Deterministic Analysis of Endless Tuning of Polarization Controllers

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***Abstract*— We describe a new, deterministic algorithm for endless, reset free polarization control (PC) based on sequences of tunable phase shifters and directional couplers which can be implemented in a variety of discrete (e.g., rotating waveplates) and integrated optic material platforms. We provide a novel quantitative formulation of what endless, reset free PC operation means, and prove in a rigorous way that our algorithm satisfies the defined requirements.**

***Index Terms*— polarization control, reset free operation, tunable thermo-optic phase shifters, control algorithms.**

I. INTRODUCTION

Polarization control (PC) is an important issue in modern fiber-optical transmission systems and opto-electronic hardware in general, and drives the ongoing search for low-cost, high-speed, and robust PC devices. Traditionally, PC is aimed at converting a variable input state of polarization (SOP) into a variable (or fixed) output SOP at places in the system where polarization matters, e.g., before the detector at the receiver end. Higher-order polarization mode dispersion (PMD) compensation for high bit-rate transmission usually requires the combination of several PC processes. Polarization multiplexing and coherent communications both depend on controlling an incoming SOP. In the first case, a wideband (i.e. equal to the signal bandwidth) response to separate orthogonal polarizations is needed. In the latter case, the incoming signal is launched on a single polarization, and must be aligned with a local oscillator's polarization at the receiver. PC consists of basically 3 important parts:

Measurement process. Commonly used are feedback signals from polarimeters where the input SOP is measured before the polarization controller or transmission through a polarizer following the polarization controller. For a modulated signal propagating through a channel with PMD, the polarization is frequency-dependent, in which case a weighted average of the SOP may easily be measured. For frequency dependent information, measurement possibilities include amplitude measurements of various radio-frequency spectral components after optical-to-electrical conversion, or direct frequency-dependent SOP detection based on tapping and transforming part of the incoming signal. In applications to PMD compensation, a variety of measurement schemes directly or indirectly related to bit error rate such as estimating degree of polarization, eye openings, or statistics of forward error correction provide indirect SOP information. In this paper, we focus on a frequency-independent polarization controller which can be used as a building block in a PMD compensator to achieve frequency-dependent polarization filtering.

PC architecture. Several tunable, mechanically or electrically driven, components are combined to achieve all possible polarization transformations. There is a large variety of component types (fiber squeezers, liquid crystrals, rotating waveplates, $LiNbO_3$/$LiTaO_3$ waveguide devices, thermo-optic phase shifters (PS)) which can be architecturally combined in many equivalent ways. The basic architectures investigated are equivalent to chains of $2-4$ linear retarders [7], [11] (e.g., PS and tunable directional couplers (DC) in an alternating order) resp. chains of 2 or more endlessly adjustable wave plates (WP) [1] (e.g., quarter (QWP) and half wave plates (HWP)), or Soleil-Babinet compensators [7], [11].

Tuning algorithm. Depending on measurement scheme and PC architecture, a control scheme needs to be designed that transfers input/output SOP information into correct parameter predictions for the PC components, and changes the settings of the physical parameters accordingly. The tuning algorithm has to satisfy two important properties: It should achieve *endless* PC, i.e., the predicted parameters stay always within the physically possible operating range of the components, and if possible be *reset free*. The latter property is not well-defined (compare [8, Sec. IIIA]), roughly speaking, beyond continuous operation it corresponds to avoiding instantaneous resets of tuning parameters when they hit their physical range limits.

The interdependencies between these issues are manifold, and depend on the particular application. E.g., calibration issues and non-ideal components can be better handled if measurements are tied to signal transmission quality, and directly translate into predictions for the tunable parameters, thus avoiding the use of analytic apparatus based on idealized component transfer functions in the control scheme. Fabrication variations may limit the achievable polarization range, thus rendering it dangerous to work with minimal architectures. Low-frequency feedback from measurements resp. time-consuming, sophisticated tuning algorithms would imply high penalties in situations of fast polarization changes, and might be prohibitive in many applications. Architectures with redundant components lead to more tuning parameters and potentially to a more complex control scheme but be overall preferable for achieving robust, endless and reset free

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operation. For more in-depth discussions of some currently proposed solutions to PC, see [7], [1], [11], [8] and the many other original papers cited therein.

The main aim of this investigation is to revisit the problem of endless, reset free PC from a mathematical point of view, and give precise definitions, explicit deterministic control algorithms, and quantitative proofs for endless, reset free operation of a PC architecture consisting of chains of WPs (more general than QWPs) with restricted rotation angle. This latter restriction allows us to implement them as PS-DC-PS or DC-PS-DC sequences whose tunable parameters also stay within prescribed bounds. The final result are provable solutions for endless, reset free PC of fixed into arbitrary (or arbitrary ino fixed) SOP with a PS-DC sequence with 4 resp. 5 tunable parameters, and of arbitrary into arbitrary SOP with 7 tunable parameters. As measurement input, we assume given input/output SOP vectors which are changing in discrete time, and the mathematical statements are formulated with respect to this information, and not with respect to intensity or other signal-related measures.

Our analysis complements existing literature, where an intuitive definition of reset free operation is followed, and control algorithms are geometrically or analytically described, and experimentally validated. Our analysis may be useful for high-end applications, and for implementing and optimizing control algorithms with guaranteed performance bounds. The paper is organized as follows. In Sec. II, we introduce basic properties polarization transformations such as PS, DC, WP, etc., comment on their endless implementation by sequences of tunable PS-DC, and give a mathematical definition of endless, reset free PC. In Sec. III, we describe the new control algorithms based on the factorization of polarization transformations into WPs with bounds on the rotation angle, and provide some simulation results. Proofs and mathematical details are given in the Appendices.

II. Notation and Definitions

We will use mostly Jones space notation but turn to Stokes space and the Poincare sphere to clarify the issues geometrically. To a large extent, our notation is consistent with the one used in [4], [3], where more background information on polarization-related issues can be found.

SOPs are represented by complex vectors $\mathbf{E} = (x,\ y)^T$ of unit length modulo an arbitrary multiplicative factor of the form $e^{j\xi}$, and polarization transformations by unitary Jones matrices U of the form

$$U := \begin{pmatrix} c - ja_x & a_z - ja_y \\ -a_z - ja_y & c + ja_x e \end{pmatrix}, \qquad (1)$$

where $c^2 + \|\mathbf{a}\|^2 = 1$ and $\mathbf{a} = (a_x,\ a_y,\ a_z)^T$. In Stokes space, $\mathbf{E}$ is represented by the unit length real vector $\mathbf{S} := (|x|^2 - |y|^2, 2\Re(xy^*), 2\Im(xy^*))^T$, and the application of U to $\mathbf{E}$ is equivalent to a rotation of $\mathbf{S}$ with axis direction $\mathbf{a}$ and angle $\phi = 2\arccos c$. This is key to the geometric interpretation of polarization transformations on the Poincare sphere $\|\mathbf{S}\| = 1$. The coordinate axes in $\mathbb{R}^3$ are denoted by x, y, z, which should not cause serious confusion.

