NUMERICAL SOLUTION OF MAXWELL'S EQUATIONS IN THE TIME DOMAIN USING IRREGULAR NONORTHOGONAL GRIDS

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Several different methods for solving Maxwell's equations in the time-domain through the use of irregular nonorthogonal grids are presented. Employing quadrilateral and/or triangular elements, these methods allow more accurate modeling of nonrectangular structures. The traditional "stair-stepping" boundary approximations associated with standard orthogonal-grid finite-difference methods are avoided. Numerical results comparing all of the methods are given. A modified finite-volume method, which is a direct generalization of the standard finite-difference method to arbitrary polygonal grids, is shown to be the most accurate.

1. Introduction

The conventional approach for numerically solving Maxwell's equations in the time domain has been the use of finite-difference methods in conjunction with orthogonal grids [1-4]. It is well known that the use of such methods can produce very accurate results (particularly when the domain is rectangular or almost rectangular). For problems involving general nonrectangular domains, two approaches have been used: (a) "stair-stepped" orthogonal approximations to irregular boundaries, or (b) transformations which map the original irregular domain into a rectangle in the transformed coordinates. The first approach can give rather poor approximations to the boundary shape unless very fine discretizations are used, and the second results in more complicated equations to solve.

Over the past several years we have been investigating methods for solving Maxwell's equations in the time domain which allow the use of irregular nonorthogonal grids. We have been motivated by the desire to be able to approximate the boundaries of irregular domains more accurately than through the use of "stair-stepped" orthogonal approximations. It goes without saying that algorithms based on irregular grids will be significantly more complex (in almost all aspects of the word) than the simple conventional finite-difference algorithm which can be described and computed using only a few variables and operations per node per time step. In spite of the known additional complexity, we feel that the greater accuracy gained through more accurate boundary approximations justify our investigations—particularly with the present trends of drastically reduced numerical computing costs.

We present in this paper a number of different numerical methods which have evolved in the course of our investigations. First, we will describe several finite element derived methods which result from the application of a Galerkin approximation procedure. The first is based on the use of equal-order interpolation on simple 4-node bilinear elements. The second and third methods use a mixed-interpolation approach with piecewise constant and piecewise bilinear elements. Finally, we will present a method which is derived using a modified finite-volume technique. This technique actually reduces to the conventional finite-difference method when applied on an orthogonal grid.
general, we have primarily used these techniques with irregular quadrilateral grids. However, these same techniques are easily applied using grids which are composed of triangles, or mixtures of quadrilaterals and triangles.

We will not present an extensive discussion of the conventional finite-difference time-domain discretizations. The reader is referred to [1–4] and the recent review article [5] for complete derivations of the FD-TD algorithms. A tutorial on a point-matched finite-element time-domain approach has also appeared recently [6]. This approach results in an algorithm very similar in accuracy and in speed to one of the mixed-interpolant methods described below.

In our presentation, we will also discuss some of the complications which arise through proper treatment of boundary conditions and material interfaces. We will also compare and contrast the various methods as to their complexity, accuracy, computer storage requirements, etc. The finite element based methods have all been generalized and implemented in a full three-dimensional setting and the comparisons we present apply equally to the 3D algorithms. An extension to 3D for the modified finite-volume method has been accomplished and validation tests are in progress.

2. The Galerkin finite-element formulation

We begin by assuming that we wish to solve Maxwell’s curl equations on an irregular two-dimensional domain $\mathcal{R}$ which has a boundary denoted by $\mathcal{S}$. We will also assume that the domain $\mathcal{R}$ has been discretized into quadrilaterals (or triangles) as depicted in Fig. 1. Maxwell’s curl equations are given by

$$\varepsilon \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H},$$

$$\mu \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E},$$

where (for our two-dimensional setting) $\mathbf{E}$, the electric field vector, and $\mathbf{H}$, the magnetic field vector, are $(E_x, E_y, 0)$ and $(0, 0, H_z)$ respectively. This choice of the TE$_z$ polarization is arbitrary and has no impact on the following results. The formulation of the TM$_z$ polarization is entirely reciprocal. The linear isotropic material properties are: $\varepsilon$, the permittivity, and $\mu$, the permeability. We will allow the material properties to be spatially varying. For the purposes of this paper, we will assume that boundary of the domain $\mathcal{R}$ is a perfectly conducting surface, i.e., $E_{\tan} = 0$. This is sufficient to guarantee that the problem is well-posed.

