Efficient Optimization of Reconfigurable Aperture Antennas

by

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A thesis submitted in partial fulfillment for the degree of Master of Science

in the

School of Engineering and Science

August 2011
Declaration of Authorship

I, Samee ur Rehman declare that this thesis titled, ‘Efficient Optimization of Reconfigurable Aperture Antennas’ and the work presented in it are my own. I confirm that:

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“The true greatness of man lies in his capacity for eternal progress.”

Abu Hamid Al-Ghazali
Reconfigurable aperture (RECAP) antennas consist of a regular array of reconfigurable elements whose state can be changed dynamically, supporting beamforming, null-steering, adaptive matching, frequency agility, etc. Finding the required state of the reconfigurable loads for a given application is difficult, due to the non-convex and non-linear nature of the optimization problem, and generic unstructured global optimization methods such as genetic algorithms and particle swarm optimization have been proposed. Although appropriate for off-line optimization, such unstructured methods are computationally intensive and may not be appropriate for dynamic in-situ optimization, indicating the need for methods that exploit the problem structure directly. This thesis presents a novel method for optimization of parasitic RECAPs for beamforming and null-steering which combines direct optimization of a reduced-order reflection model of the RECAP with efficient Newton based optimization. The developed method is orders of magnitude more efficient for beamforming and null-steering than unstructured optimization methods. The work also takes into consideration phase limited nature of reconfigurable elements and considers in-situ optimization of RECAPs.
I would like to express my heartfelt thanks and utmost gratitude to my Thesis advisor Prof. Dr. Jon Wallace for always taking out valuable time from his packed schedule for me, for bearing patiently with my questions and supporting and guiding me through his expertise and advice. I am heavily indebted to his kindness and mentoring skills. I would also like to thank Prof. Dr. Vincent Lau deeply for taking out the time to review this Thesis work. I would also like to express my gratitude and appreciation for the Computational Laboratory for Analysis, Modeling, and Visualization (CLAMV) support staff for allowing me to use the Jacobs University lab computers for my work. In the end, I would like to thank my family and friends for their continued support and encouragement.
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Chapter 1

Introduction

1.1 Motivation

Exponential growth in the field of wireless communications has resulted in the rapid saturation of the Radio Frequency (RF) spectrum. The ubiquitous use of wireless systems with wide ranging applications coupled with the increased demand of users for improved system performance has given rise to widespread interest in the development of adaptive architectures that can make better use of the limited RF resources available. The idea of using adaptive antenna systems to enhance system capacity and suppress interference levels has, therefore, become a strong focus of research in the recent past.

Smart antennas systems have been proposed as viable candidates for realization of such adaptive architectures [1]. By applying smart signal processing to an array of antenna elements these structures can support beamforming, interference suppression, spatial multiplexing and diversity [2]. The drawback with smart antennas however, is the requirement of expensive Digital Signal Processing (DSP) resources along with multiple RF chains to perform the advanced signal processing at baseband.

Reconfigurable aperture (RECAP) antennas, on the other hand, can support similar functionality as the smart antenna concept, with only a single radio frequency (RF) and digital signal processing (DSP) channel, providing reduced cost and power consumption. Furthermore, RECAPs also have the potential to achieve frequency agility and adaptive matching; operations that typically can not be supported by smart antennas.

RECAPs can be viewed as a generalization of reconfigurable antennas or parasitic antenna arrays, and consist of a single feed element surrounded by other elements loaded with programmable reactances. These reconfigurable elements (REs) or reactive loads of
the antenna array are varied to modify the antenna radiation characteristics. These arrays were first introduced by Harrington in [3] and more recently have gained significant attention [4–6].

Dipole antennas have historically been preferred in the study of parasitic arrays due to the simplicity of the wire structure and the relative ease of analysis. Recently, however, the demands of practicality have resulted in the use of planar arrays such as interconnected patch antennas as the parasitic arrays [7–10]. Reconfigurability is typically achieved in these antenna structures by using MEMS (Micro-electromechanical switches) [8, 11, 12], PIN diodes [13], FETs [14] or varactor diodes [15].

RECAPs usually consist of a large array of reconfigurable elements which helps in providing greater degree of reconfigurability. If the corresponding loss of the REs is relatively small, RECAPs can be expected to optimally manipulate antenna radiation characteristics for a particular application. Unfortunately, a potential difficulty with reactively controlled parasitic arrays in general, is that the optimization of the reconfigurable elements for a particular application is a non-linear and non-convex optimization problem, usually requiring global search methods.

Although the potential of RECAPs concept has been analyzed in detail, finding an efficient method to solve for optimal reconfigurable loads is still a work in progress. In fact, finding a solution to this problem that is not based on time consuming global search methods, is a relatively novel idea which has not been sufficiently explored. There are many outstanding questions with respect to efficient RECAP optimization, which, if answered, could help in practically implementing RECAPs functionality in an optimal manner and provide further insight into the wide ranging applications of these structures.

1.2 Problem Description

As mentioned previously, the most important challenge of RECAP structures is the problem of optimizing the reconfigurable loads, which suffers from local minima and maxima and often requires a global search of the domain space to find the best solution. Significant research has been performed in using intelligent techniques and derivative-free methods such as genetic algorithms (GAs) [16], particle swarm optimization (PSO), ant colony optimization, and various other techniques to find an optimal solution for the parasitic loads [17],[18]. Although these techniques have been successfully implemented, they typically require extensive time to find an acceptable solution. The derivative-free methods do not directly exploit the structure of the problem and follow an off-line optimization procedure which leads to greater time complexity than what might be
necessary. Furthermore there is no way to ascertain whether the solution found by these algorithms is indeed optimal or not.

Reconfigurable antennas are especially beneficial if the array elements can be optimized dynamically [19], and in such a scenario RECAPs can adapt in-situ [20] to new applications and environmental conditions that are unforeseeable in the design stage. Due to their computational complexity, techniques such as the GA and PSO can only be expected to have limited value for dynamic optimization in a real system. Although libraries of useful solutions for a parasitic array could be generated off-line and stored for real-time use, such methods do not allow the system to adapt to new conditions. The development of novel methods for RECAP optimization that are significantly more efficient than unstructured global optimization methods is therefore an important research topic.

The goal of this thesis is to develop an efficient method for RECAP antennas that is primarily dependent on local optimizers. The structure of the thesis is provided below.

Chapter 2 describes the efficient simulation of the RECAP structures and presents network analysis of the N-port antenna array. A simple first order solution is introduced in the same chapter, which is based on a reduced order expression of the antenna array response. In chapter 3, local optimization based on Newtons method and a multi-dimensional first-order Taylor series is described to steer nulls in a particular direction. This is followed by a discussion on various RECAP topologies and their relative linearity with respect to the first-order Taylor series. In general, practical REs based on tunable capacitances can have at most 180 degrees phase reconfigurability. Taking this factor into account beam forming and null steering are performed using only capacitive or inductive reconfigurable loads with reconfigurability limited to 180 degrees in Chapter 3.

Thereafter, Chapter 4 describes the various methods by which a main beam can be formed in a certain direction while maintaining nulls in other directions. It also sheds light on the limitations of trying to treat the highly non-linear RECAP problem as approximately linear.

Finally, in Chapter 5 the in-situ problem of RECAPs is introduced. Parameters, such as the embedded patterns and the scattering parameter matrix of the N-port parasitic array, that are required to perform hybrid Newton based optimization are estimated. Sensitivity analysis of these estimated antenna parameters is also presented to test the viability of using the local optimization method in a practical structure where voltages from antenna ports or reflection from the feed to the reconfigurable ports, parameters
that are required to estimate the unknown quantities, can not be measured to high accuracy.

Various antenna topologies will be used in this thesis to exhibit the flexible nature of the optimization method. In general, however, a 5×5 dipole array in a 1\(\lambda\)×1\(\lambda\) aperture is chosen to present and illustrate, with examples, the complete analysis.
Chapter 2

Background

The first challenge in the optimization of RECAPs is to devise an efficient method for simulating these structures. While we can make use of unified full-wave simulation of RECAPs for each unique states of the reconfigurable elements in order to come up with an an accurate model, such a process is very time consuming due to the high number of possible states. In this chapter we show how by taking into consideration the properties of the RECAP network we can efficiently simulate the RECAPs using a small number of Method of Moment simulations.

We thereafter introduce a first order model for the RECAP structure, which is based on a reduced order expression of the antenna array response. By means of geometric interpretation of the parameters involved, we find simple solutions for beamforming and null-steering for the Reduced Order Parasitic RECAP Model. The viability of the model is then tested by supplying the solution to the complete expression of the antenna array response and observing the corresponding beam and nulls found.

2.1 RECAPs-Hybrid Simulation Procedure

In this work we primarily study a 5×5 square dipole array, shown in Figure 2.1, where the half-wave dipoles are spaced at a distance of λ/4 so that the RECAP structure occupies a 1λ × 1λ aperture. The center element serves as the active feeding port and the surrounding elements are the reactive parasitic loads.

Performing unified full-wave simulation of the RECAP structure is computationally infeasible since each state of the reconfigurable elements would require a separate full wave simulation. As the number of possible states of the reconfigurable elements increases, the number of required full-wave simulation procedures would also become larger.
Figure 2.1: 5×5 parasitic dipole array RECAP structure

In order to keep full-wave simulations to a minimum we can model the RECAP as an equivalent circuit \([21–23]\). By doing so, we need to employ only \(N\) full-wave simulations, where \(N\) is the number of antennas in the array, followed by efficient circuit-level analysis in order to characterize the RECAP.

Figure 2.2 shows an \(N\)-port antenna array where the feed element is represented by Port 1 while Port 2 to \(N\) represent the parasitic loads. The scalars \(a_F, b_F, e_F(\hat{s})\), represent the input wave, output wave, and the single polarization embedded radiation pattern of the feed element in direction \(\hat{s}\), respectively. The vectors \(a_R, b_R, e_R(\hat{s})\), represent the input wave, output wave and the embedded radiation pattern of the parasitic ports in direction \(\hat{s}\), respectively. The diagonal input reflection matrix \(\Gamma\) is associated with the reactive reconfigurable elements (REs) with which the parasitic ports are terminated.

The input-output relationship of the RECAP is given by

\[
\begin{bmatrix}
b_F \\
b_R
\end{bmatrix} =
\begin{bmatrix}
s_{FF} & s_{FR} \\
s_{RF} & S_{RR}
\end{bmatrix}
\begin{bmatrix}
a_F \\
a_R
\end{bmatrix},
\]  

(2.1)

where \(s_{FF}\) represents the reflection at the feed port, the vector \(s_{FR}\) represents the coupling from the reconfigurable ports to the feed and the symmetric matrix \(S_{RR}\) gives the coupling of each reconfigurable port to every other reconfigurable port. The coupling from the feed port to the reconfigurable ports is of course given by \(s_{RF}\).

We therefore use a single MOM simulation for each port to obtain network characteristics and embedded radiation patterns of the antennas. Using the Numerical Electromagnetic Code (NEC) to perform the MOM simulation, we supply unit voltage to the excited antenna with all other ports short circuited, and thereby find the admittance matrix \(Y\) and the short circuit radiation patterns \(e^{sc}(\hat{s})\).
Matched \((Z_0\text{-terminated})\) embedded patterns \(e^{mc}\) are then computed by connecting a source to the \(k\)th port and terminating all other ports by the normalizing impedance \(Z_0 = 70\Omega\) to obtain

\[
e^{mc}(\hat{s}) = \frac{e^{sc}(\hat{s})}{\sqrt{Z_0}} Z (I - S).
\]  

(2.2)

The scattering matrix can be obtained by \(S = (I + Z_0 Y)^{-1} (I - Z_0 Y)\).

Network analysis using (2.1) is applied to find the array radiation pattern for arbitrary loads. By observing Figure 2.2 we see that \(a_R = \Gamma b_R\). Using (2.1) along with this information,

\[
a_R = \Gamma (I - S_{RR} \Gamma)^{-1} s_{RF} a_F.
\]  

(2.3)

The matched patterns of the ports are partitioned as \(e^{mc}(\hat{s}) = [e_F(\hat{s}) \ e_R(\hat{s})]^T\), and using superposition the radiation pattern of the complete array is

\[
e(\hat{s}) = a_F e_F(\hat{s}) + \sum_{k=1}^{N-1} a_{R,k} e_{R,k}(\hat{s}).
\]  

(2.4)

Substituting (2.3) into (2.4) we obtain

\[
e(\hat{s}) = [e_F(\hat{s}) + e_R^T(\hat{s}) \Gamma (I - S_{RR} \Gamma)^{-1} s_{RF}] a_F,
\]  

(2.5)
Equation (2.5) describes the radiated far-fields for a single polarization of the general parasitic array into direction \( \hat{s} \).

## 2.2 Reduced Order Parasitic RECAP Model

A basic problem with minimizing or maximizing radiated power given by \( |e(\hat{s})|^2 \) based on (2.5), is the non-convex dependence on \( \Gamma \), which leads to many local minima or maxima and typically necessitates a global search procedure such as a GA. This section describes a simple reduced-order representation of RECAPs that not only allows simple direct optimization, but also provides valuable insight on the operation of the array.

Consider a RECAP with \( a_F = 1 \). In order to create a main beam or a null in an arbitrary direction \( \hat{s} \) we would like to maximize or minimize \( |e(\hat{s})|^2 \) in (2.5) with respect to the diagonal reflection matrix \( \Gamma \) of the reconfigurable loads. In this work it is assumed that the reconfigurable loads are lossless reactances, meaning that the diagonal terms have unit magnitude and are given by \( e^{j\theta} \), so that only the phases of the load reflections can be varied.