The commonly used PC components have the following mathematical descriptions. Phase shifters

$$U_{PS,\phi} := \begin{pmatrix} e^{-j\frac{\phi}{2}} & 0 \\ 0 & e^{j\frac{\phi}{2}} \end{pmatrix}$$

are rotations about the x-axis with angle ϕ, directional couplers

$$U_{DC,\phi} = \begin{pmatrix} \cos(\frac{\phi}{2}) & -j\sin(\frac{\phi}{2}) \\ -j\sin(\frac{\phi}{2}) & \cos(\frac{\phi}{2}) \end{pmatrix}$$

are rotations about the y-axis with angle ϕ, and can be replaced by a PS embedded into two PS and DC with fixed rotation angle due to

$$U_{DC,\phi} = U_{PS,\pi/2} U_{DC,\pi/2} U_{PS,\phi-\pi} U_{DC,\pi/2} U_{PS,\pi/2}. \qquad (2)$$

Wave plates

$$U_{W,\phi,\theta} = \begin{pmatrix} \cos(\frac{\phi}{2}) - j\sin(\frac{\phi}{2})c_\theta & -j\sin(\frac{\phi}{2})s_\theta \\ -j\sin(\frac{\phi}{2})s_\theta & \cos(\frac{\phi}{2}) + j\sin(\frac{\phi}{2})c_\theta \end{pmatrix} \qquad (3)$$

are rotations by the angle ϕ about an axis in xy-plane in direction (c_θ, s_θ), where $c_\theta := \cos\theta$, $s_\theta = \sin\theta$, and contain PS (for $\theta = 0$) and DC (for $\theta = \pi/2$) as special cases. Other often used special cases of (3) are quarter wave plates $U_{Q,\theta} = U_{W,\pi/2,\theta}$, and half wave plates $U_{H,\theta} = U_{W,\pi,\theta}$. Any WP can be represented by a PS-DC-PS sequence as

$$\begin{aligned} &U_{W,\phi,\theta} = U_{PS,\phi_3} U_{DC,\phi_2} U_{PS,\phi_1}, \\ &\phi_1 = \phi_3 = \arctan(c_\theta \tan(\tfrac{\phi}{2})) = \arctan(\tfrac{a_x}{c}), \\ &\phi_2 = 2\arcsin(\sin(\tfrac{\phi}{2})s_\theta) = 2\arcsin(a_y). \end{aligned} \qquad (4)$$

Similar statements hold for DC-PS-DC sequences, and for other parametrized families of Jones matrices, see, e.g., [7, Formula (12)]. Sequences of alternating PS and DC are widely used in PC applications and theory, compare [7], [6].

Observe that if we restrict the rotation angle ϕ of the WP to the interval $|\phi| \le \phi_0 < \pi$ then the angles ϕ_i in (4) are also restricted to the same interval, independently of the position of the rotation axis given by θ, and are differentiable functions of ϕ, θ. This observation is key to the endless and reset free PC solution proposed below, compare also [6].

Since for U defined by (1) we have equality in

$$U = U_{PS,\phi_3} U_{DC,\phi_2} U_{PS,\phi_1} \qquad (5)$$

iff

$$\begin{aligned} c &= \cos\psi_s \cos(\tfrac{\phi_2}{2}), & a_x &= \sin\psi_s \cos(\tfrac{\phi_2}{2}), \\ a_z &= -\sin\psi_d \sin(\tfrac{\phi_2}{2}), & a_y &= \cos\psi_d \sin(\tfrac{\phi_2}{2}), \\ \psi_s &= (\phi_3 + \phi_1)/2, & \psi_d &= (\phi_3 - \phi_1)/2, \end{aligned} \qquad (6)$$

we see that any U can be matched by at least one PS-DC-PS sequence (e.g., by using the formulae $\phi_2 = 2\arccos\sqrt{c^2 + a_x^2}$, $\tan\psi_d = -a_z/a_y$, $\tan\psi_s = a_x/c$, with modifications if c or a_y vanish). However, to achieve endless operation of a PC based on such a minimal architecture would necessarily require resets, compare Sec. III for an example.

We restrict our further discussion to PC architectures equivalent to *PS-DC sequences* resulting in polarization transformations

$$U(\phi) := U_{PS,\phi_{2k+1}} U_{DC,\phi_{2k}} \dots U_{PS,\phi_3} U_{DC,\phi_2} U_{PS,\phi_1} \quad (7)$$

characterized by the vector of tuning parameters $\phi := (\phi_1, \phi_2, \dots)$ (to obtain an even number of alternating PS-DC components, the first or last PS could be dropped from the sequence). Since the PS and DC are tuned by a physical mechanism such as the electro-optic or thermo-optic effect, the tuning parameters ϕ_i have to stay within some physical limits which may be determined by the available drive power or material limitations such as maximally allowed temperature.

The commonly considered PC scenarios will be formulated with respect to discrete time which is justified in most applications.

A *PC with one SOP fixed*: Given a sequence of input SOP $\mathbf{E}_0^{in}, \mathbf{E}_1^{in}, \dots$, and a fixed output state $\mathbf{E}^{out}$, determine tuning parameters $\phi^n := (\phi_1^n, \phi_2^n, \dots)$ such that

$$\mathbf{E}^{out} = e^{j\xi_n} U_n \mathbf{E}_n^{in}, \qquad n = 0, 1, \dots, \quad (8)$$

where the phases ξ_n are undetermined, and $U_n := U(\phi^n)$ is given by (7). In Stokes space, the forefactor $e^{j\xi_n}$ disappears, and the above equation reads

$$\mathbf{S}^{out} = R_n \mathbf{S}_n^{in}, \qquad n = 0, 1, \dots,$$

where R_n denotes the 3×3 rotation matrix corresponding to U_n. Clearly, there is a symmetric problem formulation of tuning a fixed input SOP $\mathbf{E}^{in}$ into an arbitrarily changing sequence of output SOP $\mathbf{E}_n^{out},\ n = 0, 1, \dots$.

