectivity radius of the exponential map in the neighborhood of $I \in \text{SU}(N)$. Then we can write

$$U(t) = U_1(t) \dots U_K(t), \quad U_k(t) := \psi(\xi_{k-1})^* \psi(\xi_k), \quad k = 1, \dots, K,$$

where all $U_k(t)$ belong to $\text{Lip}_\alpha(\mathbb{T} \rightarrow \text{SU}(N))$, and

$$\|I - U_k\|_C \leq \|\psi(\xi_{k-1})^*\|_C \|\psi(\xi_{k-1}) - \psi(\xi_k)\|_C = \|\psi(\xi_{k-1}) - \psi(\xi_k)\|_C \leq r_N,$$

i.e., $A_k(t) = \log(U_k(t))$ is well defined and belongs to $\text{Lip}_\alpha(\mathbb{T} \rightarrow \text{su}(N))$ for all $k = 1, \dots, K$. Thus,

$$U(t) = \prod_{k=1}^K e^{A_k(t)}, \quad \|A_k\|_{\text{Lip}_\alpha} \leq C(\alpha, N, U), \quad k = 1, \dots, K. \quad (16)$$

In contrast to (3), the number of exponential factors K is not independent of α and U , and the $A_k(t)$ are general $\text{su}(N)$ -valued Lip_α -loops (and not essentially $\text{su}(2)$ -valued as in (3)).

However, as long as we accept the dependence on $U(t)$ in the constant appearing in (2), the factorization (16) is sufficient to carry out the above proof with minor changes. The reduction to the $\text{SU}(2)$ case can be circumvented by working with a similar basis $\{B_{r,k}(t)\}_{r=1, \dots, R_N, k=0, \dots, m}$ over \mathbb{R} for $\text{SU}(N)$ -valued polynomial loops of degree $\leq m$. What changes is the number R_N of subsets $\{B_{r,k}(t)\}_{k=0, \dots, m}$ of basis elements to be considered. This number depends only on N , and enters the degree estimates as a linear factor.

Remark 4. Loop approximation can be pursued for other Lie groups and manifolds as well. Work on the closely related case of $\text{SO}(N)$ -valued loops ($N \geq 3$) is ongoing [8].

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