Due to the presence of the inverse term in (2.5), direct optimization of \( \Gamma \) is difficult. In order to simplify the problem we consider the Neumann series, or

\[
(I - S_{RR}\Gamma)^{-1} = \sum_{k=0}^{\infty} (S_{RR}\Gamma)^k. \tag{2.6}
\]

The first order solution to this problem would therefore be given by replacing the inverse term \( (I - S_{RR}\Gamma)^{-1} \) representing multiple reflections by \( I \).

The low mismatch, \( S_{RR} = 0 \), leads to the radiation pattern becoming linearly dependent on the reconfigurable loads, and considering a single radiation direction \( \hat{s} \), the first-order model is

\[
e(\hat{s}) = e_F(\hat{s}) + \sum_{m=1}^{N-1} e_{R,m}(\hat{s})s_{RF,m} \gamma_m, \tag{2.7}
\]

where \( \gamma_m = \Gamma_{mm} \).

Visualizing the phasors \( \alpha \), \( \beta_m \), and \( \Gamma_m \) in the complex plane, there is a simple geometric interpretation for the dependence of radiated field in direction \( \hat{s} \) on the load reflection coefficients \( \Gamma_m \) as illustrated in Figure 2.3.
Chapter 2. Background

Changing the phases of the $\Gamma_m$ simply rotates the vectors $\beta_m$ in the complex plane, and the sum of all the rotated vectors gives the radiated field due to the reconfigurable elements. Figures 2.3(a) and (b) illustrate how the vectors can be aligned for beamforming or nullsteering for the simple case of a 2-element RECAP, and the general procedure is explained below.

2.2.1 Beamforming Solution

Beamforming is perhaps the simplest application of the RECAP, and the maximization of (2.7) is accomplished by choosing the phases of $\Gamma_m$ to align the phases of the $\beta_m$ with the phase of $\alpha$. This ensures that all waves from the REs in the direction $\hat{s}$ will add coherently with the radiated wave from the feed. The general solution is $\Gamma_m = \exp[j(\angle \alpha - \angle \beta_m)]$. Figure 2.3(a) illustrates the principle for a single feed and two REs.

The geometrical solution also yields insight on the gain limitation of the RECAP, since the magnitude of the field in direction $\hat{s}$ has a maximum value of $|\alpha| + \sum_n |\beta_n|$.

As an example application of the method, we present beamforming for the square 5×5 RECAP shown in Figure 2.2. Figure 2.4 plots the gain of the parasitic antenna in the azimuthal plane for a steering angle of $\phi_a = 45^\circ$. In this plot, “Random” denotes the gain of the best solution found with a random search of 1000 candidate RE configurations (results averaged over 100 trials). For the “Direct” solution curves, load reflection coefficients (in $\Gamma$) are found using the simple geometric solution, where the curves for “Reduced-order” and “Exact” give the patterns computed for the same solution $\Gamma$, but using (2.7) and (2.5), respectively, corresponding to the reduced-order pattern and the exact pattern.

The result shows that the direct method is able to find a better solution on average than the random search with 1000 candidates. Also, note that the run-times of the two
algorithms with a MATLAB implementation are approximately 9.5 s and 12 ms for the random search with 1000 candidates and the direct method, respectively, indicating that the direct method is much more suitable for real-time implementation than unstructured search methods.

2.2.2 Null-steering Solution

Now consider creation of a single null in direction \( \hat{s} \). In this case, the phases of the \( \Gamma_m \) should be chosen to cancel the wave due to the feed in direction \( \hat{s} \). However, care should be taken since simply aligning the vectors in direction \(-\alpha\) may lead to overshoot. Figure 2.3(b) depicts an arrangement for two REs that will generate a null. For a large number of reconfigurable elements, the best strategy for creating a null is not obvious, since there may be many possible vector-alignment solutions, and parameterizing all of these solutions appears to be non-trivial.

For the simple case illustrated in Figure 2.3 (b) the two vectors, \( \beta_1\Gamma_1, \beta_2\Gamma_2 \) need to be arranged such that their combination with \(-\alpha\) creates a triangle, with sides \( \beta_1\Gamma_1, \beta_2\Gamma_2 \) and \(-\alpha\).

In the first step of vector alignment we assume that \( |\beta_1|\Gamma_1, |\beta_2|\Gamma_2 \) need to cancel \( |\alpha| \). \( |\Gamma| = 1 \) since the loads are lossless. In this scenario the cancellation is given by Figure 2.5 where \( a = |\beta_1|, b = |\beta_2| \) and \( c = |\alpha| \) can be assumed to be three sides of a triangle and \( A \) and \( B \) are the corresponding angles needed to achieve the alignment.

The angles \( A \) and \( B \) required to align the vectors \( a \) and \( b \) such that a triangle is formed with side \( c \) are given by the cosine rule. The formation of such a triangle would mean that an exact cancellation takes place when

\[
A = \cos^{-1} \left( \frac{b^2 + c^2 - a^2}{2bc} \right) \quad (2.8)
\]
The phases on the loads $\Gamma$ to align the vectors for null-steering would then be given by

$$\angle \Gamma_m = \angle (\alpha - \beta_m) + \pi + \theta_m. \quad (2.10)$$

where $\theta_m = [B - A]$ in the case illustrated in Figure 2.5.

It can now be visualized that $(M - 1)/2$ triangles such as the one in Figure 2.5 would be required to cancel $\alpha$ for a $N \times N = M$ RECAP size. Let the three sides of the triangles be given by $a_m$, $b_m$ and $c_m$. The vectors $\beta_m$ are first sorted according to descending magnitude and then divided into $(M - 1)/2$ blocks of two vectors $[|\beta_{a_m}|, |\beta_{b_m}|]$ each. These blocks then form triangles with sides $a_m = |\beta_{a_m}|$, $b_m = |\beta_{b_m}|$ and $c_m$ given by

$$c_m = \begin{cases} 
\frac{a_m + b_m}{(\sum_{n=1}^{M-1} |\beta_n|)/|\alpha|} & \text{if } m < (M - 1)/2 \\
\frac{a_m + b_m}{\alpha - \sum_{n=1}^{M-1} |\beta_n|} & \text{if } m = (M - 1)/2
\end{cases}$$

The corresponding angles $A_m$ and $B_m$ are again found through the cosine rule. The phases on the loads $\Gamma$ to align the vectors for null-steering would then be given by

$$\angle \Gamma_m = \angle (\alpha - \beta_m) + \pi + \theta_m. \quad (2.11)$$

where $\theta_m = [B_m - A_m]$.

Note that like beamforming, the geometric interpretation lends insight on the limits of the RECAPs for the null-steering application. A perfect null can be created in a single
Figure 2.6: Null formation for the reduced order expression and the exact expression when the optimized reduced order $\gamma$ values are supplied to both
direction as long as $\sum_n |\beta_n| \geq |\alpha|$, which indicates the minimum number of elements that are needed for null formation.

When the phases $\Gamma_m$ obtained from the first-order solution are supplied to the exact expression of the objective function in (2.5) we see that the null is often completely destroyed. Figure 2.6 illustrates this behavior. A perfect zero is formed at $\phi_0 = 160^\circ$ for the reduced order expression. But using the same optimized $\Gamma_m$ values from the first order solution in the exact expression leads to a radiation pattern with a relatively high gain value of about $-7$ dB at $\phi_0 = 160^\circ$. This is obviously not a very low null value.

Figure 2.3 also shows why we should expect nulls to be so sensitive to any error. Requiring a high non-zero value, in case of the beam, puts less constraint on the system as compared to finding a zero, in case of the null. When the effect of the error from the higher order terms is taken into consideration, in-phase addition for beam-forming is somewhat insensitive to small phase perturbations. So while in Figure 2.3, an approximate in phase addition of $\alpha$ and $\beta_m \Gamma_m$ would also give a reasonably good value for the beam, for the null, the effect of the error from the higher order terms cannot be neglected since the goal is to obtain zero radiated field in direction $\hat{s}$. Nulls are therefore very sensitive to small changes in the value of $\Gamma$.

Summarizing, when multiple reflections due to $S_{RR}$ being non-zero are taken into account, the value of the main beam is seen to be robust to these errors but nulls are often completely destroyed.
2.3 Conclusion

In this chapter, we first introduced our RECAP of choice, a $5 \times 5$ square structure. Thereafter we showed how, by analyzing the network in terms of the $S$ parameter matrix, we could efficiently simulate the RECAPs using only a single MOM simulation for each port.

The Reduced Order Parasitic RECAP Model was also introduced for beamforming and null-steering. It was shown how a simple geometrical interpretation of the dependence of radiated field in direction $\hat{s}$ on the load reflection coefficients $\Gamma_m$ enabled us to find beamforming and null-steering solutions for the Reduced/First Order model.

The beamforming solution turned out to be a very good solution for the exact expression of the objective function, indicating that the maximization procedure is relatively insensitive to the effect of higher order terms. The beamforming results were also compared to the best gain obtained from a random search of 1000 candidate RE configurations. It was shown that the beamforming solution found from the first-order expression performed better both in terms of gain as well as time complexity.

On the other hand, for null-steering, it was shown that by making use of the cosine rule, we can end up with perfect nulls for the first order case. But when the same optimized first order $\Gamma$ values are used in the exact expression of the objective function, the null is often completely destroyed. It was observed that nulls, requiring almost a perfect zero, are much more sensitive to small phase perturbations compared to main beams.
Chapter 3

Newton Based Optimization

An obvious drawback of the first-order solution is that multiple reflections cause (2.7) to not be precisely equal to (2.5). Although beamforming is somewhat insensitive to this effect, even small amounts of error can destroy null formation. Here we provide a local optimization approach based on Newton’s method that allows exact nulls to be found in an efficient manner.

Our goal is to develop a first-order approximation of (2.5) within the neighborhood of the reflection coefficient matrix $\mathbf{\Gamma}$, which can be efficiently accomplished using a multi-dimensional Taylor series. Considering lossless reconfigurable elements, $\Gamma_{\ell\ell} = \exp(j\theta_{\ell})$, and using the identities for matrix derivatives

$$\frac{\partial \mathbf{A}^{-1}}{\partial x} = -\mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial x} \mathbf{A}^{-1},$$

$$\frac{\partial \mathbf{a}^T \mathbf{b}}{\partial x} = \mathbf{a}^T \frac{\partial \mathbf{A}}{\partial x} \mathbf{b},$$

the derivative of the objective function in (2.5) with respect to the phase of the $\ell$th load can be evaluated as

$$\frac{\partial e(\hat{s})}{\partial \theta_{\ell}} = -\mathbf{e}_R^T (\mathbf{R}^{-1} - \mathbf{S}_{RR})^{-1} \frac{\partial (\mathbf{R}^{-1} - \mathbf{S}_{RR})}{\partial \theta_{\ell}} (\mathbf{R}^{-1} - \mathbf{S}_{RR})^{-1} \mathbf{s}_{RF}. \quad (3.3)$$

Evaluating the inner derivative in (3.3),

$$\left[ \frac{\partial (\mathbf{R}^{-1} - \mathbf{S}_{RR})}{\partial \theta_{\ell}} \right]_{mn} = \frac{\partial (\mathbf{R}^{-1} - \mathbf{S}_{RR})_{mn}}{\partial \theta_{\ell}} \quad (3.4)$$

$$= -j \delta_{m\ell} \delta_{n\ell} \exp(-j\theta_{\ell}) \quad (3.5)$$
where \(1_{ik}\) is an elementary matrix that is all zeros except for a 1 for the \(ik\)th element. Combining (3.3) and (3.19),

\[
\frac{\partial e(\hat{s})}{\partial \theta_\ell} = je_R^T(\hat{s})(\Gamma^{-1} - S_{RR})^{-1}1_\ell(\Gamma^{-1} - S_{RR})^{-1}s_{RF}e^{-j\theta_\ell},
\]  

(3.6)

and the \((N-1)\times1\) gradient vector \(d\) is formed with elements \(d_\ell = \partial e(\hat{s})/\partial \theta_\ell\). A rigorous approach is to model \(e(\hat{s})\) with the first-order multi-dimensional Taylor series for real and imaginary parts of (2.5) by expanding about the point \(\theta_n\) to obtain the solution at a new point \(\theta_{n+1}\).

\[
e_{re}(\hat{s}, \theta_{n+1}) = e_{re}(\hat{s}, \theta_n) + d_{re}^T(\theta_n)(\theta_{n+1} - \theta_n)
\]

(3.7)

\[
e_{im}(\hat{s}, \theta_{n+1}) = e_{im}(\hat{s}, \theta_n) + d_{im}^T(\theta_n)(\theta_{n+1} - \theta_n).
\]

(3.8)

Below we describe two methods for efficiently finding a null based on the first order multi-dimensional Taylor series.

### 3.1 Separate Newton Step

The goal in the optimization is to make both \(e_{re} + e_{im} = 0\). Newton steps can be applied separately to move in the direction of vanishing \(\text{Re}\{e(\hat{s})\}\) and \(\text{Im}\{e(\hat{s})\}\) according to

\[
\theta_{n+1} = \theta_n - \frac{e_{re}(\hat{s}, \theta_n)d_{re}(\theta_n)}{\|d(\theta_n)\|^2} - \frac{e_{im}(\hat{s}, \theta_n)d_{im}(\theta_n)}{\|d(\theta_n)\|^2}
\]

(3.9)

where \(e_{re}(\hat{s}) = \text{Re}\{e(\hat{s})\}\), \(e_{im}(\hat{s}) = \text{Im}\{e(\hat{s})\}\), \(d_{re} = \text{Re}\{d\}\), \(d_{im} = \text{Im}\{d\}\), and \(\theta_n = [\theta_{1,n} \ldots \theta_{N,n}]^T\). Using the first-order solution as a starting point, it is expected that we should be near a null, and only a few Newton steps should be required. Alternatively, we can check whether the first order solution provides a useful starting point by instead seeding the iteration with a random starting point. The iteration in (3.9) is stopped when the null is “deep enough” for a given application or tolerance.