B *Arbitrary PC*: Given both a sequence of input SOP $\mathbf{E}_0^{in}$, $\mathbf{E}_1^{in}, \dots$, and a sequence of output SOP $\mathbf{E}_0^{out}$, $\mathbf{E}_1^{out}, \dots$, determine tuning parameters ϕ^n such that similarly

$$\mathbf{E}_n^{out} = e^{j\xi_n} U_n \mathbf{E}_n^{in}, \quad U_n = U(\phi^n), \qquad n = 0, 1, \dots, \quad (9)$$

It is well known that any solution for PC scenarios **A** and **B** at least requires a 2-parameter PC architecture, e.g., a single WP would be sufficient to transform an arbitrary input SOP into an arbitrary output SOP, see [7, p. 1204]. However, such *minimal* architectures do not admit robust control algorithms in the sense discussed below, and all existing proposals for **A** and **B** are based on architectures with more tuning parameters. Note, that determining tuning parameters ϕ^n such that $U(\phi^n) = U_n$ for a given sequence of Jones matrices $U_0, U_1, \dots$ represents a third PC scenario which would obviously require at least 3 tuning parameters.

Definition. A given PC architecture with transfer matrix (7) allows for *endless, reset free* operation in the above PC scenarios **A** resp. **B** if the tuning parameters $\phi^n = (\phi_1^n, \phi_2^n, \dots)$ can always be chosen such that

i) the relations in (8) resp. (9) are satisfied,
ii) the parameters ϕ_i^n, do not exceed the physically imposed limits, and
iii) the tuning parameters change at a rate comparable with the changes of the input data, i.e., there is an absolute constant $0 < C_{PC} < \infty$ such that

$$\Delta\phi^n := \max_{i=1,\dots,M} |\phi_i^n - \phi_i^{n-1}| \le C_{PC} \|\mathbf{E}_n^{in} - \mathbf{E}_{n-1}^{in}\|,$$

for **A** resp.

$$\Delta\phi^n \le C_{PC}(\|\mathbf{E}_n^{in} - \mathbf{E}_{n-1}^{in}\| + \|\mathbf{E}_n^{out} - \mathbf{E}_{n-1}^{out}\|),\ n \ge 1,$$

for **B** (other distance measures for SOPs, e.g., arclength distance between the corresponding Stokes vectors, could be used as well).

Requirements i) and ii) alone correspond to what is traditionally understood as endless operation while iii) is the mathematically precise formulation for reset free operation as suggested in [8, Sec. IIIA]. The above definition of $\Delta\phi^n$ assumes that the tuning parameters are appropriately normalized. Then the size of the constant C_{PC} becomes a meaningful quantitative measure for **endless, reset free operation**.

The requirements in this definition are probably too restrictive to adequately describe all practical applications because often only a functional (such as intensity measurements which represent condensed information about the position of $\mathbf{E}_n^{out}$ relative to $\mathbf{E}_n^{in}$) of the input/output SOP is available which would necessitate changes in i) and iii). Also, due to measurement errors exact tuning i) should probably be replaced by tuning within a tolerance related to the achievable measurement accuracy. However, for the purpose of a precise mathematical analysis of the issue of endless, reset free PC operation, the above definition is still useful and will be accepted without further comment.

As far as we can judge from the literature, researchers are implicitly aware of these issues since, beginning with the early papers on PC solutions with resets [7], [11], redundant components were introduced to achieve so-called *error-tolerant operation*. Roughly speaking, whenever a reset occurs, some of the redundant components take over the basic function of the PC device allowing the reset to be less visible and stretched over a longer time interval if necessary. This is evidently related to achieving a better control of C_{PC}. However, no detailed investigation of requirement iii) nor estimates for the constants involved were given. In particular, no analysis in the sense of iii) has appeared for the two most popular architectures for practical PC control, the 4-parameter PS-DC-PS-DC architecture [7], [10] and the 4-parameter WP-WP architecture (there are several equivalent implementations of the latter, e.g., [7], [11], [8]). Our contribution is the analysis of this issue using the $2k+1$-parameter architecture described by (7) with $k = 2, 3$, which we approach via intermediate factorization into WPs with limited ϕ range, see Sec. III. This is close in spirit to considerations in [2], [6].

For lack of space, we cannot dwell on the influence of the actual value C_{PC} on PC quality in a practical control environment (**based on simulations detailed below, numerical values $C_{PC} \le 15$ have been found for the PC architectures and control scheme proposed**). We just remark that it should be considered in concert with other aspects such as achievable tuning speed, reliability and frequency of available measured SOP information, and processing speed of the control unit, in particular, the complexity of the computational algorithm producing the next tuning vector ϕ^n. In other words, the overall required speed of the control scheme remains the ultimate challenge. However, our proof that a given control scheme satisfies iii) with some input-independent constant C_{PC} seems to be of independent interest.

III. Control Algorithm

We will explain our algorithm geometrically, using Stokes space terminology. To achieve a certain rotation of $\mathbf{S}^{in}$ into $\mathbf{S}^{out}$, we will use several WP rotations. Our goal is to find the WP factors in such a way that their rotation angles ϕ always satisfy $|\phi| \le \phi_0$ for some fixed $0 < \phi_0 < \pi$ (for short, such a WP is called *admissible*), and the reset free property iii) holds, i.e., changes in the WP parameters ϕ, θ remain uniformly bounded by the changes in the input resp. output SOP. If this can be achieved then formula (4) applied to each of the involved WP shows that this translates into iii) for the tuning parameters of an equivalent PS-DC-PS chain, and, by concatenation, into the proof of i)–iii) with a PS-DC sequence (7) whose length depends on the number of WP rotations involved.

The construction of the auxiliary WP rotations is based on the following observations, see App. I for proofs. Fix ϕ_0, β according to

$$2\pi/3 < \phi_0 < \pi, \qquad (\pi - \phi_0)/2 \le \beta < \phi_0 - \pi/2. \quad (10)$$

Consider the circle $\mathcal{C} := \{\mathbf{S} = (S_x, S_y, S_z = \sin\beta)^T\}$ on the Poincare sphere with radius $\cos\beta$ and center on the z-axis.

Observation 1. For any two SOP $\mathbf{S}', \mathbf{S}'' \in \mathcal{C}$, there is an admissible WP that rotates $\mathbf{S}'$ into $\mathbf{S}''$, and whose parameters ϕ and θ depend smoothly on these two SOP.

Observation 2. If $\mathbf{S}$ belongs to the sperical caps $\{\mathbf{S} : \ S_z \le -\sin(\phi_0 - \beta)\}$ (called region A) resp. $\{\mathbf{S} : \ S_z \ge \sin(\phi_0 + \beta)\}$ (region B) with centers at the poles $(0, 0, \pm 1)^T$ on the z-axis, then any $\mathbf{S}' \in \mathcal{C}$ can be reached from $\mathbf{S}$ by an admissible WP.