In order to apply a Galerkin procedure to obtain discrete approximations for Maxwell’s equations, we assume that the electric and magnetic fields can be approximated as piecewise polynomials:

$$\mathbf{E} = \sum_{i=1}^{N_E} E_i(t) \phi_i(x),$$

$$\mathbf{H} = \sum_{i=1}^{N_H} H_i(t) \psi_i(x)$$

where $\phi_i$ and $\psi_i$ are basis functions associated with the $i$th node or element of the domain discretization. If the basis functions $\phi_i$ and $\psi_i$ are polynomials of the same order, we refer to this as
equal-order interpolation. When \( \phi_i \) and \( \psi_i \) are polynomials of differing orders, we refer to this as mixed-interpolation.

The discrete Galerkin equations are obtained by multiplying the two curl equations, (1)-(2), by \( \phi_i \) and \( \psi_i \) respectively, and then integrating the resulting equations over the domain \( R \). The Galerkin equations are

\[
\int_R e \phi_i \frac{\partial E}{\partial t} = \int_R \phi_i \nabla \times H, \quad i = 1, \ldots, N_E; \\
\int_R \mu \psi_i \frac{\partial H}{\partial t} = -\int_R \psi_i \nabla \times E, \quad i = 1, \ldots, N_H.
\]

Substituting the basis function expansions (3) and (4) into eqs. (5) and (6) gives the following coupled set of ordinary differential equations:

\[
\frac{dE_x}{dt} = C_x H_z,
\]

\[
\frac{dE_y}{dt} = -C_x H_z,
\]

\[
\frac{dH_z}{dt} = B_x E_x - B_y E_y
\]

where the nodal amplitude coefficients \( E_x \) and \( E_y \) are vectors of dimension \( N_E \) and \( H_z \) is a vector of length \( N_H \). The matrices \( M_E, M_H, C_x, C_y, B_x, \) and \( B_y \) are referred to as the mass and gradient matrices. The entries of these matrices are integrals of products of the various basis functions and derivatives. Specifically,

\[
[M_E]_{ij} = \int e \phi_i \phi_j, \\
[M_H]_{ij} = \int \mu \psi_i \psi_j, \\
[C_x]_{ij} = \int \phi_i \partial_x \psi_j, \\
[C_y]_{ij} = \int \phi_i \partial_y \psi_j, \\
[B_x]_{ij} = \int \psi_i \partial_x \phi_j, \quad \text{and} \quad [B_y]_{ij} = \int \psi_i \partial_y \phi_j.
\]

Normally, they would be evaluated by use of Gaussian quadrature following transformation to local orthogonal coordinates via an isoparametric mapping. However, for the linear Maxwell’s equations they may be evaluated as analytic functions of the nodal coefficients and coordinates.

Equations (7)-(9) are the discrete Galerkin approximating equations for Maxwell’s curl equations in two space dimensions. In addition to these equations, exterior boundary conditions need to be specified. As the specific procedure can vary significantly depending upon the particular finite-element approximation chosen, we will discuss the implementation of boundary conditions with each different method presented.

3. Time integration method

Since the Galerkin procedure discretizes only the spatial variables, the question of solving the time-dependent ordinary differential equations (7)-(9) needs to be considered. The time integration is complicated by the presence of the mass matrices, \( M_E \) and \( M_H \), which multiply the time derivatives of the field variables. In order to be able to efficiently use explicit time integration techniques, we always choose to use a “lumped mass” approximation technique. The net result of mass lumping is that mass matrices are replaced by approximating diagonal matrices (usually formed by simply summing the matrix entries on each matrix row). It is known that mass-lumping can degrade the propagation characteristics of the algorithm. However, we have found that the efficiency gains have always more than compensated for the loss of accuracy.

In practice, we have used a variety of numerical integration algorithms, ranging from variable-order variable-stepsize multistep predictor-corrector algorithms to the simple fixed-step “leapfrog” method. We have found that the efficiency and second-order accuracy of the explicit leap-frog algorithm is completely adequate for most problems. This algorithm is defined as follows:

\[
E^{n+1/2} = E^{n-1/2} + dt M_E^{-1} f(H^n),
\]

\[
H^{n+1} = H^n + dt M_H^{-1} g(E^{n+1/2}).
\]

where \( f \) and \( g \) represent the right-hand sides of equations (5) and (6). As this is an explicit time-stepping algorithm, the timestep size \( dt \) must be chosen small enough to guarantee stability. We have observed that \( dt \) must be chosen no larger than the smallest interior distance between two nodes of the grid divided by the local propagation speed \( \frac{1}{\sqrt{\mu e}} \).
4. ELHL equal-order interpolation method

The most conventional Galerkin approximation comes from using equal-order interpolation with low-order piecewise polynomials. Our first algorithms are obtained by assuming that over each quadrilateral element of the grid, the electric and magnetic fields are represented by piecewise bilinear functions. That is, \( \phi_i = \psi_i \) is a bilinear function which is 1 at node \( i \) and is 0 at all of the other nodes associated with elements which share node \( i \) (a linear function in the case of triangular elements). We will subsequently refer to this method as the ELHL method (E-Linear, H-Linear) [7, 8]. In this case, there are three unknown nodal amplitude coefficients associated with each node in the grid, \( E_{x,i}, E_{y,i}, \) and \( H_{z,i} \), which must be computed (see Fig. 2(a)). In the typical case that node \( i \) is surrounded by four quadrilateral elements, the Galerkin equation for \( E_{x,i} \), for instance, will involve the nodal magnetic field coefficients at the nine nodes which define the four elements. The computation of the other components, \( E_{y,i} \) and \( H_{z,i} \), at node \( i \) is similar. The total number of unknowns to be computed is simply three times the number of nodes which define the grid.