### 3.2 Joint Newton Step

An alternative method is to move in the jointly optimal and minimum norm direction to make both the real and imaginary radiated fields in the specific direction tend to 0. We do this by setting \(e_{re}(\hat{s}, \theta_{n+1}) = e_{im}(\hat{s}, \theta_{n+1}) = 0\),

\[
\begin{bmatrix} e_{re}(\hat{s}, \theta_n) & e_{im}(\hat{s}, \theta_n) \end{bmatrix}^T = -\begin{bmatrix} d_{re}(\theta_n) & d_{im}(\theta_n) \end{bmatrix}^T(\theta_{n+1} - \theta_n)
\]

(3.10)
which can be solved as
\[ \theta_{n+1} = \theta_n - (A^T)^+ b^T, \]  
\[(3.11)\]
where \((.)^+\) is the pseudo-inverse.

Having developed two methods for finding a single null, the methods can be naturally extended to find multiple nulls. In \((3.9)\) we simply extend the Newton step to be
\[ \theta_{n+1} = \theta_n - \sum_k \left[ \frac{e_{\text{re}}(\hat{s}_k)d_{\text{re}}(\hat{s}_k)}{\|d(\hat{s}_k)\|^2} + \frac{e_{\text{im}}(\hat{s}_k)d_{\text{im}}(\hat{s}_k)}{\|d(\hat{s}_k)\|^2} \right], \]  
\[(3.12)\]
where \(e(\hat{s})\) and \(d\) are always evaluated at the current step \(\theta_n\). Alternatively, in \((4.23)\) additional columns are added to \(A\) and \(B\) for the additional directions. In this case, it is clear that if too many directions are added, an exact solution to \((3.10)\) may not be possible, but the method will still find the best step in the least square sense.

Here, it should also be noted that when multiple nulls are considered, the first order solution, according to its current formulation, does not provide a very useful initial seed. This is because the first order solution tries to optimize the solution for only the case of a single null. Therefore in the case of the multiple nulls the use of random seeding makes more sense since first order computation requires relatively more time and does not result in a more efficient null optimization.

### 3.3 Numerical Examples

In this section we first test how well our Newton based optimization performs for the single null case as well as the multiple null case. A comparison of the two methods, the Separate Newton and the Joint Newton, is also presented to find out which method is more efficient in practice. We also compare whether seeding \(\Gamma\) with the first order null solution is useful compared to random seeding when forming nulls using the Newton based optimization. Later we study how joint beamforming and null-steering performs when the first order solution is used for the main beam and the Newton based optimization is used for the nulls. Finally, a comparison of the Newton based optimization with the Genetic Algorithm is presented.

#### 3.3.1 Single Null

First we consider the case of finding a single null in a particular direction. Figure 3.1 considers a \(1 \times 9\) uniform linear array with \(\lambda/4\) spacing and compares the number of iterations needed by the joint and separate Newton methods with random seeds as well
Chapter 3. *Newton Based Optimization*

Figure 3.1: Gain in null direction $\phi = 180^\circ$ versus number of Newton steps for the joint Newton method and separate Newton method with either first-order solution or random seeding. The search is terminated when the goal of $-25$ dB is obtained.

Table 3.1: Performance of Direct-Newton Method

<table>
<thead>
<tr>
<th>$N_{\text{null}}$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_b$</td>
<td>8.7 dB</td>
<td>6.0 dB</td>
<td>-1.4 dB</td>
<td>-17.5 dB</td>
</tr>
<tr>
<td>$G_b-n$</td>
<td>88 dB</td>
<td>88 dB</td>
<td>87 dB</td>
<td>87 dB</td>
</tr>
<tr>
<td>$M$</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>45</td>
</tr>
<tr>
<td>$t$</td>
<td>39 ms</td>
<td>53 ms</td>
<td>85 ms</td>
<td>489 ms</td>
</tr>
</tbody>
</table>

as first-order solution as starting points, where the goal is to find a null at a steering angle $\phi = 180^\circ$. As can be seen the combination of the joint Newton method with first-order solution seeding is the fastest to converge with a null at $-55$ dB after only three iterations. The slowest convergence rate is exhibited by the separate Newton method with random seeding which requires 10 iterations to find a null lower than the threshold of $-25$ dB. It is also interesting to note that the separate Newton method with the first-order solution as a starting point outperforms the joint Newton method with random seeding, thus illustrating the value of using the first-order approximation over a random starting point. Again, it needs to be pointed out that the first-order approximation can only be used for the single null case.

3.3.2 Multiple Nulls

Next, using our $5 \times 5$ element structure with $\lambda/4$ spacing, from 1 to 8 nulls are formed by applying (4.23) iteratively until a minimum main-beam to null gain ($G_{b-n}$) separation of 60 dB is achieved. The first-order solution is used to obtain the main beam in the specified direction. The method was run for 200 realizations consisting of a random main beam and random null positions having a minimum angular separation of $10^\circ$, and average performance is given in Table 3.1 for different numbers of nulls.
The result indicates that near exact nulls can be created with relatively few iterations \((M)\) of the algorithm and low average run times per realization \((t)\), but that for increasing numbers of nulls the gain obtained from the first-order solution in the specified main-beam direction \((G_b)\) is significantly degraded.

The table suggests that when the amount of reconfigurability is high relative to the number of nulls, there are numerous roots of \((2.5)\) available in search space and “excess reconfigurability” allows nulls to be found very efficiently with minimal impact on the first-order solution for the main beam. However, as the number of nulls increases and approaches the limits of the structure, the number of required iterations increases dramatically and the impact on the main beam is significant.

### 3.3.3 Comparison with a Genetic Algorithm

Genetic algorithms are well suited for difficult optimization problems with multiple goals such as RECAP optimization. Here, we compare the joint beamforming and nullsteering solution from the previous section with solutions obtained by a genetic algorithm (GA).

The GA here is similar to that employed in [24], and operates on a population of 100 candidate individuals, where the best 10 individuals are retained in each iteration (elitism). The chromosome vector consists of genes that are the real-valued phases \(\theta\), where the \(5\times5\) array is scanned in a raster-like fashion. For each iteration, parents are selected randomly from the 10 best individuals, and the child chromosome is generated by copying element-by-element from one parent, where for each copy there is a probability of 0.2 of switching (crossing over) to the other parent, allowing multiple cross-overs to occur. Finally, each gene of the children is mutated with a probability of 0.2, and a mutation causes that element to assume a random phase uniformly distributed on \([0, 2\pi]\). The fitness function is the minimum gain separation from the main beam to nulls, given by

\[
G_{b-n} = G_{dB}(\phi_{beam}) - \max_k G_{dB}(\phi_{null,k}),
\]

(3.13)

where \(\phi_{beam}\) is the direction of the desired main beam, \(\phi_{null,k}\) is the direction of the \(k\)th desired null, and the genetic algorithm is stopped when \(G_{b-n}\) reaches 60 dB or 1000 iterations have been performed.

Figures 3.2(a)-(d) compare the direct solution with a single run of the genetic algorithm for a desired main beam at \(45^\circ\) and from 1 to 8 nulls at fixed angles \(100^\circ, 130^\circ, \ldots, 310^\circ\). The direct solution uses the same approach as in the previous section, where a main beam is found with the first-order solution and refined with the Newton-based method to find the nulls.
From 1 to 4 nulls, there is remarkable similarity in the solutions found by the two distinct methods, with the direct method providing slightly higher gain and deeper nulls. For 8 nulls, the performance of the two methods diverges, since the GA is not able to find a suitable solution after 1000 iterations.

Table 3.2 gives the numerical performance of the genetic algorithm and direct method. As indicated, for relatively few nulls, the performance of the two methods is similar, except that the direct method is over 1000 times faster than the GA.
Table 3.2: Comparison of genetic algorithm and direct method

<table>
<thead>
<tr>
<th>$N_{null}$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_b$</td>
<td>GA</td>
<td>8.7 dB</td>
<td>5.9 dB</td>
<td>7.7 dB</td>
</tr>
<tr>
<td></td>
<td>Direct</td>
<td>11.4 dB</td>
<td>10.0 dB</td>
<td>9.1 dB</td>
</tr>
<tr>
<td>$G_{b-n}$</td>
<td>GA</td>
<td>67 dB</td>
<td>60 dB</td>
<td>48 dB</td>
</tr>
<tr>
<td></td>
<td>Direct</td>
<td>96 dB</td>
<td>95 dB</td>
<td>73 dB</td>
</tr>
<tr>
<td>$t$</td>
<td>GA</td>
<td>33 s</td>
<td>133 s</td>
<td>554 s</td>
</tr>
<tr>
<td></td>
<td>Direct</td>
<td>0.033 s</td>
<td>0.049 s</td>
<td>0.084 s</td>
</tr>
</tbody>
</table>

3.4 Topology Comparison - Taylor Series

Having shown we can approximate the objective function linearly in a small neighborhood to establish a useful null optimization method, it now becomes relevant to look at different RECAP topologies and establish which one exhibits the most linear/non-linear behavior while also possessing ample reconfigurability.

To this end array topologies of different sizes (9 elements or 25 elements), types (linear, square, circular) and with different inter-element spacing $\lambda/10$, $\lambda/4$, $\lambda/2$, etc. were studied. The idea was to find a particular structure that exemplifies the most optimum use in terms of linearity, time complexity and reconfigurability when applying the first order multi-dimensional Taylor series.

A certain steering angle is chosen, for instance $\phi = 180^\circ$ and each topology is modeled with the multi-dimensional Taylor series for that direction. A Monte Carlo analysis is then performed. $M = 100$ random samples of $\theta$ are chosen, where, for each realization, the Taylor series expression is recomputed with discrete changes in the value of $\theta$ from $\Delta = 0^\circ - 57^\circ$ with a step size of 0.01$^\circ$. Let $\delta_i$ represent the $i$th variation in the value of $\Delta$

$$e_{taylor}(\hat{s}, \theta + \delta_i) = e(\hat{s}, \theta) + \delta_i \frac{\partial e(\hat{s})}{\partial \theta}$$ (3.14)

The result of $e_{taylor}(\hat{s}, \theta + \delta_i)$ is then compared to the actual $e(\hat{s}, \theta + \delta_i)$ obtained from the objective function in (2.5) and the difference between the values, representing the error is noted.

Let $\psi_i$ represent the error over 100 iterations at each variation point $\delta_i$

$$\psi_i = \frac{\sum_{j=1}^{M} |e_{taylor}(\hat{s}) - e(\hat{s})|}{M}$$ (3.15)

In order to illustrate the comparison between each topology better we scale $\psi_i$ of each topology with the mean value of $\psi_i$ over the $M$ random realizations.
Chapter 3. **Newton Based Optimization**

Figure 3.3: Plot of the average error $\Psi$ against peak RE variation $\Delta$. The results are plotted for different topologies: linear, square, circular with different numbers of elements $N$ and different inter-element spacing $\lambda$.

For each topology, the average error $\Psi$ is then plotted against $\Delta$. Figure 3.3 shows the result for the different topologies compared. It can be seen from the figure, that the greater the inter-element spacing, the more linear the topology is. Hence, the lowest error $\Psi$ is exhibited by the topologies with greater inter-element spacing.

It is interesting to note, that the performance of $0.1\lambda$ inter-element spaced square and circular RECAPs perform significantly worse than all the other topologies. This can be attributed to the fact that when inter-element spacing is so small, the reconfigurable elements are strongly coupled to one another and therefore exhibit even stronger non-linear behavior. This gives rise to greater error when the linearly approximated Taylor series is compared to the objective function in (2.5).

From the plots, it can also be surmised, that in general, the 9 element RECAPs have much lower error. But this particular advantage is nullified when taking into account the lack of enough reconfigurability in these relatively few element structures. The small size of this topology means that maintaining high number of nulls is not possible. But,
in case we are looking for only a single null along with a main beam, the 9 element RECAPs ought to be preferred.

The linear RECAPs generally have a very low $\Psi$ (error) value. But this topology does not perform very efficiently when searching for nulls in the end-fire direction. Therefore, their application is severely limited.

Our topology of choice, the 25 element square RECAP with inter-element spacing of $0.25\lambda$ also exhibits low value for the error $\Psi$. It has also been shown in the previous section that there is enough reconfigurability in the structure in order to create as many as 8 nulls.

As a final note, it must be reminded that these plots only describe the behavior at a distinct steering angle $\phi = 180^\circ$. In order to make a definitive statement on the globally best topology all $\phi$ directions should be taken into account. Furthermore, the eventual result of the Taylor series expression greatly depends on the randomly chosen $\theta$ values. We have tried to average the result over a certain number of iterations $M$, but the fact is that $\theta$ can take a very wide range of values between 0 and $360^\circ$ and the linearity/non-linearity of the structure is highly affected by the particular $\theta$ used.

### 3.5 Phase Limited Reconfigurable Elements

Thus far in our discussion of RECAPs and their optimization behavior we have assumed that the REs are lossless and can flexibly change phase between $0^\circ$ and $360^\circ$. This assumption has allowed us to choose any phase on the loads that we require to optimize the RECAPs. However, in reality the reconfigurable elements in a practical setup would certainly experience limited phase variation. Reconfigurability in these structures is typically achieved by using MEMS (Micro-electromechanical switches), PIN diodes, FETs or Varactor diodes.