Observation 3. For any $\mathbf{S} \neq (0, 0, \pm 1)^T$, denote by $\hat{\mathbf{S}}$ its closest point on $\mathcal{C}$. Then $\mathbf{S}$ can be rotated into $\hat{\mathbf{S}}$ by an admissible WP vice versa. This WP has the minimal rotation angle ϕ among all WP that rotate $\mathbf{S}$ onto $\mathcal{C}$, and changes smoothly with $\mathbf{S}$.

Observation 4. If $\mathbf{S}$ is transformed into $\mathbf{S}' \in \mathcal{C}$ by an admissible WP, and $\mathbf{S}$ is not the reflection of $\mathbf{S}'$ w.r.t. the xy-plane then the underlying WP transformation is uniquely determined by $\mathbf{S}, \mathbf{S}'$, and its changes are bounded by the respective changes in $\mathbf{S}, \mathbf{S}'$.

Following Observation 1 and 3, the naive approach would be to transform an arbitrary $\mathbf{S}^{in}$ first into the point $\mathbf{S}' := \hat{\mathbf{S}}^{in} \in \mathcal{C}$, then $\mathbf{S}'$ into the point $\mathbf{S}'' := \hat{\mathbf{S}}^{out}$, and finally $\mathbf{S}''$ into $\mathbf{S}^{out}$. All these WP are admissible, and can easily be determined from the four SOP involved. Unfortunately, there is one difficulty: The mapping $\mathbf{S} \to \hat{\mathbf{S}}$ has no continuous extension to the poles, and small changes in $\mathbf{S} \approx (0, 0, \pm 1)^T$ can lead to large variations in $\hat{\mathbf{S}}$, thus, leading to violations of property iii).

However, since this problem occurs only near the poles, where according to Observation 2 any point on $\mathcal{C}$ can be reached by an admissble WP transformation, we can improve upon this naive algorithm by replacing the rigid definition of $\mathbf{S}'$ and $\mathbf{S}''$ by the following strategy. Given the positions of $\mathbf{S}^{in}_{n-1}$ and $\mathbf{S}'_{n-1}$ known from the previous input SOP (for which we assume that an admissible WP transformation exists), we rotate the new input SOP $\mathbf{S}^{in}_n$ into the point $\mathbf{S}'_n \in \mathcal{C}$ which is closest to $\hat{\mathbf{S}}^{in}_n$ among the points satisfying

$$\|\mathbf{S}'_n - \mathbf{S}'_{n-1}\| \le A\|\mathbf{S}^{in}_n - \mathbf{S}^{in}_{n-1}\|, \quad (11)$$

where A is a fixed constant. This constrained minimization problem is easy to solve. Indeed, if the inequality

$$\|\hat{\mathbf{S}}^{in}_n - \mathbf{S}'_{n-1}\| \le A\|\mathbf{S}^{in}_n - \mathbf{S}^{in}_{n-1}\|$$

holds then obviously $\mathbf{S}'_n := \hat{\mathbf{S}}^{in}_n$ is the solution. Otherwise, we determine the two points $\mathbf{S}' \in \mathcal{C}$ that satisfy

$$\|\mathbf{S}' - \mathbf{S}'_{n-1}\| = A\|\mathbf{S}^{in}_n - \mathbf{S}^{in}_{n-1}\|,$$

and choose the one closer to $\hat{\mathbf{S}}^{in}_n$ as $\mathbf{S}'_n$. It is intuitively clear that a large enough A will allow us to quickly reduce the distance between $\mathbf{S}'_n$ and $\hat{\mathbf{S}}^{in}_n$ as $\mathbf{S}^{in}_n$ moves away from the poles, and achieve $\mathbf{S}'_n = \hat{\mathbf{S}}^{in}_n$ before the exceptional situation mentioned in Observation 4 can occur. This then automatically implies property iii) for the first WP. Similarly, we proceed when finding $\mathbf{S}''_n$ from $\mathbf{S}^{out}_n$ to determine the third WP transformation.

What we claim (for the rigorous proof, see Appendix II) is that for appropriately chosen β and ϕ_0, large enough A, and with an appropriate initialization for $n = 0$, this algorithm leads to auxiliary sequences $\{\mathbf{S}'_n\}$, $\{\mathbf{S}''_n\}$ satisfying

$$\begin{aligned} \|\mathbf{S}'_n - \mathbf{S}'_{n-1}\| &\le C\|\mathbf{S}^{in}_n - \mathbf{S}^{in}_{n-1}\|, \\ \|\mathbf{S}''_n - \mathbf{S}''_{n-1}\| &\le C\|\mathbf{S}^{out}_n - \mathbf{S}^{out}_{n-1}\|, \end{aligned} \quad (12)$$

and such that all three resulting WP transformations are admissible, and change in a manner controlled by the input/output SOP changes as well.

As indicated above, see (4), this leads to the desired solution for arbitrary PC with 3 WPs resp. a PS-DC sequence (7) with $k = 3$, i.e., with 7 tunable parameters. As to PC with one fixed SOP, without loss of generality we can assume that the fixed SOP belongs to $\mathcal{C}$ (if not, one needs to implement the equivalent of a fixed WP) and drop either the third or the first WP from the consideration. Thus, we actually need only two WPs or a PS-DC sequence with $k = 2$, i.e., 5 tunable parameters, for PC scenario **A**. A slightly more economic solution is possible if one augments Observation 1 by the following

Observation 1'. There is an endless, reset free way of transforming an arbitrary $\mathbf{S}' \in \mathcal{C}$ into the fixed SOP $(0, 1, 0)^T$ by using a 2-parameter DC-PS chain.

This can be realized by explicit analytic computations (see Appendix III). To get to the overall 4-parameter solution for PC scenario **A**, we apply first the above described algorithm to transform $\mathbf{S}^{in}_n$ into a suitable $\mathbf{S}'_n \in \mathcal{C}$. The underlying admissible WP is factored into a DC-PS-DC chain which

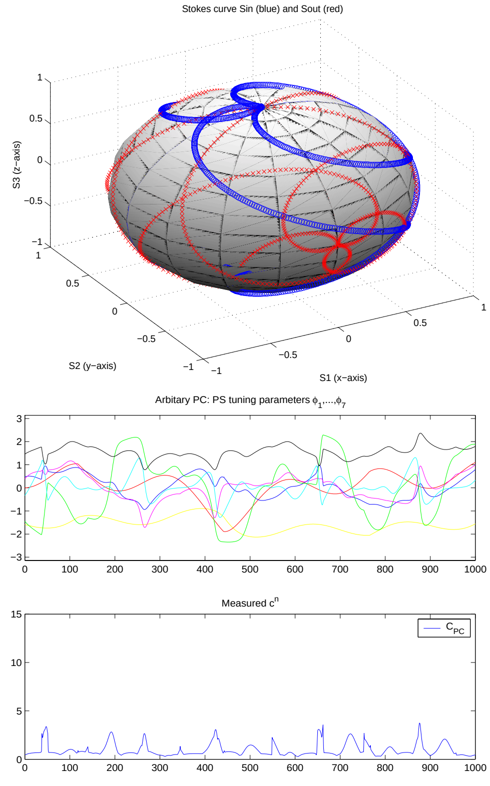

Fig. 1. Simulation for arbitrary PC

can be concatenated with the DC-PS chain obtained from Observation 1' with $\mathbf{S}' = \mathbf{S}'_n$!