Certain complications arise when one considers how to impose the exterior (perfect electric conductor) boundary conditions. These conditions involve the tangential component of the electric field and not (in general) the nodal values \( E_x \) or \( E_y \). In order to impose the essential boundary condition that \( E_{tan} = 0 \), we must change variables at the boundary. However, before this can be done, a proper choice of normal and tangential directions needs to be specified. Following arguments similar to those of Engelman et al. [9], it can be demonstrated that the proper average tangent direction for our piecewise linear boundary approximation is determined from the vector which connects the boundary node before node \( i \) to the boundary node which follows node \( i \) (see Fig. 2(b)). With this choice of tangent direction, the eqs. (7) and (8) can be rotated and transformed so that they involve the normal electric field component, \( E_n \), and the tangential electric field component, \( E_{tan} \). The equation for \( E_{tan} \) is then replaced by the essential boundary condition \( E_{tan} = 0 \) and the equation for \( E_n \) is retained. This complication means that additional information for nodes on or immediately adjacent to the boundary is required and that different computations are required. While the amount of extra storage and computation is not great when compared with the total computational effort, it does complicate the code design logistics and its interface to code users.

Similar considerations are required to properly treat material interfaces where discontinuities in \( E_n \) can occur, for instance, when materials with different dielectric constants \( \varepsilon \) are adjacent to one
another. Here, not only are normal and tangential components required, but also the known discontinuities in the normal electric field must be accounted for. Internal to the code, the discontinuity can be avoided by switching to the continuous electric displacement variable $D_n = \varepsilon E_n$. However, either on input or output, one has to deal with the usually discontinuous $E_x$ and $E_y$ components at the material interface locations. Again, these material interfaces really serve only as code logistics annoyances and not as significant technical obstacles.

5. ELHC mixed-interpolation method

Our first mixed-interpolation scheme is derived by assuming that (as with the ELHL method) the electric field components are approximated by piecewise bilinear functions, $\phi_i$, over the grid elements. However, the magnetic field will be approximated by piecewise constant functions, $\psi_i$, over each grid element. We will subsequently refer to this method as the ELHC method (E-Linear, H-Constant) [9]. We also note that the constant basis function $\psi_i$ is more logically associated with the $i$th element and therefore the number of unknown magnetic field coefficients, $N_H$, becomes equal to the number of elements in the grid. The total number of unknowns for this ELHC mixed-interpolation method is twice the number of nodes plus the number of elements. It should also be noted that mixed-interpolation representations can be viewed as generalizations of staggered-grid finite-difference methods to distorted meshes.

Because the magnetic field is being approximated by a discontinuous piecewise constant function, it is necessary to invoke Stokes' theorem (or integration by parts) to evaluate the integral on the right-hand side of eq. (5). We obtain

$$
\int_R \mu \psi_i \frac{\partial \mathbf{H}}{\partial t} = - \int_R \psi_i \mathbf{\nabla} \times \mathbf{E}, \quad i = 1, \ldots, N_H
$$

(11)

which is referred to as the “weak” form. In eq. (10), $\mathbf{n}$ is an outward unit normal to the boundary $S$. As before, we note that all of the integrals can be analytically evaluated in terms of nodal or barycenter (location in an element defined as the average of the locations of the nodes defining that element) quantities for the elements. For a typical node, which has four quadrilateral elements associated with it, the computation of $E_{x,i}$ and $E_{y,i}$ at the $i$th node will involve only the four magnetic field values of the elements surrounding the node. Correspondingly, the computation of the magnetic field coefficient associated with the $i$th element will involve only the electric field coefficients associated with the nodes which define the element (see Fig. 3). The ELHC algorithm can be similarly derived for triangular grids.

![Fig. 3. ELHC mixed-interpolation field component stencil.](image)

Since the basis functions for the electric field components are the same as for the ELHL method, all of the previously mentioned complications related to boundary conditions and material interfaces apply to this ELHC method. The line integral around the domain boundary $S$ of $\phi_i(\mathbf{n} \times \mathbf{H})$ in equation (10) actually never enters into the approximation. This is because $\mathbf{n} \times \mathbf{H}$ is tangent to the boundary $S$, and we replace the Maxwell
equation for $E_{tan}$ (the only equation where the line integral would be involved) by the essential condition that $E_{tan} = 0$.