Let's take the example of a reconfigurable element created using a varactor diode. A varactor diode is a diode that performs as a variable capacitor. The variable capacitance leads to a change in the reactance value. The changing reactance value gives rise to a phase change as the biasing voltage is varied. But such a circuit based on a varactor diode can not provide $360^\circ$ phase change. In general, we can expect a phase change of only about $180^\circ$. It is this phase limitation that we would like to take into account in our analysis in this chapter. In the best scenario, we should be able to achieve all our optimization goals using these phase limited REs.
3.5.1 Simulating Phase Limitation - Capacitive REs

We have to limit the RE phase to a total shift of not more than 180°. While we could potentially choose any interval within the range of 0° to 360° to accommodate the limitation, we first take into consideration only capacitive loads. This would mean a phase change of the reflection coefficient between −180° and 0°.

Implementing such a phase limitation in our simulation is not entirely trivial. It is unclear as to what would be the optimal manner of dealing with an element which is overshooting the phase limit. We perform this limitation using the following equation

$$\theta' = -\frac{\pi}{2} (1 + \cos(\theta)) \quad (3.16)$$

This transformation simply scales the randomly chosen \( \theta \) values so that we are always within the limitation of −180° and 0°.

In order to perform the Newton based optimization we now have to include the effect of (3.16) and reformulate \( \frac{\partial e(\hat{s})}{\partial \theta_\ell} \). \( \Gamma_l \) is no longer given by \( e^{j\theta_\ell} \) but by \( e^{-j\frac{\pi}{2} (1 + \cos(\theta_\ell))} \).

Now we know from Chapter 3 that the derivative of the objective function in (2.5) with respect to the phase of the \( \ell \)th load can be evaluated as

$$\frac{\partial e(\hat{s})}{\partial \theta_\ell} = -e_R^T (\Gamma^{-1} - S_{RR})^{-1} \frac{\partial (\Gamma^{-1} - S_{RR})}{\partial \theta_\ell} (\Gamma^{-1} - S_{RR})^{-1} s_{RF} \quad (3.17)$$

But the evaluation of the inner derivative is different this time since there is an added factor given by \( \frac{\partial (-\frac{\pi}{2} (1 + \cos(\theta_\ell)))}{\partial \theta_\ell} = \frac{\pi}{2} \sin(\theta_\ell) \)

$$\left[ \frac{\partial (\Gamma^{-1} - S_{RR})}{\partial \theta_\ell} \right]_{mn} = \frac{\partial (\Gamma^{-1} - S_{RR})_{mn}}{\partial \theta_\ell} = -\frac{j\pi}{2} \sin(\theta_\ell) \delta_m \delta_n \exp(-j\theta_\ell) \quad (1_{\ell\ell})_{mn} \quad (3.18)$$

Substituting (3.18) into (3.17) we get

$$\frac{\partial e(\hat{s})}{\partial \theta_\ell} = \frac{j\pi}{2} \sin(\theta_\ell) e_R^T (\Gamma^{-1} - S_{RR})^{-1} 1_{\ell\ell} (\Gamma^{-1} - S_{RR})^{-1} s_{RF} e^{-j\theta_\ell} \quad (3.20)$$

The rest of the Newton based optimization is the same.

$$\theta_{n+1} = \theta_n - (A^T)^+ b^T \quad (3.21)$$

After the above update step, we substitute \( \theta \) into (3.16). The result is then substituted into (2.5) to get the updated value for \( e(\hat{s}) \).
Table 3.3: Comparison of capacitive and switched inductive/capacitive REs multiple null performance

<table>
<thead>
<tr>
<th>$N_{null}$</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>5</td>
<td>6</td>
<td>12</td>
<td>-</td>
</tr>
<tr>
<td>$t$</td>
<td>0.13 s</td>
<td>0.23 s</td>
<td>0.90 s</td>
<td>-</td>
</tr>
<tr>
<td>$M_{switched}$</td>
<td>3</td>
<td>4</td>
<td>18</td>
<td>72</td>
</tr>
<tr>
<td>$t_{switched}$</td>
<td>0.09 s</td>
<td>0.17 s</td>
<td>1.30 s</td>
<td>6.89 s</td>
</tr>
</tbody>
</table>

3.5.2 Switched Inductive/Capacitive REs

As mentioned previously, phase is not restricted to only shifts between $-180^\circ$ and $0^\circ$. We can also assume that our REs are inductive instead of capacitive. This assumption allows us to shift the phase from $0^\circ$ to $180^\circ$. In such a scenario, it can be seen that the negative sign in (3.16) will go away. Consequently, there will also be an additional negative sign in equation (3.20) for the derivative $\frac{\partial e(\hat{s})}{\partial \theta}$. In order to be more flexible in our approach to optimize the $5 \times 5$ RECAP structure we switch between capacitive and inductive REs such that we assume every odd RE to be capacitive and every even RE to be inductive. In this manner the combination of the 24 REs is able to scan phases encompassing the whole $360^\circ$ range. The hope is that since the combined effect of any two REs is to try to cover all phase angles available, the degree of reconfigurability of the structure is not reduced and that the optimization process will not be detrimentally affected by the individual limitation of each RE to a phase shift of only $180^\circ$.

3.5.3 Numerical Examples - Phase limited REs

In order to exhibit and compare the applicability of both the capacitive and switched inductive/capacitive REs we try to find multiple nulls using both approaches. Figure 3.4 shows the maximum number of nulls that can be formed by the capacitive and the switched capacitive/inductive REs. For each case, the null steering angles were chosen randomly at a minimum phase difference of $30^\circ$. The additional degree of reconfigurability by using the switched capacitive/inductive REs is at once apparent since with this method we are able to easily and consistently realize 7 "deep" nulls while the capacitive REs barely manage to find 5 nulls. At the same time, the limitations of both methods become apparent when we see that the switched capacitive/inductive REs are, in general, unable to find a solution for more than 7 nulls even if we let the algorithm run for 1000 iterations. On the other hand, the capacitive REs often can not even realize 6 nulls within the same number of iterations. Nevertheless, the use of the switching method
means that we do not lose too much on the reconfigurability in the structure due to the phase limitation.

Table 3.3 shows a comparison of the capacitive and the switched inductive/capacitive methods for finding 1, 2, 5 and 7 nulls. The average behavior of both algorithms was taken into consideration to confirm that the methods were consistently able to obtain these nulls for any random set of initial phase values $\theta$. The results have been averaged over 50 runs of both the switched inductive/capacitive method and capacitive method.

It can be seen that 1 and 2 nulls are found in relatively equal amount of time $t$ and $t_{\text{switched}}$ and iterations $M$ and $M_{\text{switched}}$ for both methods. It is interesting to note that when searching for 5 nulls, the capacitive method does better than the switched inductive/capacitive method in terms of iterations and time complexity.

As mentioned, the capacitive method is unable to find 7 nulls, which is indicated by '-' in Table 3.3. Finally, our search for 7 nulls requires a relatively much greater number of iterations and time compared to the trend from 1 to 5 nulls. This can be attributed to the fact that the reconfigurability in the structure is reaching a saturation point when
we search for as many as 7 nulls. Finally, it ought to be noted that neither method can provide us the flexibility of the $360^\circ$ phase shift based REs, using which, we were able to form 8 nulls.

3.6 Conclusion

In this chapter, we introduced the use of a hybrid direct Newton approach for null steering. The Separate Newton Method was based on the use of the vector of derivatives \( \frac{\partial e(\hat{s})}{\partial \theta} \) to drive \( \text{Re}\{e(\hat{s})\} \) and \( \text{Im}\{e(\hat{s})\} \) to zero separately. The Joint Newton Method, on the other hand, relied on modeling \( e(\hat{s}) \) with the first-order multi-dimensional Taylor Series and jointly moving in the minimum norm direction to find a null.

It was shown that when searching for a single null, using the null solution found from the reduced order parasitic RECAP model gave us a better result than seeding with a random initial \( \theta \). It was also shown that the Joint Newton Method was superior to the Separate Newton Method, in terms of time required to find a suitable null.

The Joint Newton Method with random seeding was then used to create up to 8 nulls and this was integrated with the reduced order main beam result to get joint beam-null optimization. It was shown that when searching for few nulls we were able to maintain a high beam gain value, but as we searched for greater number of nulls the beam value degraded. Results were also compared with a GA and it became apparent that when searching for up to 4 nulls the results of the Newton based optimization and the GA were remarkably similar.

Thereafter, a comparison of different topologies was presented, to analyze the relative linearity of each structure and to see which topology would work best with the first order Taylor series. It was shown that with greater inter-element spacing, the linearity generally increased. Nevertheless, it was hard to prove rigorously based on this analysis if a certain topology was globally optimal or not.

Finally, it was pointed out that REs are typically limited to a phase shift of only $180^\circ$ and therefore our Newton based optimization procedure had to take this into consideration. Two different shift ranges for REs, the capacitive and switched inductive/capacitive phase limited REs, were introduced and it was shown that using these REs we could easily search for at least 5 nulls at sufficiently separated steering angles.
Chapter 4

Constraining Main Beam

It was shown in the numerical examples of Chapter 3 that as the Newton based optimization searches for greater number of nulls in the solution space, the gain value in the direction of the main beam is significantly degraded. This behavior is exhibited when the amount of reconfigurability in the structure is relatively less and the number of nulls being searched for is high. In such a scenario there are limited number of roots available in the search space and in order to obtain these nulls, the Newton based optimization changes the value of $\Gamma$ in such a way that there is also an impact on the main beam.

The primary reason that the main beam suffers during this search for the null solution is that there is no constraint on the gain value performance in the main beam direction. Furthermore, the first order solution described in Chapter 2 provided a good starting estimate of the main beam gain value, but this solution is still a suboptimal one. In order to accurately gauge how well this solution performs in terms of optimality we have to first formulate a method to compute the optimal main beam performance. The first section of this chapter describes how to obtain such a measure. Thereafter, several methods are presented which can help to achieve this solution by placing a constraint on the main beam direction.

4.1 Optimal Main Beam Performance

We start by studying the case of a theoretically optimal gain in the main beam direction when we are not searching for any nulls. We show that by using an impedance based circuit model of the RECAP we can conveniently find the theoretically optimal main beam. We then add the constraint of simultaneously searching for multiple nulls and study how the theoretically optimal main beam performance deteriorates as we search for greater number of nulls.
4.1.1 Optimal Gain without nulls

Consider Figure 4.1 which shows a circuit model of the generic RECAP structure in terms of the input current $i_1$ at the feed and the currents $i_2$ at the reconfigurable loads. The voltage-current relationship can be described in terms of the impedance matrix $Z$.

\[
\begin{bmatrix}
  v_F \\
  v_R
\end{bmatrix} =
\begin{bmatrix}
  Z_{FF} & z_{FR} \\
  z_{RF} & Z_{RR}
\end{bmatrix}
\begin{bmatrix}
  i_F \\
  i_R
\end{bmatrix}.
\]  

(4.1)

We can obtain network characteristics and open circuit radiation patterns using network analysis similar to how it was performed in Chapter 1.

Now, let $e^{oc}(\hat{s}_b)$ be the open circuit far field radiation pattern in the beam direction $\hat{s}_b$. Then the far field radiation pattern $e(\hat{s}_b)$ would be given by

\[
e(\hat{s}_b) = e^{ocT}(\hat{s}_b)i
\]  

(4.2)

Since $|e(\hat{s}_b)|$ must be maximized with respect to the current $i$, a realistic constraint is required to avoid a trivial solution. We consider the input power constraint on the RECAP structure, given by

\[
i^H \text{Re} \{Z\} i = 1
\]  

(4.3)

which can be decomposed into

\[
i^H M^\frac{1}{2} i^H M^\frac{1}{2} i = 1
\]  

(4.4)

Taking the inverse, the current $i$ can be written as

\[
i = M^{-\frac{1}{2}} i'
\]  

(4.5)
Now substituting (4.5) into (4.2) we get

\[ e(\hat{s}_b) = e^{oc} M^{-\frac{1}{2}} i' \]  

(4.6)

where \( i'^H i' = 1 \). Next we take the outer product of \( e' \) to obtain a square matrix

\[ E_b = e'_b e'_b^H \]  

(4.7)

Since we want to find \( i' \) that maximizes \( i'^H E_b i' \), our next step is to perform the eigenvalue decomposition of \( E_b \). The eigenvector corresponding to the highest (and the only non-zero) eigenvalue gives us the optimal value for \( i' \). Finally, the optimal current value \( i \) can be found by substituting \( i' \) into (4.5). The corresponding maximum radiation pattern \( e(\hat{s}_b) \) is obtained by substituting \( i \) into (4.2).

It should be noted that in the derivation we have considered above, our RECAP structure is lossless. One danger of analyzing lossless structures with an input power constraint is the possibility of supergain solutions that can not be practically achieved. In order to include loss in the structure we must include a positive real part in the diagonal of the impedance matrix \( Z \). This is equivalent to putting a resistance at each of the \( N \) reconfigurable elements in the RECAP.

Figure 4.2 shows the optimal far field gain produce in direction \( \hat{s} = 45^\circ \) for a lossless RECAP and two slightly lossy RECAP structures. For these lossy cases, 0.1Ω and 0.5Ω loss is added to the diagonal of the impedance matrix \( Z \) as mentioned.

The optimal gains for lossless, 0.1Ω and 0.5Ω loss RECAPs in direction \( \hat{s} = 45^\circ \) are 17.95 dB, 15.73 dB and 15.03 dB respectively.

### 4.1.2 Optimal Gain with Nulls

In the derivation above we assumed that the \( N \) element RECAP was solely devoted towards creating a main beam in a particular direction and no nulls were being searched for. When we search for nulls in addition to finding the main beam, there is a constraint on the system which needs to be taken into account.