We conclude with some remarks and simulation results. We start with arbitrary PC. To exhibit the behavior of the algorithm, we have created synthetic input/output SOP sequences winding around the Poincare sphere and passing through the critical pole regions, a typical example is shown in Fig. 1. To visualize the behavior of the algorithm, we show plots of the resulting 7 tuning parameters (here, the DCs in (7) have been replaced by PS using (2)), and of the quantities $c^n := \Delta\phi^n/(\|\mathbf{S}_n^{in} - \mathbf{S}_{n-1}^{in}\| + \|\mathbf{S}_n^{out} - \mathbf{S}_{n-1}^{out}\|)$ which give an indication of the size of the constant C_{PC} in iii). **Suitable parameters β and A have been found via extensive numerical testing which showed that values $\beta \approx \pi/6 \ldots \pi/8$, and $A \approx 5$ lead to favorable values for C_{PC} in the range ≤ 10. For Fig. 1, $\beta = \pi/8$, and $A = 4.9$ were used (this A is much smaller than the very pessimistic value appearing in the proof given in Appendix II).** The proposed algorithm works stably. **As expected from the above description of our control scheme, the peaks observed in the c^n plots can be traced back to situations where the input resp. output SOP sequences pass the pole regions.**

For PC with fixed output SOP, we demonstrate the 4-parameter DC-PS-DC-PS solution (with $\mathbf{S}^{out} = (0, 1, 0)^T$) first on a very particular input SOP sequence given by

$$\mathbf{S}_n^{in} = (0, \cos hn, \sin hn)^T, \quad n = 0, \ldots, N,$$

which moves in constant steps given by the small parameter h along the circle on the Poincare sphere cut out by the yz-plane. Fig. 2 depicts (in blue) such a circular movement with $h = 0.01$ and $N = 6000$. It repeatedly passes the critical pole regions, and exhibits the properties of the algorithm and the role of the parameters β and A. The plots in Fig. 3 show the 4 PS-DC tuning parameters and the values $c^n := \Delta\phi^n/\|\mathbf{S}_n^{in} - \mathbf{S}_{n-1}^{in}\|$ produced by our algorithm with $\beta = \pi/8$ and $A = 4$. Each time $\mathbf{S}_n^{in}$ passes a critical pole, the algorithm detects that $\hat{\mathbf{S}}_n^{in}$ switches to the opposite side of the circle $\mathcal{C}$, far away from the previous $\mathbf{S}'_{n-1}$. Controlled by (11), the algorithm now starts slowly moving with $\mathbf{S}'_n$ towards $\hat{\mathbf{S}}_n^{in}$. After $\mathbf{S}'_n$ has caught up with $\hat{\mathbf{S}}_n^{in}$ (which should happen before $\mathbf{S}_n^{in}$ reaches the critical position mentioned in Observation 4), the auxiliary point $\mathbf{S}'_n = \hat{\mathbf{S}}_n^{in}$ on $\mathcal{C}$ remains at either $(0, \cos\beta, \sin\beta)^T$ or $(0, -\cos\beta, \sin\beta)^T$ until the input SOP sequence reaches a pole again. During this period, the DC-PS from Observation 1' does not change, i.e., the value ϕ_4^n remains constant. The result is a value $C_{PC} = \max c^n \approx 13$ in this case (note the difference in the definition of c^n compared to arbitrary PC). **Numerical tests with values** $A < 3.5$ lead to a violation of iii), while for larger A the obtained constant C_{PC} grows linearly with A, as expected. For $\beta \approx \pi/8 \ldots \pi/6$, the observed minimal values of C_{PC} were comparable while for $\beta < \pi/8$ and $\beta > \pi/6$ they tended to become larger.

This example also shows that the 3-parameter PS-DC-PS architecture given by (5) does not guarantee endless, reset

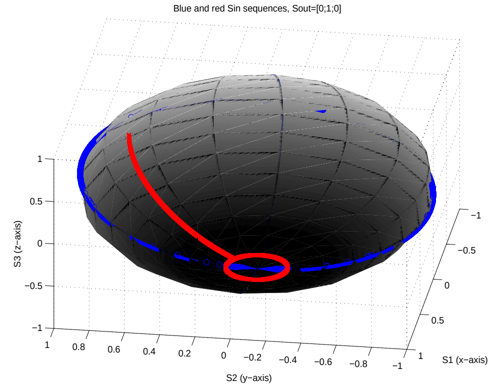

Fig. 2. Special $\mathbf{S}^{in}$ sequences for evaluating C_{PC} for the 4-parameter PC-architecture

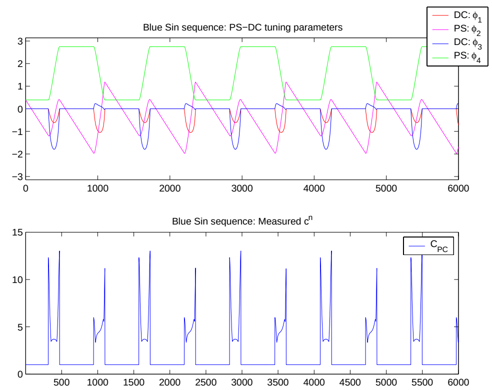

Fig. 3. Tuning parameters and measured C_{PC} values (blue $\mathbf{S}^{in}$ sequence)

