Recursive definition of uniform B-splines

\[ B_m(x) = \frac{1}{m} \left( x \cdot B_{m-1}(x) + (2-x) \cdot B_{m-1}(x-1) \right) \]

\[ B_m(x-i) = \frac{1}{m} \left( (x-i) \cdot B_{m-1}(x-i) + (i+1-x) \cdot B_{m-1}(x-i-a) \right) \]
Properties of B-splines

• piecewise polynomial
• unit integral
  \[ \int_{-\infty}^{+\infty} B_m(x) \, dx = 1 \]
• non-negative
  \[ B_m(x) \geq 0 \]
• partition of unity
  \[ \sum_i B_m(x - i) = 1 \]
• support
  \[ B_m(x) \neq 0, \ x \in [0, m] \]
A B-spline can be written as a linear combination of dilates and translates of itself.

\[ B_0(x) = B_0(2x) + B_0(2x-1) \]

\[ B_1(x) = \frac{1}{2} \left( B_1(2x) + 2B_1(2x-1) + B_1(2x-2) \right) \]

\[ B_n(x) = \frac{1}{2^n} \sum_{k=0}^{n+1} \binom{n+1}{k} B_n(2x-k) \]
Refinement masks
Definition:

A curve in B-spline representation is a linear combination of B-splines:

\[ \gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \sum_i B(t-i) p_i = B(t)p \]

with \( B(t) = \begin{bmatrix} \ldots & B(t+1) & B(t) & B(t-1) & \ldots \end{bmatrix} \)

and control points

\[ p = \begin{bmatrix} \vdots \\ p_0 \\ p_1 \\ \vdots \end{bmatrix} \]
Curve in B-spline representation
Refinement of curve

Refine each B-spline in linear combination:
Refinement of curve

\[ \gamma(t) = \sum_i p_i B(t - i) \]

\[ = \sum_i p_i \left( \sum_k s_k B(2(t - i) - k) \right) \]

\[ = \sum_j B(2t - j) \sum_i s_{j-2i} p_i \]

refined bases

refinement of control points
Refinement of curve

Linear operation on control points:

\[ \gamma(t) = \sum_i p_i B(t - i) = B(t)p \]

\[ B(t) = \sum_k s_k B(2t - k) = B(2t)S \]

\[ \gamma(t) = B(t)p = B(2t)Sp \]
Refinement of curves

Bases and control points:

$$\gamma(t) = B(t)p = B(2t)Sp$$

$$B(t) = B(2t)S$$

$$p^1 = Sp^0$$
Subdivision operator

\[ B(t) = B(2t)S \]

\[ \frac{1}{8}(1, 4, 6, 4, 1) \]

- \( S \) is called subdivision matrix
- Subdivision is stationary
Subdivision

Apply successive subdivision of control polygon formed by sequence of control points:

\[ p^{j+1} = S p^j \]

\[ P^j(t) \]
\[ P^{j+1}(t) \]
\[ P^{j+2}(t) \]
Convergence

• Subdivision with the presented subdivision matrix converges to a curve in cubic B-spline representation.
• Hence, one can draw the control polygon instead of the curve itself.
• The subdivision matrix defines the behavior of the resulting curve in terms of convergence, smoothness, and approximation/interpolation property.
Chaiken's corner cutting algorithm (1974)

\[ Q_0 = \frac{1}{4} P_0 + \frac{3}{4} P_1 \]
\[ Q_1 = \frac{3}{4} P_0 + \frac{1}{4} P_1 \]
\[ Q_2 = \frac{1}{4} P_1 + \frac{3}{4} P_2 \]
\[ Q_3 = \frac{3}{4} P_1 + \frac{1}{4} P_2 \]
\[ Q_4 = \frac{1}{4} P_2 + \frac{3}{4} P_3 \]
\[ Q_5 = \frac{3}{4} P_2 + \frac{1}{4} P_3 \]
Corner cutting
Corner cutting

A control point

The limit curve

The control polygon
4-point scheme

(Dyn, Gregory, Levin 1987)
4-point scheme

\[ p_{2i}^{k+1} = p_i^k \]

\[ p_{2i-1}^{k+1} = -\frac{1}{16} p_{i-3}^k + \frac{9}{16} p_{i-2}^k + \frac{9}{16} p_{i-1}^k - \frac{1}{16} p_i^k \]
4-point scheme
The 4-point scheme is interpolating, while the corner cutting scheme is approximating.
Subdivision Curves/Surfaces

Approach limit curve/surface through an iterative refinement process:

Refinement 1

Refinement 2

Refinement $\infty$
1.3.2 Subdivision Surfaces
Tensorproduct surfaces

\[ u(x, y) = \sum_{i,j} \rho_{ij} B(x-i) B(y-j) \]

\[ t(y) = \sum_{i} a_i B(y-i) \]

\[ s(x) = \sum_{i} p_i B(x-i) \]

\[ p_{00}, p_{01}, p_{11}, p_{12}, p_{21}, p_{11} \]

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Tensorproduct surfaces

- Tensorproduct surfaces are defined over regular quadrilateral meshes.

- Sometimes semi-regular quadrilateral meshes are allowed, which have a few, sufficiently separated vertices of valence not equal to 4.
Tensorproduct surfaces

- Semi-regular meshes are necessary, e.g., to represent the boundary surface of a genus 0 object.

- In geometric modeling, patches of regular meshes are traditionally used and stuck together.

- It becomes difficult to handle patch density.
Subdivision scheme $S$ can be described by

- a doubling operator $D$
- an averaging operator $A$

by $S = A^n \cdot D$.

$S$ is called the midpoint operator.
Midpoint operator on tensorproduct surfaces
Midpoint operator on tensorproduct surfaces

• $S = A^2 \cdot D$ is the subdivision scheme by Doo & Sabin. It converges towards a $C^1$-continuous surface.

• $S = A^3 \cdot D$ is the subdivision scheme by Catmull & Clark. It converges towards a $C^2$-continuous surface.