Let the vector \( e_k \), of size \( N \times 1 \) represent the open circuit radiation patterns in the \( k \)th direction in which a null is being searched for. For each null direction \( k \), we would want \( i'^H e_k e_k^H i' \) to be zero for perfect null performance. However, since these are non-negative
constraining the sum of them to be zero.

\[
\sum_{k=1}^{T} i_k^H e_k e_k^H i' = 0
\]  

(4.8)

This can be written as

\[
i^H \left( \sum_{k=1}^{T} D_k \right) i' = 0
\]  

(4.9)

Any vector \( i' \) that lies in the null-space of the matrix \( D \) will satisfy this equation. Let \( U_D \Lambda_D U_D^H \) represent the eigenvalue decomposition of \( D \). We can identify the null space of \( D \) as the \( N - k \) eigenvectors \( U_{D0} \) corresponding to the \( N - k \) zero eigenvalues \( \Lambda_{D0} \).

Let us define a new vector \( i'' \) that always lies in the null-space of \( D \). An arbitrary vector \( i' \) can be projected onto the null-space of \( D \) using the operation

\[
i'' = U_{D0} U_{D0}^H i'
\]  

(4.10)

Using this as the input current to the array, we can now maximize \( i'^H Q^H E_b Q i' \) instead of \( i'^H E_b i' \) to obtain the optimal main beam with the perfect nulls in the desired directions. Let \( E_{b'} = Q^H E_b Q \). Taking the eigenvalue decomposition of \( E_{b'} \), the eigenvector
Table 4.1: Theoretically Optimal Beam Performance

<table>
<thead>
<tr>
<th>$N_{null}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Loss</td>
<td>17.95 dB</td>
<td>17.90 dB</td>
<td>17.84 dB</td>
<td>17.68 dB</td>
<td>17.55 dB</td>
</tr>
<tr>
<td>0.1Ω Loss</td>
<td>15.73 dB</td>
<td>15.65 dB</td>
<td>15.48 dB</td>
<td>15.35 dB</td>
<td>15.27 dB</td>
</tr>
<tr>
<td>0.5Ω Loss</td>
<td>15.03 dB</td>
<td>14.93 dB</td>
<td>14.78 dB</td>
<td>14.53 dB</td>
<td>14.33 dB</td>
</tr>
</tbody>
</table>

corresponding to the highest (and the only non-zero) eigenvalue gives us the optimal value for $i'$.

The optimal current value $i$ can now be found by substituting $i'$ into (4.5). The corresponding maximum radiation pattern $e(\hat{s}_b)$ is obtained by substituting $i$ into (4.2).

Table 4.1 shows the optimal gain $e(\hat{s}_b)$ in direction $\hat{s} = 45^\circ$ for 1, 2, 4 and 8 nulls while taking into consideration for lossless, 0.1Ω loss, 0.5Ω and 1Ω loss RECAP structures. It can be noted from the table that the added constraint of searching for nulls does not impose a heavy toll on the $5 \times 5$ RECAP structure being considered. Mathematically this can be understood by the fact that in equation (4.8) $D$ can have a maximum size of $8 \times N$, which is the case when we are searching for 8 nulls. This means that the null space of $D$ spans a minimum of $N - 8$ columns. Since this null space is sufficiently big in comparison to the size of the whole matrix, the cost on the optimal value of the main beam while searching for 8 nulls is not too high. This also suggests that theoretically there is enough reconfigurability in the structure to create a main beam of 13.85 dB, for the 1Ω loss case, along with the creation of 8 nulls.

4.2 Beamforming without nulls

Having found the theoretically optimal main beam gain values, we now turn our attention towards achieving these results by applying different beamforming optimization algorithms. We show how we can use the gradient ascent method and a Newton based beam optimization method to achieve near optimal results. In the case of the Newton based beam optimization we model the magnitude squared of the beam direction $|e_b(\hat{s})|^2$ using the Taylor series. In the end we show why we can not use the second derivative of $|e_b(\hat{s})|^2$ to find the optimal beamforming solution.

4.2.1 Gradient Ascent

While we could potentially maximize both the real and imaginary parts of $e_b(\hat{s})$, where $e_b(\hat{s})$ is the radiation pattern we are trying to optimize in direction $\hat{s}$, it makes much more
sense to maximize $|e_b(\hat{s})|^2 = e_b(\hat{s})^*e_b(\hat{s})$. By maximizing the magnitude squared only, we achieve the same result while putting less constraint on the system. Let $P = e_F(\hat{s})$ and $Q = e_R^T(\hat{s})\Gamma(I - S_{RR}\Gamma)^{-1}s_{RF}$. Now $|e_b(\hat{s})|^2$ can be written as

$$|e_b(\hat{s})|^2 = (P + Q)^*(P + Q) \quad (4.11)$$

We next take the derivative of $|e_b(\hat{s})|^2$ with respect to $\theta_\ell$, the phase on the $\ell_{th}$ reconfigurable element, in order to perform the gradient search.

$$\frac{\partial |e_b(\hat{s})|^2}{\partial \theta_\ell} = \frac{\partial (P + Q)^*(P + Q)}{\partial \theta_\ell} \quad (4.12)$$

Now the product rule is applied on (4.12)

$$\frac{\partial |e_b(\hat{s})|^2}{\partial \theta_\ell} = (P + Q)^*\frac{\partial (P + Q)}{\partial \theta_\ell} + (P + Q)\frac{\partial (P + Q)^*}{\partial \theta_\ell} \quad (4.13)$$

which can be written as

$$\frac{\partial |e_b(\hat{s})|^2}{\partial \theta_\ell} = 2\text{Re} \left\{ (P + Q)^* \frac{\partial (P + Q)}{\partial \theta_\ell} \right\} \quad (4.14)$$

Substituting $e_b(\hat{s}) = P + Q$ we have

$$\frac{\partial |e_b(\hat{s})|^2}{\partial \theta_\ell} = 2\text{Re} \left\{ (e_b(\hat{s}))^* \frac{\partial (e_b(\hat{s}))}{\partial \theta_\ell} \right\} \quad (4.15)$$

Finally, following the same operation for finding the derivative of $\frac{\partial (e_b(\hat{s}))}{\partial \theta_\ell}$ as was applied in Chapter 3 for the Newton based null optimization, we get

$$\frac{\partial |e_b(\hat{s})|^2}{\partial \theta_\ell} = 2\text{Re} \left\{ (e_b(\hat{s}))^* j e_R^T(\hat{s})(\Gamma^{-1} - S_{RR})^{-1}\mathbf{1}_{\ell}(\Gamma^{-1} - S_{RR})^{-1}s_{RF}e^{-j\theta_\ell} \right\} \quad (4.16)$$

The $(N - 1) \times 1$ gradient vector $\mathbf{d}$ is formed with elements $d_\ell = \partial |e_b(\hat{s})|^2 / \partial \theta_\ell$.

Since in trying to move towards the main beam, we are approaching the local maximum of the function, we have to take steps in the direction of the positive of the gradient. This can be done by using the typical gradient ascent method so that we update $\theta$ in the following way

$$\theta_{n+1} = \theta_n + \delta \mathbf{d} \quad (4.17)$$

Where $\delta$ represents the step-size employed in the gradient search.

$\theta_1$ can be assigned the phase solution values found through the first order parasitic RECAP model described in Chapter 2. Alternatively, a randomly chosen set of $\theta_1$ values could also be fed as the initial seeding to the gradient ascent problem.
4.2.2 Newton Based Beam Optimization

We model the magnitude squared of the beam direction $|e_b(\hat{s})|^2$ using the first-order multi-dimensional Taylor series similar to how it was employed in Chapter 3.

\[
|e_b(\hat{s}, \theta_{n+1})|^2 = |e_b(\hat{s}, \theta_n)|^2 + d^T(\theta_n)(\theta_{n+1} - \theta_n) \tag{4.18}
\]

In Chapter 3, we were searching for the root to satisfy the null problem, therefore $e_{re}(\hat{s}, \theta_{n+1}) = e_{im}(\hat{s}, \theta_{n+1}) = 0$. In the search for the optimal beam, we can set $|e_b(\hat{s}, \theta_{n+1})|^2 = |e_{b,\text{opt}}(\hat{s}, \theta_{n+1})|^2$ where $|e_{b,\text{opt}}(\hat{s}, \theta_{n+1})|^2$ is the known magnitude squared of the optimal beam value that we found from the rigorous derivation shown in the last section.

\[
\frac{|e_{b,\text{opt}}(\hat{s}, \theta_{n+1})|^2 - |e_b(\hat{s}, \theta_n)|^2}{h} = d^T(\theta_n)(\theta_{n+1} - \theta_n) \tag{4.19}
\]

which can be solved as

\[
\theta_{n+1} = \theta_n + (g^T)\cdot h \tag{4.20}
\]

Since the beamforming problem is very non-linear we would like to move in small step sizes towards the solution. We, therefore, again use a chosen step-size $\delta$ such that the equation can be rewritten as

\[
\theta_{n+1} = \theta_n + \delta(g^T)\cdot h \tag{4.21}
\]

4.2.3 Second Derivative optimization

Intuitively it makes sense to try to find the second derivative of $|e_b(\hat{s})|^2$ in order to find the local maximum of the objective function. In other words, knowing that we are close to an optimal solution when we solve for the main beam using the first order RECAP model, we can simply force the derivative at the first order solution point to zero and we should hopefully thereby reach the local maximum main beam result. The idea of the Taylor series can again be made use of in this case. We pose the maximization problem in the following way. Let $p_n$ and $Q_n$ represent the first and second derivatives of $|e_b(\hat{s})|^2$ at the $n^{th}$ step of the Taylor series based update respectively

\[
p_{n+1} = p_n + Q_n(\theta_{n+1} - \theta_n) \tag{4.22}
\]

Setting $p_n = 0$

\[
\theta_{n+1} = \theta_n - (Q_n^T)\cdot p_n^T, \tag{4.23}
\]
\( Q_n \) was solved for numerically at each step. Using this approach we could easily find a set of \( \theta \) values to force \( p = 0 \), but surprisingly this did not mean we reached a local maximum for the main beam or even attained a better solution than the one obtained from the first order RECAP model. The expectation was that since we are close to an optimal solution when using the first order model, the maximum point would be very close by and could be reached in a linear manner.

Finding random solutions that are far worse than even the first order RECAP model means that there are lots of local maxima and minima even when we are very close to the maximum point solution and trying to move in a linear manner towards that solution does not always work.

### 4.3 Beamforming with nulls

In order to keep null steering robust while creating a suitable beam, we explore an approach where nulls and beam are optimized in a separate manner. The first order multi-dimensional Taylor series expression for nulls was derived in Chapter 3 as

\[
\begin{bmatrix}
  e_{re}(\hat{s}, \theta_n) \\
  e_{im}(\hat{s}, \theta_n)
\end{bmatrix}^T = - \begin{bmatrix}
  d_{re}(\theta_n) \\
  d_{im}(\theta_n)
\end{bmatrix}^T (\theta_{n+1} - \theta_n)
\quad \text{(4.24)}
\]

We make use of the null space of the fat matrix \( A \) to isolate the beam optimization from the null optimization so that they do not detrimentally affect each other. When optimizing the main beam we want to move in directions that do not change the nulls. To do this, we must find an optimal step direction to improve our main beam and then project this direction on to the null space of \( A \) so that the nulls remain unaffected by the beam optimization.

The null space of \( A \) is projected on to \( \theta \). The main beam optimization step is applied on this null space projected version of \( \theta \). After the beam optimization step is completed, the Newton based null optimization is applied on the beam optimized \( \theta \) to complete the joint beamforming and null-steering optimization. Since only the range space of \( A \) is used to perform the null optimization, the two procedures do not affect each other.

\( A \) is a fat matrix, since we look for a maximum of 8 nulls and we have \( M = 24 \) parasitic elements in our \( 5 \times 5 \) RECAP structure, so that \( A \) has a maximum size of \( 16 \times 24 \). The rank of \( A \) is \( \min(m,n) \) where \( m \) and \( n \) are the rows and columns respectively. The rank of \( A \) is also given by \( m = 2 \times N_{null} \) where \( N_{null} \) are the number of nulls we are searching for. This means that the null space of \( A \) spans a size of \( N_{RE} - 2N_{null} \) where \( N_{RE} \) is the number of reconfigurable elements.
Now, we go into the details of the optimization procedure. We first perform the main beam optimization by projecting the optimal beam direction on to the null space of $A$.

We apply the singular value decomposition on the $m \times n$ $A$ matrix to identify the null space of $A$

$$A = USV^H$$  \hspace{1cm} (4.25)

The null space of $A$ can be identified as the rows of $V^H$ corresponding to the $n - m$ zero columns of the singular value matrix $S$. Let $V_{null}^H$ represents the null space singular vector matrix of $A$. Let the gradient vector of the magnitude squared of the main beam, $|e_b(\hat{s})|^2$, be denoted by $g$. In order to isolate the main beam optimization, we project $V_{null}^H$ on to $\theta$ and $g$ and apply the Newton beam optimization step on these projected versions.

Let $\theta_c$ and $\theta_b$ be the projection of $V^H$ and $V_{null}^H$ on to $\theta$.

$$\theta_c = V^H\theta$$  \hspace{1cm} (4.26)

$$\theta_b = V_{null}^H\theta$$  \hspace{1cm} (4.27)

It should be noted that the last $n - m$ values of $\theta_c$ are the same as $\theta_b$ or $\theta_c[m+1 \ldots M] = \theta_b$. This information will be helpful when we want to convert the updated $\theta_b$ back to $\theta$.