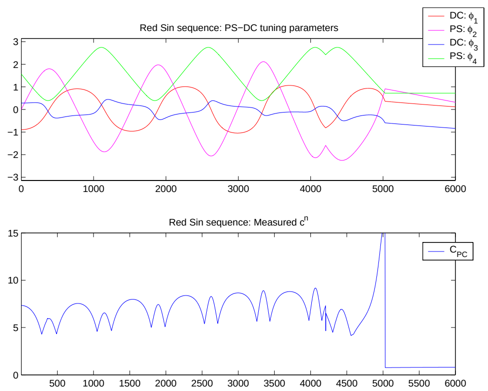

Fig. 4. Tuning parameters and measured C_{PC} values (red $\mathbf{S}^{in}$ sequence)

free operation in the sense of the definition given in Sec. II, even for PC scenario **A** (it does not matter that we have fixed $\mathbf{S}^{out} = (0,1,0)^T$, this example also works for any other fixed $\mathbf{S}^{out}$). On the contrary, assume that properties i)-iii) can be achieved with the architecture (5) with some $C_{PC} < \infty$ and some fixed bounds on the tuning parameters ϕ_i. Take $h > 0$ small enough. For $n = 0$, where $\mathbf{S}_0^{in} = \mathbf{S}^{out}$, we may assume without loss of generality that $\phi_1^0 = \phi_2^0 = \phi_3^0 = 0$. Then, using the fact that $\|\mathbf{S}_n^{in} - \mathbf{S}_{n-1}^{in}\| = 2\sin(h/2)$ is fixed and small, and the interpretation of PS and DC as rotations about the x- resp. y-axis, it is easy to realize that iii) implies that always $\phi_2^n = 0$ and $\phi_1^n + \phi_3^n = -nh$ (there is no way that any of these angles jumps by π in any of the steps). But the latter equality shows that, no matter how we choose ϕ_1^n, ϕ_3^n, by letting n grow we eventually contradict property ii), i.e., at least one of these tuning parameters will violate the imposed range limitations. Similar considerations are possible for other ≤ 3-parameter architectures suggested in the literature, such as, e.g., combinations of quarter and half wave plates, and we are confident that enforcing endless, reset free operation in the sense of our definition for PC scenario **A** requires a minimum 4-parameter solution. On the other hand, for PC scenario **B** the 7-parameter solution demonstrated above is probably not minimal.

The second input SOP sequence depicted in red in Fig. 2 winds around the pole $[0;0;-1]$ on a small circle before switching to a circular path, similar to the previous example. By doing this, **our simulation runs reveal that for the control algorithm to satisfy the requirement iii) slightly larger values of $A \approx 5$ are required**, and that $C_{PC} \approx 15$ is a more realistic minimal value for the constant in iii) achievable with this 4-parameter architecture. Indeed, by carefully choosing the small circle and taking a very small step-size $h > 0$, we can force the algorithm to produce an $\mathbf{S}'_n$ that is located on $\mathcal{C}$ almost exactly on the opposite side from $\hat{\mathbf{S}}_n^{in}$. This way, if we now turn away from the pole on a circular path, similarly to the previous example, there remains a smaller distance for turning $\mathbf{S}'_n$ back into $\hat{\mathbf{S}}_n^{in}$ which makes taking larger A a necessity. The resulting DC-PS parameters and c^n values for $A = 4.9$, $\beta = \pi/8$ are shown in Fig. 4.

Finally, we have also tested the algorithm on randomly generated input SOP sequences. Fig. 5 shows an input SOP distribution with $N = 2000$ points, and the results of our control algorithm (again with $\beta = \pi/8$ and $A = 4.9$). The observed c^n values are typically smaller than those obtained in the above worst-case scenarios.

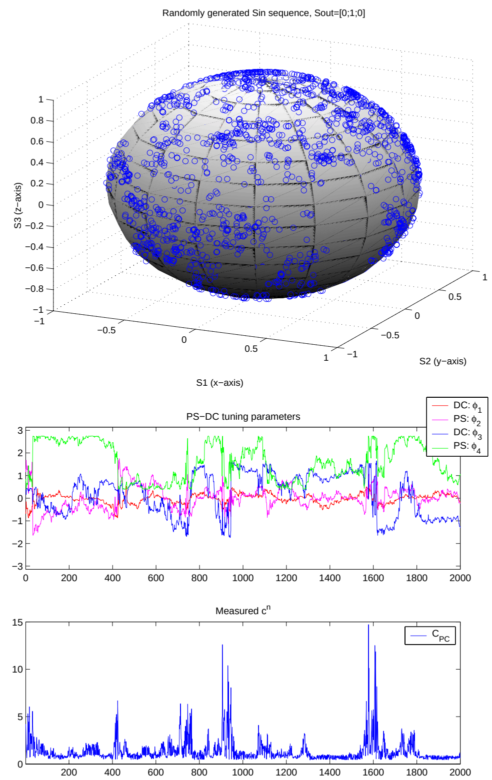

Fig. 5. Test with random input SOP sequence

IV. Conclusions

We have described a factorization algorithm of an arbitrary polarization transformation into WP transformations with restricted rotation angle resulting in endless, reset free PC using PS-DC sequences of PS driven by phase shifters. The number of tuning parameters for arbitrary PC is 7, if either input or output SOP are fixed then 4 or 5 tunable parameters are sufficient. The control algorithm which is based on input/output SOP information is simple and fully deterministic. Rigorous proof is provided that changes in the tuning parameters are bounded by an absolute constant times changes in input/output SOP changes. **Simulations support our theoretical findings.**

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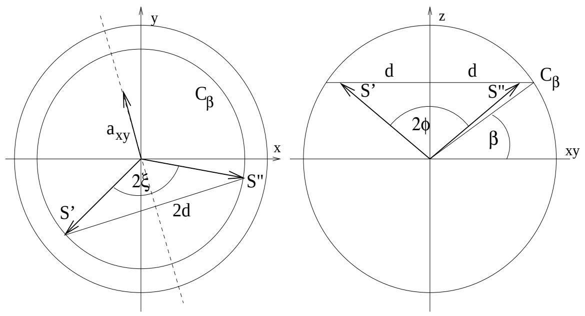

Fig. 6. Notation for the proof of Observation 1

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APPENDIX I
JUSTIFICATION OF OBSERVATIONS 1-4

Observation 1. The geometric argument is evident from Fig. 6 (the viewing angle for the picture on the right is in the direction of the rotation axis shown by a dashed line in the left picture). Because the rotation axis of a WP has to be in the xy-plane, there are two possible choices for $\mathbf{a} = (a_x, a_y, 0)$, and we take the one which is in the direction of the projection of $(\mathbf{S}' \times \mathbf{S}'')$ into xy-plane, and has length $\sqrt{1-c^2}$, where

$$c = \cos(\frac{\phi}{2}) = \frac{\sin\beta}{\sqrt{d^2 + \sin^2\beta}}.$$

Here $2d = \|\mathbf{S}' - \mathbf{S}''\|$ is the Euclidean distance between the two Stokes vectors. Since scalar and vector product as well as distance squared are smooth functions of their arguments, by the chain rule $\mathbf{a}$ and c and, thus, the PS-DC-PS parameters in (4) are smooth functions of $\mathbf{S}'$, $\mathbf{S}''$ which shows the claims made in Observation 1. A moment's reflection shows that there is no ambiguity in the definition of $\mathbf{a}$ when $d \to 0$ because then $c \to 1$ and $\mathbf{a} \to \mathbf{0}$. Note that $0 < \cos(\phi_0/2) = \sin\beta \le c = \cos\frac{\phi}{2} \le 1$ under the assumptions made, thus the WP is admissible.