Let $g_b$ be the projection of $V_{null}^H$ on to $g$

$$g_b = V_{null}^Hg$$  \hspace{1cm} (4.28)

Now we can apply the Newton based beam optimization on $\theta_b$ using the derivatives $g_b$.

$$\frac{|e_b^{opt}(\hat{s},\theta_{n+1})|^2 - |e_b(\hat{s},\theta_n)|^2}{h} = d^T(\theta_b,n)(\theta_{b,n+1} - \theta_{b,n})$$  \hspace{1cm} (4.29)

which can be solved as

$$\theta_{b,n+1} = \theta_{b,n} + (g_b^T)^+h$$  \hspace{1cm} (4.30)

We can now update the last $n - m$ values of $\theta_c$ according to the updated $\theta_{b,n+1}$ or $\theta_c[m+1 \ldots M] = \theta_{b,n+1}$. The orthogonality of $V$ allows us to conveniently convert $\theta_c$ to find the updated beam optimized version of $\theta$.

$$\theta = V\theta_c$$  \hspace{1cm} (4.31)
After the above beam optimization step, we can now apply the Newton based null optimization. We apply (4.24) to update $\theta$ to accommodate the different null directions.

$$\theta_{n+1} = \theta_n - (A^T)^+ b^T, \quad (4.32)$$

Following the above procedure we are able to optimize the beam and nulls separately in a few Newton steps.

### 4.4 Beamforming - Numerical Examples

In the following section, we compare the different beamforming methods, introduced in the last section, with the theoretically optimal beam solution. First we consider the case of beamforming without nulls. Thereafter, the different results for main beam optimization with 1, 2, 4 and 8 nulls are compared with the first-order solution as well as the theoretically optimal beam solution.

#### 4.4.1 Beamforming - no nulls

Figure 4.3 shows the maximum antenna gain found in the direction $\phi = 45^\circ$ using the gradient ascent and the Newton based beam optimization. We keep the result of the
Table 4.2: Comparison of Beam performance - no nulls

<table>
<thead>
<tr>
<th>Optimal Beam (1Ω Loss)</th>
<th>Gradient Ascent</th>
<th>Newton Beam optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.66 dB</td>
<td>14.30 dB</td>
<td>14.64 dB</td>
</tr>
</tbody>
</table>

Table 4.3: Comparison of Beam performance - with nulls

<table>
<thead>
<tr>
<th>$N_{\text{null}}$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal Beam (0.5Ω Loss)</td>
<td>14.93 dB</td>
<td>14.78 dB</td>
<td>14.53 dB</td>
<td>14.33 dB</td>
</tr>
<tr>
<td>Optimal Beam (1Ω Loss)</td>
<td>14.54 dB</td>
<td>14.36 dB</td>
<td>14.09 dB</td>
<td>13.85 dB</td>
</tr>
</tbody>
</table>

optimized beam with 1Ω loss as a reference. Using appropriate values for the step-size, $\delta = 0.001$, for both methods, we let the respective algorithms run for 3000 iterations.

Table 4.2 shows the value of the main beam in direction $\phi = 45^\circ$. This is the the value that is also plotted in Figure 4.3. As we can see the results for both the gradient ascent as well as the Newton based beam are almost equal to the optimal solution. This indicates that both these methods work extremely well and seem to be near optimal.

4.4.2 Beamforming with nulls

In Figure 4.4 four different plots are shown for joint beamforming and null-steering with 1, 2, 4 and 8 nulls for the Newton based beam and the First Order beam with multiple nulls. We put a threshold of -20 dB on the null values and run the Newton based beam algorithm for 1500 iterations again having chosen $\delta = 0.001$ as the step size for the optimization. The same threshold of -20 dB is also set for the First order joint beam and null formation. It can be seen that in all the plots the Newton based beam stays at quite a high value at a steering angle $\phi = 45^\circ$ and that the randomly chosen nulls are maintained on or below -20 dB in each case. The comparison with the First order joint beam and null formation is also quite enlightening. Obviously, in all cases the First order beam performs significant worse than our Newton based beam. The First Order beam deteriorates quite a lot when we search for more and more nulls. Consequently the gain difference in the beam direction between the two methods increases as we search for a greater number of nulls.

Keeping the optimal beam results for 0.5Ω and 1Ω as reference, we note from Table 4.3 that the values for the main beam are almost equal to or quite close to the optimal beam values. In each case our Newton based optimization operates at a maximum value which is within a 0.5 dB of the optimal results of the 1Ω loss. In fact for the 1, 2 and 4 nulls case, the results are almost equal and when we search for 8 nulls, the Newton
based beam finds a maximum value of 13.55 dB in the search direction which is 0.3 dB short of the optimal beam for 1Ω loss (13.85 dB).

4.5 Conclusion

In this chapter we made an effort to constrain the main beam to an optimal value, both for the case where we were searching for a main beam on its own and the case where we searched for joint beam and nulls. To this end, we first computed the theoretically optimal values for the main beam in a certain direction $\hat{s}$. This was done by means of analyzing the RECAP in terms of the currents at the feed and the REs.

After having found these optimal values, we described the gradient ascent and the Newton beam optimization methods for the optimization of the main beam without nulls. Thereafter, it was also explained why the method of using the second derivative of $e(\hat{s})$ to find the maximum value in the radiation direction $\hat{s}$ did not work well. It was shown that while we could easily make use of the second derivative of $e(\hat{s})$ to force the first
derivative of $e(\hat{s})$ to zero, this mostly did not result in a strong beam value. In search of the maximum point we ended up at a smaller local maximum than even the first order solution.

Thereafter we showed that by using the null space of $A$, the matrix that represents the real and imaginary parts of $d_\ell = \partial |e_\ell(\hat{s})| / \partial \theta_\ell$, we can optimize the beam and null directions separately. Since the $m \times n$ $A$ matrix was a fat matrix, the rank was given by $\min(m,n)$ which allowed us to use a null space spanning $n-m$ columns. Using this property we were able to isolate the beam optimization from the null optimization.

Finally, the different results for main beam optimization with 0, 1, 2, 4 and 8 nulls were shown. These results were compared to the theoretically optimal main beam performance results and it was shown that the performance matched very well in all cases.
Chapter 5

In-situ problem - Unknown Parameters Estimation

Pattern synthesis is the crux of this work and using the $5 \times 5$ RECAP, we have so far been able to find multiple nulls as well as an optimized main beam in chosen directions. Now the question arises: given that we are able to successfully apply beam and nulls optimization on the RECAP structure what further do we have to include in the problem domain to realize these solutions in a practical structure.

There are several issues to be considered, since only limited information of the system is known a priori in a practical system. For instance, so far we have assumed that we have perfect knowledge of our scattering matrix $S$ as well as of the radiated far-fields at the feed $e_F(\hat{s})$ and at the REs $e_R(\hat{s})$. We have further made the assumption that there is no noise in the radiated far-field values. These variables represent the principle of our objective function to be optimized in equation (2.5).

We have not treated the RECAP structure as a black-box and have therefore not optimized it blindly, but we have instead depended on network analysis to accurately describe the objective function in (2.5). While this was a more rigorous way of understanding the underlying problem and of optimizing the structure, we now end up with the most challenging and difficult issue: trying to estimate the radiated far-field directions and $S$ parameters.

Therefore, before we can even consider the optimization process, we have to estimate the $25 \times 25$ elements $S$ parameters matrix as well as $e(\hat{s}) = [e_F(\hat{s}) \ e_R(\hat{s})]^T$. In this chapter, we first provide a description of how these parameters can be estimated. We then provide a comparison of how well the estimated parameter compare with the actual values of these parameters. Thereafter, a short work on sensitivity analysis is presented.
where we add certain amount of error in known quantities and observe how that affects the estimation procedure.

### 5.1 S-parameters and Radiated far-fields estimation

We find the first-order Taylor series of the input reflection coefficient $\Gamma_{in}$ with respect to initially randomly chosen $S$ parameters in order to estimate the actual $S$ parameters of the RECAP. By measuring $\Gamma_{in}$ several times based on random $\Gamma$ we are able to form a linear system having the required number of equations to solve for the unknown $S$ parameters. To find the radiated far-fields at the feed and the load, we assume that we can measure the total radiated far-field $e(\hat{s})$. Again we model the problem using the first-order Taylor series of the total radiated far-field with respect to the unknown and initially randomly chosen radiated far-fields at the feed and the load. $e(\hat{s})$ is measured for several random $\Gamma$ and a linear system of equations is formed to find the unknown parameters. The complete procedure is described below.

#### 5.1.1 S-parameters

In a practical structure we can only measure the input reflection coefficient $\Gamma_{in}$ at the feed port. Let us derive this quantity by reusing the network analysis in Chapter 1. Equation (5.1) gives the input-output relationship of the RECAP

$$
\begin{bmatrix}
  b_F \\
  b_R
\end{bmatrix} =
\begin{bmatrix}
  s_{FF} & s_{FR} \\
  s_{RF} & S_{RR}
\end{bmatrix}
\begin{bmatrix}
  a_F \\
  a_R
\end{bmatrix}.
\tag{5.1}
$$

In Chapter 1 we found that the incident wave at the REs $a_R$ can be written in terms of the incident wave at the feed $a_F$ as

$$
a_R = \Gamma(I - S_{RR}\Gamma)^{-1}s_{RF}a_F. \tag{5.2}
$$

Using the above equation and the fact that $b_F = s_{FF}a_F + s_{FR}a_R$, we can write $b_F$ purely in terms of $a_F$ as

$$
b_F = s_{FF} + s_{FR}\Gamma(I - S_{RR}\Gamma)^{-1}s_{RF}a_F. \tag{5.3}
$$

Now, $s_{FR} = s_{RF}^T$, therefore $\Gamma_{in}$ is

$$
\Gamma_{in} = s_{FF} + s_{RF}^T(\Gamma^{-1} - S_{RR})^{-1}S_{RF}
\tag{5.4}
$$
It can be seen that the only variable that we can change on the right side of the equation is the matrix $\Gamma$ with non-zero diagonal elements representing the phase on each RE. In order to be able to estimate the $S$ parameters we will investigate an approach where the phases $\theta$ of the REs are varied in a random fashion and $\Gamma_{in}$ is obtained, allowing the input-output relationship to estimated. In order to not have an under-determined system, the total number of $\Gamma_{in}$ values obtained must be equal to or more than the number of unique $S$ parameter values being searched for. Here it is pertinent to point to out that for the case of reciprocal antennas the scattering parameter matrix is symmetric. Therefore, if the RECAP is made up of $N \times N = M$ elements, then the $S$ matrix is given by $M \times M = P$ elements where the number of unique elements would be given by $\frac{P+M}{2}$, meaning all the elements above and including the diagonal.

We can yet again use the first order multi-dimensional Taylor series to estimate $S$. We must first randomly assign the $\frac{P+M}{2}$ unique values of $S$. Let these unique values be given by $s^u$.

$$s^u = [s_{FF}, s_{RF}, \text{vec}(S_{RR})] \quad (5.5)$$

Where $\text{vec}(S_{RR})$ denotes the columns of the unique parts of the symmetric matrix $S_{RR}$ stacked into a vector.

To begin the estimation procedure, each element of $s^u$ is chosen as a complex number with magnitude between 0 and 1. We have to take care though, that the randomly chosen elements of $s^u$ represent a passive (non-amplifying) network otherwise instability may result. The overall power constraint on the RECAP structure means that the singular values of the actual $S$ parameters, denoted by $S_{\text{act}}$ must never be greater than 1 otherwise the structure has potential gain.

In order to enforce a passive network we must first take the singular value decomposition of the matrix version $S^u$ of the randomly chosen $s^u$ vector. Let $U\Sigma V^T = \text{svd}(S^u)$. The orthogonal matrices $U_S$ and $V_S$ are then transformed by multiplying them with a new set of randomly generated singular values $\Sigma_1$ that are strictly between 0 and 1 to get $S^u = U\Sigma_1 V^T$. In such a way we make sure that our initial choice of $S^u$ would not represent a RECAP that makes the algorithm potentially unstable.

Having a current estimate or guess of $S^u$ our goal is to find a better estimate for $S^u$ given observations of $\Gamma_{in}$. This is accomplished by finding the first-order Taylor series of $\Gamma_{in}$ with respect to $S^u$ which requires the derivative of $\Gamma_{in}$ with respect to $s_{FF}$, $s_{RF}$ and $S_{RR}$. Note that only the $\frac{P+M}{2}$ unique values of $S_{RR}$ on the diagonal and upper triangular are considered in this procedure.
Chapter 5. *In-situ problem* - *Unknown Parameters Estimation*

The solution to \( \frac{\partial \Gamma_{\text{in}}}{\partial \gamma_{\text{FF}}} \) is trivial and is given by

\[
\frac{\partial \Gamma_{\text{in}}}{\partial \gamma_{\text{FF}}} = 1.
\] (5.6)

To solve for \( \frac{\partial \Gamma_{\text{in}}}{\partial \gamma_{\text{RF}}} \) let us first remind ourselves that \( A = \Gamma^{-1} - S_{\text{RR}} \). We now use the following identity for matrix derivatives

\[
\frac{\partial x^T B x}{\partial x} = (B + B^T)x,
\] (5.7)

which means that \( \frac{\partial \Gamma_{\text{in}}}{\partial \gamma_{\text{RF}}} \) is given by

\[
\frac{\partial \Gamma_{\text{in}}}{\partial \gamma_{\text{RF}}} = (A^{-1} + (A^{-1})^T)S_{\text{RF}}
\] (5.8)

Now, to solve for \( \frac{\partial \Gamma_{\text{in}}}{\partial S_{\text{RR}}} \), we assume that \( S_{\text{RR}} \) is not symmetric. This assumption allows us to make use of the following identity for matrix derivatives

\[
\frac{\partial a^T X^{-1} a}{\partial X} = -X^{-T}a a^T X^{-T}
\] (5.9)

Using the above property, we can write \( \frac{\partial \Gamma_{\text{in}}}{\partial S_{\text{RR}}} \) as

\[
\frac{\partial \Gamma_{\text{in}}}{\partial S_{\text{RR}}} = A^{-T}S_{\text{RF}}S_{\text{RF}}^T A^{-T}
\] (5.10)

It should be noted that the negative sign goes away when we apply the identity in (5.9) to (5.10) because there is an additional negative next to \( S_{\text{RR}} \) in \( A \), with which the negative of the identity gets cancelled.