Observation 2. To see this, look at the point on $\mathcal{C}$ farthest away from $\mathbf{S}$ which is opposite to $\hat{\mathbf{S}}$, observe that reaching it from $\mathbf{S}$ by a WP transformation results in the largest rotation angle, and check that under the restrictions on S_z valid for regions A and B, this maximal ϕ satisfies the admissibility condition $|\phi| \le \phi_0$.

Observation 3. Since $\mathbf{S}$ and $\hat{\mathbf{S}}$ belong to a large circle on the Poincare sphere through the poles $(0,0,\pm 1)^T$, a simple geometric argument shows that the axis of the associated WP transfomation should be normal to the plane of this circle. Computations show that the parameters of this WP are given by

$$c = \sqrt{\frac{1 + \mathbf{S}\cdot\hat{\mathbf{S}}}{2}}, \quad \mathbf{a} = \frac{\mathbf{S}\times\hat{\mathbf{S}}}{2c}.$$

Since $\hat{\mathbf{S}}$ is closest to $\mathbf{S}$ on $\mathcal{C}$, we have

$$\mathbf{S}\cdot\hat{\mathbf{S}} \ge \cos(\beta + \pi/2) = -\sin\beta > -1/2$$

which shows that $c > 0$ and $\mathbf{a}$ are well-defined, and depend smoothly on $\mathbf{S}$ (as long as $\mathbf{S}$ stays away from the poles $(0,0,\pm 1)$).

Observation 4. First, note that in terms of Jones matrices, uniqueness is understood up to an arbitrary sign choice (i.e., U and $-U$ represent the same polarization transformation). Because the axis $\mathbf{a}$ of a WP transformation rotating $\mathbf{S}$ into $\mathbf{S}'$ by definition belongs to the xy-plane, and should satisfy $\mathbf{a}\cdot(\mathbf{S}' - \mathbf{S}) = 0$, the only ambiguity in determining this WP from $\mathbf{S}$ and $\mathbf{S}'$ occurs when $\mathbf{S}' - \mathbf{S}$ is orthogonal to the xy-plane, i.e., if $\mathbf{S}$ is the reflection of $\mathbf{S}'$ about the xy-plane which is the exceptional situation excluded. In all other cases, $\mathbf{a}$ and c are uniquely determined (up to sign choices) by

$$\mathbf{a} = (a_x, a_y, 0),\ a_x^2 + a_y^2 + c^2 = 1,\ \mathbf{a}\cdot(\mathbf{S}' - \mathbf{S}) = 0,\ c = \cos\frac{\phi}{2},$$

where $|\phi| \le \phi_0$ is obtained from $\cos\phi = \mathbf{S}\cdot\mathbf{S}'$.

APPENDIX II
PROOF OF iii)

It is sufficient to consider the algorithm for determining $\{\mathbf{S}'_n \in \mathcal{C}\}$ from $\{\mathbf{S}^{in}_n\}$ described in Sec. III (to simplify notation, we will write $\mathbf{S}_n = \mathbf{S}^{in}_n$). For simplicity, we consider the particular parameters $\phi_0 = 5\pi/6$, $\beta = \pi/6$, and an A specified below. **The arguments can easily be extended to other** β, ϕ_0 **related by (10), resulting in different values for** A. Thus, $\mathcal{C}$ has radius $\sqrt{3}/2$, and the boundary circle of region A (the spherical cap centered at the pole $\mathbf{P} := (0,0,-1)^T$), and defined in Observation 2) has radius $1/2$. We call the reflection of region A w.r.t. the xy-plane region A* (a spherical cap contained in region B bounded by a circle of radius $1/2$), and region C the set of all points $\mathbf{S}$ outside regions A and A*. Finally, for any $\mathbf{S}$ we denote by $\tilde{\mathbf{S}}$ resp. $\tilde{\mathbf{S}}^*$ the point on the boundary circle of region A resp. A* which is closest to $\mathbf{S}$.

By induction, we will show that the algorithm yields the following properties: If $\mathbf{S}_n$ belongs to region C, we always have $\mathbf{S}'_n = \hat{\mathbf{S}}_n$, while for $\mathbf{S}_n$ from regions A or A' the point $\mathbf{S}'_n$ satisfies

$$\tau_n := \tau(\mathbf{S}'_n, \hat{\mathbf{S}}_n) \leq B d_n, \tag{13}$$

where $\tau(\mathbf{S}', \mathbf{S}'')$ denotes the arclength distance between $\mathbf{S}' \in \mathcal{C}$ and $\mathbf{S}'' \in \mathcal{C}$ along the circle $\mathcal{C}$, $d_n := \min(\|\tilde{\mathbf{S}}_n - \mathbf{S}_n\|, \|\tilde{\mathbf{S}}^*_n - \mathbf{S}_n\|)$, and the constant B will be defined later. Note that arclength and Euclidean distance are related by $\|\mathbf{S}' - \mathbf{S}''\| \leq \tau(\mathbf{S}', \mathbf{S}'') \leq \pi\|\mathbf{S}' - \mathbf{S}''\|/2$. These properties, together with Observations 2 and 3 imply that the WP transforming $\mathbf{S}_n$ into $\mathbf{S}'_n$ is always admissible, and Observations 2 and 4 ensure that WP changes are controlled by the changes in $\mathbf{S}_n$ and $\mathbf{S}'_n$ (note that the exceptional situation of Observation 4 does not occur because it belongs to the interior of region C where Observation 2 applies). By definition of the algorithm, see (11), the changes in $\mathbf{S}'_n$ are in turn controlled by the changes in $\mathbf{S}_n$, so that the reset free property iii) follows.

For simplicity, let us assume that initially $\mathbf{S}'_0 = \hat{\mathbf{S}}_0$. Then $\tau_0 = 0$, and the induction assertion is true for $n = 0$. Suppose, it holds after $n-1$ steps. We will first consider the case when $\mathbf{S}_n$ belongs to region C. If $\mathbf{S}_{n-1}$ was also from region C then by induction assumption $\mathbf{S}'_{n-1} = \hat{S}_{n-1}$ and

$$\begin{aligned} \epsilon_n &:= \|\hat{\mathbf{S}}_n - \mathbf{S}'_{n-1}\| = \|\hat{\mathbf{S}}_n - \hat{\mathbf{S}}_{n-1}\| \\ &= \sqrt{3}\|\tilde{\mathbf{S}}_n - \tilde{\mathbf{S}}_{n-1}\| \leq \sqrt{3}\delta_n, \end{aligned}$$

where $\delta_n := \|\mathbf{S}_n - \mathbf{S}_{n-1}\|$. Thus, if $A \geq \sqrt{3}$ then we indeed choose $\mathbf{S}'_n = \hat{\mathbf{S}}_n$. If $\mathbf{S}_{n-1}$ was from region A (the case of region A' is covered by a symmetry argument) then the triangle estimate gives

$$\epsilon_n \leq \|\hat{\mathbf{S}}_n - \hat{\mathbf{S}}_{n-1}\| + \|\hat{\mathbf{S}}_{n-1} - \mathbf{S}'_{n-1}\|,$$

where the first term is bounded from above by $2\sqrt{3}\delta_n$, and the second by induction assumption for $n-1$ by $\leq B d_{n-1} \leq B\delta_n$. Thus, if $A \geq 2\sqrt{3} + B$ then we have $\epsilon_n \leq A\delta_n$ which implies the induction hypothesis for n also in this case.