Now in order to include the property that \( S_{RR} \) is a symmetric matrix, we stack only those elements from each column of the matrix \( \frac{\partial \Gamma_{\text{in}}}{\partial S_{\text{RR}}} \) which are on the diagonal and upper triangular, into a vector. Let this vector be denoted by \( \text{vec}(\frac{\partial \Gamma_{\text{in}}}{\partial S_{\text{RR}}}) \). Now we can write all the derivatives we have computed into a single vector \( \mathbf{d} \)

\[
\mathbf{d} = \left[ \frac{\partial \Gamma_{\text{in}}}{\partial \gamma_{\text{FF}}}; \frac{\partial \Gamma_{\text{in}}}{\partial \gamma_{\text{RF}}}; \text{vec}(\frac{\partial \Gamma_{\text{in}}}{\partial S_{\text{RR}}}) \right].
\] (5.11)

Using the above equations we have found \( \mathbf{d} \) for a particular value of \( \Gamma \) and \( S^u \). We have to perform this operation at least \( K = \frac{P+M}{2} \) times, since we need that many unique equations to be able to form a linear system to solve for all \( S \) parameter values.

Let \( \Gamma_{\text{in}} = [\Gamma_{\text{in},1}, \Gamma_{\text{in},2}, \ldots, \Gamma_{\text{in},K}] \) be the measured reflection coefficient of the RECAP for the unique RE loading \( \Gamma_1, \Gamma_2, \ldots, \Gamma_K \).
Let the corresponding set of derivatives of $\Gamma_{in}$ for the consecutive loadings be given by $R = [d_1, d_2, \ldots, d_K]$. Now the Taylor series can be used to find a linear relationship relating $\Gamma_{in}$ to $S^u$.

$$\Gamma_{in}^{(n+1)} = \Gamma_{in}^{(n)} + \frac{[d_1, d_2, \ldots, d_K](S^u,(n+1) - S^u,(n))}{R}$$ (5.12)

This relationship allow $S^u$ to be found using an iterative strategy. Let $S^u,(n)$ be the current estimate of $S^u$ and $\Gamma_{in}^{(n)}$ be the corresponding input reflection coefficient for a specific RE loading, given by (5.4). For the same loading, the actual input reflection is measured as $\Gamma_{in}^{act} = \Gamma_{in}^{(n+1)}$. The relationship can now be inverted to obtain an updated estimate of $S^u$, or

$$S^u,(n+1) = S^u,(n) + R^{-1}(\Gamma_{in}^{act} - \Gamma_{in}^{(n)})$$ (5.13)

Unfortunately, direct application of this procedure led to stability problems, which appears to result from the derivative matrix $R$ being nearly singular. The first thought therefore was to look at the singular value decomposition (SVD) of the matrix $R$ being inverted.

The SVD of $R$ revealed that a very high number of singular values $\Sigma$ were very close to zero. This means that when we perform the pseudo-inverse, these singular values, representing the 'near-null-space' of $R$, begin to dominate and pull the solution towards instability or infinity.

One option is to zero the relatively small singular values $\Sigma_0$ using some sort of tolerance or threshold value before taking the pseudo-inverse. This way the solution will always be stable and will likely converge to a result close to $S^{act}$. The downside, however, is that a threshold value would likely have to be changed and readjusted for each new and different type of RECAP topology. While we are concentrating on the $5 \times 5$ RECAP, we would still like a general solution for the problem that works without revision on any RECAP topology.

An alternative approach to solving the issue of ill-conditioned $R$ is to increase the number of equations that we have in our system so that the solution for $S$ is over-determined and consequently can be solved in a least square sense. After trying out different values, it was found that letting the number of equations be $N_{cond} = 8 \times \text{length}(S^u)$ lead to sufficient improvement in the conditioning of the problem since the lowest singular values were no longer close to zero but had a relatively high non-zero value. It must be noted that now we have to measure $\Gamma_{in}^{act}$ $N_{cond}$ times.

It may be argued that over-determining the system so much leads to much greater time complexity. But it must also be noted that improving conditioning on $R$ can lead
to faster convergence in the iterative algorithm. Therefore, if fewer iterations of the algorithm are required, the time increase may be small or even negative.

Using the described algorithm we are able to find the vector $s^u$ that minimizes $\Gamma_{in}^{act} - \Gamma_{in}^{(n)}$. However, a comparison of the individual elements of the matrix version $S^u$ with the actual $S$ parameters $S^{act}$ reveals some surprising behavior. We observe that while the value of $s_{FF}^{act}$ is very close to $s_{FF}^{act}$, both $s_{RF}^{act}$ and $S_{RR}^{act}$ were completely different from $s_{RF}^{act}$ and $S_{RR}^{act}$ even though the error $\Gamma_{in}^{act} - \Gamma_{in}$ was almost exactly zero. This shows that (5.13) updates $s^u$ in such a way that $\Gamma_{in}^{act} - \Gamma_{in}^{(n)}$ is minimized, but does not guarantee that the correct values of $S$ are obtained.

A hint to why this happens can also be found in the structure of (5.4) where the second part of the equation contains $s_{RF}$ and $s_{RF}^T$ embedding the term $A^{-1} = (\Gamma^{-1} - S_{RR})^{-1}$. The result of $s_{RF}^T A^{-1} s_{RF}$ is quite dependent on what value $s_{RF}$ takes, whose presence on both sides means that it can powerfully affect the direction in which the solution is moving. The $s_{RF}$ value directly depends on how $S_{RR}$ is being optimized which means that the two quantities are strongly coupled to each other. This coupling and especially the embedded structure of the second part of (5.4) means that a wide range of $s_{RF}$ and $S_{RR}$ values would satisfy the scalar result for $\Gamma_{in}$. But this behavior does not hurt our application since we do not care about individual values of $S$ as long as $\Gamma_{in}$ has the correct dependence on RE load values $\Gamma$.

### 5.1.2 Radiated Far-fields

It is not enough to be able to identify $S^{act}$, since in order to have complete information of the optimization system we need the embedded radiation patterns at the feed port and the reconfigurable ports. To find these, we must now consider the objective function in (2.5). To estimate embedded radiation patterns in direction $\hat{s}$, we assume that the system can measure the total radiation far-field $e(\hat{s})$. This is reasonable, since if we are trying to create a null in the direction of another user at $\hat{s}$, that user can feedback the signal it measures.

Having made this assumption, we can now perform the same operation to find $e_F$ and $e_R$ that was used to find $S$. Again, the first step is to assign random initial values for the real and complex parts of $e_F(\hat{s})$ and $e_R(\hat{s})$, the radiated far fields at the feed port and the reconfigurable ports in direction $\hat{s}$. We then take the derivative of $e(\hat{s})$ in (2.5) with respect to both these variables.

$$\frac{\partial e(\hat{s})}{\partial e_F} = 1 \quad (5.14)$$
\[
\frac{\partial e(\hat{s})}{\partial e_R} \text{ is given by}
\]
\[
\frac{\partial e(\hat{s})}{\partial e_R} = A^{-1} s_{RF}
\] (5.15)

Now, we mentioned in the previous section, that \( S^{\text{est}} \) does not have the same values as \( S^{\text{act}} \). But \( S^{\text{est}} \) gave the correct result for \( \Gamma_{in} \). We need to update \( S^{\text{est}} \) found from the previous section using (2.5) so that \( e(\hat{s}) \) is also satisfied. At the same time, we do not need to update the value of \( s_{RF}^{\text{est}} \) since we are already applying the optimization on \( e_R \).

By keeping \( s_{RF}^{\text{est}} \) constant and allowing \( e_R \) and \( S_{RR}^{\text{est}} \) to be updated, we can apply the Taylor series update process to get the correct result for \( e_R A^{-1} s_{RF}^{\text{est}} \) and consequently for \( e(\hat{s}) \).

We therefore again assume that \( S_{RR} \) is unstructured and take the derivative of \( e(\hat{s}) \) with respect to \( S_{RR}^{\text{est}} \) using the identity for matrix derivative
\[
\frac{\partial a^T X^{-1} b}{\partial X} = -X^{-T} a b^T X^{-T}
\] (5.16)

To get
\[
\frac{\partial e(\hat{s})}{\partial S_{RR}^{\text{est}}} = A^{-1} e_R s_{RF}^{\text{est}} A^{-1}
\] (5.17)

Again, in order to include the symmetric property of \( S_{RR} \), we stack only the unique elements (diagonal and upper triangular) from each column of the matrix \( \frac{\partial e(\hat{s})}{\partial S_{RR}^{\text{est}}} \) into a vector. Let this vector be denoted by \( \text{vec}(\frac{\partial e(\hat{s})}{\partial S_{RR}^{\text{est}}}) \). Now we can write all the derivatives we have computed into a single vector \( d_e \)
\[
d_e = \left[ \frac{\partial e(\hat{s})}{\partial e_F}, \frac{\partial e(\hat{s})}{\partial e_R}, \text{vec}(\frac{\partial e(\hat{s})}{\partial S_{RR}^{\text{est}}}) \right].
\] (5.18)

Let \( Q = [d_{e,1}, d_{e,2}, \ldots, d_{e,K}] \) Using the same approach as was adopted in the previous section, we obtain the Taylor series expansion for \( e_n = [e_F, e_R, \text{vec}(S_{RR}^{\text{est}})] \)
\[
e^{u,(n+1)} = e^{u,(n)} + Q^+(\bar{\Gamma}_{in}^{\text{act}} - \bar{\Gamma}_{in}^{(n)})
\] (5.19)

Again, we make use of an over-determine system by having \( N_{\text{cond}} = 8 \times \text{length}(e^u) \) equations and solve the problem in a least square sense to find \( e^u \). Using this approach based on the Taylor series we can estimate the required unknown parameters, \( S, e_F \) and \( e_R \).

We now show a comparison of how the estimated parameters compare to the exact parameters when searching for a single null at a particular steering angle for a 9 element linear array with \( \lambda/10 \) spacing as well as for the 5 \( \times \) 5 square RECAP with \( \lambda/4 \) spacing.

First, the \( S \)-parameter estimation algorithm described in the first section is run for 200 iterations. The resulting \( s_{FF}, s_{RF} \) and \( S_{RR} \) are then supplied to the far field radiation
Figure 5.1: Performance of actual and estimated parameters when searching for a null at steering angle $\phi = 80^\circ$ for a 9 element linear array with $\lambda/10$ spacing

Figure 5.2: Performance of actual and estimated parameters when searching for a null at steering angle $\phi = 80^\circ$ for a $5 \times 5$ element square RECAP with $\lambda/4$ spacing

and $S_{RR}$ estimation described above. This algorithm is also run for 200 iterations. The estimated parameters are then used to perform the Newton based optimization.

Figures 5.1 and 5.2 show plots of a single null at a steering angle $\phi = 80^\circ$ for the exact parameters as well as the estimated parameters for the linear and the square array respectively. Only a solitary point at $\phi = 80^\circ$ is plotted for the estimated parameters
case since we only estimate the radiation far fields $e_F, e_R$ for $\phi = 80^\circ$ direction. The Newton Based optimization was performed on the estimated parameters since these parameters are the ones that we would optimize in a practical setup. The Newton based search was halted, in both the linear and square array cases, once a threshold of $-30^\circ$ was reached. The same $\Gamma$ was then fed to the actual parameters to see how the gain in the null direction is affected in this case. We note that there is only a slight difference between the actual parameters result and the estimated parameters result in both figures. The estimated parameters have a stronger null in Figures 5.1 and 5.2, but when $\Gamma$ is fed to the actual parameters, the result is still below $-30$ dB for both cases.

5.2 Sensitivity Analysis

We have so far shown that we can correctly estimate the unknown parameters $S, e_F$ and $e_R$ when we have perfect knowledge of $\Gamma_{in}$ and the radiated far-field in the direction $\hat{s}$ given by $e(\hat{s})$. But in a practical setup there is bound to be some amount of error in our measurement of these values. Therefore, it would make sense to incorporate these errors and try to learn the unknown parameters, given these limitations.

5.2.1 Variations in Scattering parameters

In this section, we observe how error in $S$ parameters affects the input reflection coefficient $\Gamma_{in}$ and the Newton based null optimization procedure. For both cases we perform a Monte-Carlo analysis, where we add steadily increasing error to the magnitude and phase of $S$ and study how sensitive $\Gamma_{in}$ and the Newton based null optimization are to these changes. Such an exercise is beneficial for the case of sensitivity analysis of $\Gamma_{in}$ with respect to change in $S$ since it gives us a reference point for how much change in $\Gamma_{in}$ would lead to unacceptably high error in $S$. A criterion for how much error is acceptable in $S$ is found by studying the behavior of the Newton based null optimization with respect to increasing error in $S$ and noting where the nulls start to go above the threshold of $-20$ dB.

5.2.1.1 Effect on Input reflection coefficient $\Gamma_{in}$

Before we look at how our estimation procedure performs when error in $\Gamma_{in}$ is taken into account, it is instructive to consider the reverse relationship. Meaning we observe how the value of $\Gamma_{in}$ changes before and after we add a random error to $S^{act}$. 
To this end, we perform a Monte-Carlo analysis with 1000 iterations. For each iteration
we first choose a random set of $\theta$ between $0^\circ$ and $360^\circ$ which is converted into $\Gamma$ using
$\Gamma = e^{j\theta}$. $\Gamma$ and $S^{\text{act}}$ are then substituted into (5.4) to find $\Gamma_{\text{in}}^{\text{act}}$. Thereafter, a random
error in magnitude and phase is added to each element of $S$. This random error is chosen
from a range which initially lies between $R = 0 \text{ dB} - 0.05 \text{ dB}$. $R$ is then increased in
small step sizes of $0.05 \text{ dB}$, until at the last step we can choose an error between a range
of $R = 0 \text{ dB} - 1 \text{ dB}$.

After the random error in magnitude and phase is added to each element of $S$, the
corresponding $\Gamma_{\text{in}}$ is recalculated while taking this variation in $S$ into consideration.
This procedure is repeated for 1000 iterations for each unique value of $R$. The root
mean squared error (RMSE) between $\Gamma_{\text{in}}^{\text{act}}$ and $\Gamma_{\text{in}}$ for the phase is then computed. For
the magnitude case, the RMSE is computed between the dB values of $\Gamma_{\text{in}}^{\text{act}}$ and $\Gamma_{\text{in}}$. In
such a way we vary the magnitude of $S$ from 0 to 1 dB and the phase from $0^\circ$ to $1^\circ$ and
are able to observe the corresponding change in $\Gamma_{\text{in}}$.

Figure 5.3 shows plots of the Root Mean Squared Error (RMSE) for magnitude and phase
of $\Gamma_{\text{in}}$ against the magnitude and phase error in $S$ for the 9 element linear array and the
5×5 square RECAP. We observe that the variation for both topologies is quite similar.
As we start to introduce error in the magnitude of $S$ there is relatively less change in
the corresponding RMSE of $|\Gamma_{\text{in}}|$, especially in the case of the square RECAP. In the
case of the phase error in $S$, we observe that the RMSE phase of $\Gamma_{\text{in}}$ relatively changes
quite a bit more. This suggests that $\Gamma_{in}$ is much more sensitive to changes in phase in $S$ rather than changes in magnitude.

If we read the plot the other way round, meaning we assume that we changed the RMSE phase and magnitude $\Gamma_{in}$ and then observe the corresponding change in $S$, then this gives us an approximate indication of how sensitive $S$ is to error in $\Gamma_{in}$. Therefore, using Figure 5.3 we get a better understanding of how much change in $S$ is expected when measuring $\Gamma_{in}$ while taking error into account.

5.2.1.2 Effect on Newton Based Null optimization

We would now like to gauge how introduction of error in phase and magnitude of $S$ affects the average worst null performance of the Newton based null optimization. We perform the analysis while searching for 2, 4, 6 and 8 nulls and as mentioned, the average worst null performance for each case is found. In order to clearly identify the effects of each, phase error and magnitude error in $S$ are introduced separately. We first discuss the case of error in magnitude of $S$.

We steadily add a random magnitude error to each element of $S$. We start from 0 dB and slowly increase the range $R$ within which this random error can be chosen, in small step sizes, until we reach 1 dB. At each step we perform the Newton based null optimization for a maximum of 40 iterations using $S$, where $S$ has some random error in magnitude added to it. We repeat this procedure 1000 times for each unique value of $R$. The performance of the worst null, meaning the null with the highest value in the direction of the randomly chosen steering angle, is averaged over these 1000 iterations.

The average worst null performance for the case of error in phase of $S$ is also computed in exactly the same way. The only difference now is that the phase error is increased from $0^\circ$ to $10^\circ$ while the magnitude error of $S$ was limited between 0 dB and 1 dB. We can observe from Figure 5.3 that even if the error in $\Gamma_{in}$ is high, we do not expect the phase error in $S$ to go up to any more than $2^\circ - 3^\circ$ in a practical system. However we employ a relatively big range for phase error in $S$ so that we can observe the trend in the worst null performance.

Figure 5.4 shows average worst null performance plotted in dB against magnitude error in $S$ and phase error in $S$. As expected, when the error increases in the magnitude and phase of $S$ the corresponding null performance deteriorates for all number of nulls being searched for. It is interesting to note that the worst null performance is quite insensitive to increase in the number of nulls searched for. This is exhibited by the fact that all the lines in both plots are very close to one another.
There is a sharp increase in the average worst null value when we move from 0 to 0.5° phase error in $S$ but thereafter there is relatively slow increase in the worst null value. As mentioned previously we do not expect the phase error in $S$ to increase beyond 3° and the corresponding worst null performance is below -20 dB for all nulls searched. Likewise, there is a sharp increase in the average worst null value when we move from 0 dB magnitude error to 0.1 dB magnitude error in $S$ but afterwards the increase in average worst null value is slow. In the same way, as long as we are below a magnitude error of 0.3 dB in $S$, all of the worst nulls are below -20 dB, which is still quite deep value.

The fact that the worst null performance does not deteriorate exponentially as the phase and magnitude error in $S$ is increased bodes well for our Newton based optimization. This means that even if the error in the estimated $S$ is high, due to greater error in measurements of $\Gamma_{in}$ and radiated far-fields $e(\hat{s})$, nulls can still be expected to be relatively strong.

### 5.3 Variations in Input reflection-coefficient $\Gamma_{in}$ and radiated far-field $e(\hat{s})$

We have so far looked at how variations in the S-parameter matrix might effect $\Gamma_{in}$ and the Newton based null optimization. So far in our discussion we have not included the effect of any errors on our unknown parameters estimation procedure. In a practical
system our ultimate goal would be to be able to realize near perfect Newton based null optimization performance even when there are errors in the input reflection-coefficient $\Gamma_{in}$ and radiated far-field at a chosen steering angle $e(\hat{s})$.

To do so, we must study how our estimation procedure behaves when exposed to error prone $\Gamma_{in}$ and $e(\hat{s})$ values. Again, we treat the phase error and magnitude error separately. In the following, we assume that we are searching for a null at a steering angle of $\phi = 80^\circ$.

We steadily add a random magnitude error to $\Gamma_{in}$ and perform the first step of our estimation procedure to get $S^{est}$. Thereafter we supply $S^{est}$ to equation (2.5), describing the radiated far-field $e(\hat{s})$ and add a random magnitude error to $e(\hat{s})$ in the specified direction $\phi = 80^\circ$. We start from 0 dB and slowly increase the range $R$ within which this random error can be chosen, in small step sizes, until we reach 1 dB. At each step we run the two parts of our estimate procedure, each for 500 iterations, using the $\Gamma_{in}$ and $e(\hat{s})$ with random error in magnitude added to them. Using the parameters estimated, we perform Newton based null optimization for the steering angle $\phi = 80^\circ$ and note how the null performance deteriorates as the magnitude error is increased.

This estimation procedure is repeated 50 times for each unique value of the range $R$ and the corresponding null performance is averaged over these iterations. We employ the exact same method to analyze how null performance is affected by changes in phase in $\Gamma_{in}$ and $e(\hat{s})$. The phase error is changed from a range of $R = 0^\circ - 16^\circ$.

Figure 5.5 shows how the antenna gain in dB (steering angle $\phi = 80^\circ$) for the 9 element linear RECAP and the $5 \times 5$ element square RECAP changes with respect to magnitude error in $\Gamma_{in}$ and $e(\hat{s})$. When we look at the zero error case, it is interesting to note that after 500 iterations of the estimation procedure the linear RECAP has a much stronger null than the square RECAP. In both cases, the null performance slowly deteriorates till we are at about -20 dB for the square RECAP for 1 dB magnitude error and -50 dB for the linear RECAP at the same error point.

The performance for the square RECAP tends to be a bit unsteady in comparison to the linear RECAP. This can be attributed to the fact that in the case of the square RECAP we are searching for about a 100 times more parameters than for the linear RECAP. This means that while in the case of the linear RECAP, we tend to find a suitable solution in all cases, for the square RECAP the solution tends to converge very slowly for a small number of cases and this leads to a more random nature in null performance that is not always entirely dependent on the magnitude error but is dependent on the convergence of the algorithm too.
It is however quite encouraging to see that our estimation procedure performs more than adequately even when exposed to up to 1 dB magnitude error, with the antenna gain in the null direction remaining below -20 dB for both the RECAP topologies.

Figure 5.6 shows how the antenna gain in dB (steering angle $\phi = 80^\circ$) for the 9 element linear RECAP and the $5 \times 5$ element square RECAP changes with respect to phase error in $\Gamma_{in}$ and $\epsilon(\hat{s})$. The first aspect that strikes us upon looking at this plot is the surprising similarity in the trend of both the lines representing the linear RECAP and the square RECAP. We note that again in this case, the linear RECAP has a much stronger null when compared to the square RECAP for no error. For both plots, we see that for 0 to 2 degrees of phase error, the null performance does not deteriorate but in fact gets better. We could surmise from this, that the estimation procedure is not affected much at all by such a relatively small phase error and at this stage, the result is influenced by the rate of convergence of the algorithm. Therefore the performance may even be seen to improve when the average convergence of the algorithm is faster.

The trend we see, is a slow deterioration in null performance as phase error is increased and by the time we reach a very significant amount, $16^\circ$ of phase error, the antenna gain is still at quite a low value below -20 dB in both cases.

Another aspect of interest is the fact that even though the two topologies are significantly different in size and type, the behavior of both with respect to increasing error is pretty much the same. In both cases, by the time we reach $16^\circ$ phase error, the null value has
increase from about 15 dB to 20 dB from the initial zero error case. We note that the same behavior is present in the magnitude error case as well.

5.4 Conclusion

In this chapter we took into consideration the in-situ problem and delved into some of the limitations in a practical system. It was first mentioned that many of the parameters present in the objective function such as the radiated far-fields at the feed and the reconfigurable loads $e_F(\hat{s})$, $e_R(\hat{s})$ and the scattering parameters $S$ are unknown to us. We needed to estimate these parameters with a certain level of accuracy to assure that the Newton based null and beam optimization works efficiently and correctly.

Making use of the parameters that we could measure in a real setup, such as the input reflection coefficient $\Gamma_{in}$ and the radiated far-field $e(\hat{s})$ we were able to devise a method based on the first order Taylor series to estimate the unknown parameters. It was shown that when the system of equations was equal to the number of unknown parameters, the solution to the Taylor series was unstable. The problem became well-conditioned when the number of equations was 8 times the number of unknown parameters searched for. The over-determined system was then solved in a least square sense to solve for the radiated far-fields at the feed and the REs as well as the scattering parameters.
We observed that even if the individual values of the estimated parameters did not agree with the actual parameters, the final result for $\Gamma_{in}$ as well as $e(\hat{s})$ were remarkably close to the actual values. It was argued that this behavior was exhibited since the optimization procedure was not concerned with the individual values as long as the difference between $\Gamma_{in}^{actual}$ and $\Gamma_{in}$ was reduced.

After the parameter estimation, an analysis was presented on the possible variations in the results due to phase and magnitude error in $S$ and the resulting effects on $\Gamma_{in}$ and the average worst null when searching for multiple nulls. It was shown that the results, while not insensitive to change in $S$ parameters, were not highly affected by the errors and that the null optimization performed adequately even when errors were added to the estimated parameters. We also observed that the error in $\Gamma_{in}$ was more sensitive to change in phase of $S$ as opposed to change in magnitude of $S$.

Finally, we studied how variations in $\Gamma_{in}$ and $e(\hat{s})$ detrimentally affect the estimation procedure. It was shown that when magnitude and phase error were added to both these components the performance of the Newton based null optimization deteriorated. But we saw that even in the case of high error, the null always remain below the threshold of -20 dB.
Chapter 6

Conclusion

In this work we have presented an efficient method for optimization of reconfigurable aperture antennas using a multi-dimensional first order Taylor series. It was first shown how, by employing network analysis, we can efficiently simulate the RECAPs using a single MOM simulation for each reconfigurable port. A Reduced Order RECAP model was then introduced which employed a first order expression of the antenna array response to come up with strong beams in the chosen steering angle.

Newton based null optimization was introduced in Chapter 3 where it was shown that we can easily find multiple nulls in chosen directions by means of linearizing the objective function in a small neighborhood. The optimization procedure was compared with a genetic algorithm and the results compared favorably in terms of null optimization. But it was shown that the gain of the reduced order parasitic model main beam deteriorated as greater number of nulls were searched for. Thereafter, a comparison of the relative linearity of different RECAP topologies was presented to find out which one would work best with the Newton based optimization. Reconfigurable elements with phase shift limited to 180° were then simulated and the Newton based null optimization was performed with these phase limited loads.

In chapter 4, we first pointed out that the main beam obtained from the reduced order RECAP model was a suboptimal solution. We then computed the RECAP theoretically optimal solution for the main beam in a particular direction $\hat{s}$. Afterwards, we developed a Newton based beam optimization method and employed the null space of the matrix of derivatives to optimize the main beam and null separately.

Finally in Chapter 5, we looked at the in-situ problem and estimated the unknown parameters: the radiated far-fields at the feed and the reconfigurable loads $e_F(\hat{s})$, $e_R(\hat{s})$ and the scattering parameters $S$. We again made use of a first order Taylor series of the
radiated far field $e(\hat{s})$ and the input-reflection coefficient $\Gamma_{in}$ to come up with a solution. It was shown that in order to stabilize the system of equations and to have a problem that was well-conditioned we had to over-determine the system by a ratio of 8 equations for each parameter to be estimated. In the end, we performed sensitivity analysis to find out how the Newton based null and beam optimization would perform if there were errors in the estimated parameters due to measurement errors in $\Gamma_{in}$ and $e(\hat{s})$. 
Bibliography