If $\mathbf{S}_n$ belongs to region A (region A' is covered by symmetry), then we have to prove (13). If $\mathbf{S}_{n-1}$ is from region A' then obviously $\delta_n \geq \sqrt{3} \geq \epsilon_n$, and $A \geq 1$ implies $\mathbf{S}'_n = \hat{S}_n$, and (13) holds trivially. If $\mathbf{S}_{n-1}$ was from region C, then by arguments already used above, we have

$$\epsilon_n = \sqrt{3}\|\tilde{\mathbf{S}}_n - \tilde{\mathbf{S}}_{n-1}\| \leq 2\sqrt{3} d_n,$$

which again trivially implies (13) if $A \geq 2\sqrt{3}$.

In the remaining, most involved case, where both $\mathbf{S}_{n-1}, \mathbf{S}_n$ belong to region A, we start with the geometrically obvious estimate

$$\begin{aligned} \tau_n &\leq \tau_{n-1} + \tau(\hat{\mathbf{S}}_n, \hat{\mathbf{S}}_{n-1}) - \tau(\mathbf{S}'_{n-1}, \mathbf{S}'_n) \\ &\leq B d_{n-1} + \frac{\pi\sqrt{3}}{2}\|\tilde{\mathbf{S}}_n - \tilde{\mathbf{S}}_{n-1}\| - A\delta_n, \end{aligned}$$

which holds under the assumption that $\mathbf{S}'_n \neq \hat{\mathbf{S}}_n$ (otherwise there is nothing to prove). We now set $B := \sqrt{3}\pi/(4\sin(\pi/24)) \approx 10.4$. This allows us to assume further that $d_n < 2\sin(\pi/24)$ because otherwise $B d_n \geq \sqrt{3}\pi/2$ and (13) automatically holds. In other words, $\mathbf{S}_n$ is closer to the boundary circle of region A than to the pole $\mathbf{P}$. This implies first of all that

$$\|\tilde{\mathbf{S}}_n - \tilde{\mathbf{S}}_{n-1}\| \leq 4\delta_n,$$

the worst case occurs if $d_n \to 2\sin(\pi/24)$ and $\mathbf{S}_n$ and $\mathbf{S}_{n-1}$ go in opposite directions from the pole $\mathbf{P}$. Finally, if $d_{n-1} - d_n > 0$ then

$$d_{n-1} - d_n \leq t \leq \delta_n.$$

Substituting all this into the above inequality, we get

$$\begin{aligned} \tau_n &\leq B d_n + B(d_{n-1} - d_n) + \frac{\pi\sqrt{3}}{2}\|\tilde{\mathbf{S}}_n - \tilde{\mathbf{S}}_{n-1}\| - A\delta_n \\ &\leq B d_n + (B + 2\sqrt{3}\pi - A)\delta_n = B d_n, \end{aligned}$$

if we set $A := B + 2\sqrt{3}\pi$. I.e., we have established (13). This value of $A \approx 21.3$ satisfies all previous requirements, which concludes the proof that our algorithm guarantees the reset free property iii) with some finite C_{PC}.

Appendix III
DC-PS transformation from $\mathcal{C}$ into $(0,1,0)^T$

For the suggested DC-PS transformation, we have to solve $\mathbf{E}^{out} = U_{PS,\phi_2} U_{DC,\phi_1} \mathbf{E}$, where $\mathbf{E}$ corresponds to an arbitrary SOP $\mathbf{S} \in \mathcal{C}$, and $\mathbf{E}^{out} = [1,\ 1]^T/\sqrt{2}$ to the fixed output SOP $(0,1,0)^T$. Equivalently, this equation reads

$$\begin{aligned} \mathbf{E} &= U_{DC,-\phi_1} U_{PS,-\phi_2} \mathbf{E}^{out} \\ &= \frac{1}{\sqrt{2}} U_{DC,-\phi_1} \begin{pmatrix} e^{j\phi_2/2} \\ e^{-j\phi_2/2} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{j\phi_2/2}\cos\frac{\phi_1}{2} + j e^{-j\phi_2/2}\sin\frac{\phi_1}{2} \\ e^{-j\phi_2/2}\cos\frac{\phi_1}{2} + j e^{j\phi_2/2}\sin\frac{\phi_1}{2} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{\phi_2-\phi_1}{2} + j\sin\frac{\phi_2+\phi_1}{2} \\ \cos\frac{\phi_2+\phi_1}{2} - j\sin\frac{\phi_2-\phi_1}{2} \end{pmatrix}. \end{aligned}$$

From this, we get a formula for the Stokes space representation $\mathbf{S}$ of $\mathbf{E}$, and use $\mathbf{S} \in \mathcal{C}$:

$$\begin{aligned} \cos\beta\cos t &= \sin\phi_1 \sin\phi_2 \\ \cos\beta\sin t &= \cos\phi_2 \\ \sin\beta &= \cos\phi_1 \sin\phi_2. \end{aligned}$$

From the second equation, we find

$$\phi_2 = \arccos(\cos\beta\sin t) \in [\beta,\ \pi - \beta], \tag{14}$$

which also guarantees that $1 \geq \sin\phi_2 \geq \sin\beta > 0$, uniformly in t. Thus, transforming the remaining two equations into

$$\sin\phi_1 = \frac{\cos\beta\cos t}{\sin\phi_2}, \qquad \cos\phi_1 = \frac{\sin\beta}{\sin\phi_2} \in [\sin\beta,\ 1],$$

we obtain a unique solution

$$\phi_1 = \arcsin\left(\frac{\cos\beta\cos t}{\sin\phi_2}\right) \in [-\frac{\pi}{2} + \beta,\ \frac{\pi}{2} - \beta], \tag{15}$$

where the bound on ϕ_1 is obvious from the equation for $\cos\phi_1$. Note that the right-hand sides in (14), (15) are continuously differentiable, 2π-periodic functions of $t \in \mathbb{R}$. This establishes the claims in Observation 1'.

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