6. ECHL mixed-interpolation method

The second mixed-interpolation method arises if we simply reverse the roles played by the basis functions in the ELHC method. The two electric field components are now approximated by piecewise constant functions, $\phi_i$, over the grid elements. The magnetic field will be approximated by piecewise bilinear functions, $\psi_i$. We will refer to this method as the ECHL method (E-Constant, H-Linear) [7, 8]. The constant basis function $\phi_i$ is associated with the $i$th element barycenter and the bilinear basis function $\psi_i$ with the $i$th node. The total number of unknowns for the ECHL method is twice the number of elements plus the number of nodes in the grid. This method would appear to be very similar to the ELHC method. However, it has some interesting subtleties that make it more attractive than the ELHC method.

Because the electric field is being approximated by discontinuous piecewise constant functions, Stokes’ theorem (or integration by parts) is required to evaluate the integral on the right-hand side of eq. (6). We obtain the “weak” form

$$\int_R \varepsilon \phi_i \frac{\partial E}{\partial t} = \int_R \phi_i \nabla \times H,$$

$$\quad i = 1, \ldots, N_E;$$

$$\int_R \mu \psi_i \frac{\partial H_z}{\partial t} = -\oint_S \psi_i E_{tan} + \int_R (\nabla \psi_i \times E) \cdot \hat{e};$$

$$\quad i = 1, \ldots, N_H. \tag{13}$$

For a typical node, which has four quadrilateral elements associated with it, the computation of $H_z$ at the $i$th node will involve only the electric field values associated with elements which surround the node. Also, the computation of the electric field values associated with the $i$th element will involve only the magnetic field values associated with the nodes which define the element (see Fig. 4). Again, no complications arise if grids composed of triangular elements are used.

When one considers imposing boundary conditions for the ECHL method, it becomes obvious that things are quite different than for the ELHC method. We note that none of the electric field quantities (now associated with the element barycenters) are located on the domain boundary $S$ where we wish to specify $E_{tan}$. However, the tangential electric field naturally appears in the boundary line integral for the magnetic field nodal equation (13). To approximate a perfect electric conductor boundary ($E_{tan} = 0$), we simply set this line integral to zero (i.e., ignore it). If we want to specify the tangential electric field so as to excite or drive a problem, we have only to evaluate the line integral using the desired values for $E_{tan}$. The boundary conditions have become “natural” to the formulation through the choice of basis functions. In direct contrast to the previous methods, no rotations or transformations of the electric field components $E_x$ and $E_y$ to $E_r$ and $E_{tan}$ are required. Many code logistics problems related to boundary conditions are eliminated with this method.

![ECHL mixed-interpolation field component stencil.](image)
are associated with element barycenters (away from the material interfaces which occur at element sides) and the material interfaces naturally take care of themselves in the formulation. Again, no rotations to normal and tangential local coordinates are required. The material interface can be handled by simply specifying different material constants for the different elements.

7. Modified finite-volume method

Finite-volume techniques are usually derived by integrating the basic equations to be solved over a grid element and then discrete approximations are formed for the resulting integral quantities. Our finite volume technique will make use of the basic quadrilateral grid and also its "dual" grid. For each quadrilateral element, we define the barycenter and construct the dual grid by connecting barycenters with straight lines passing through each of the interior sides of the original grid. We make the assumption that the grid is sufficiently regular that the join of two barycenters intersects only the side which the two elements share. Figure 5(a) shows a grid and its dual. Our discrete solution components will be associated with the sides and barycenters of the original grid. The quantity associated with an element side is the projection of the electric field vector onto that side, i.e., $E \cdot s$,
where $s$ is a unit vector in the direction of the side. The magnetic field values are associated with the cell barycenters. Figure 5(b) depicts these associations.

We begin by decomposing the electric field vector $E$ into two orthogonal components which are determined from the dual grid. We have

$$E = (E \cdot s^*)s^* + (E \cdot s^*)s_*^*$$  \hspace{1cm} (14)

where $s^*$ is a unit vector in the dual side direction and $s_*^*$ is a unit vector orthogonal to $s^*$ so that $(s_*^*, s^*, \xi)$ forms a right-handed system (see Fig. 5(c)). For a particular element side $s$, the quantity to be computed is $E \cdot s$. Using (14) we have

$$\frac{d(E \cdot s)}{dt} = \left( \frac{dE}{dt} \cdot s^* \right) (s^* \cdot s) + \left( \frac{dE}{dt} \cdot s_*^* \right) (s_*^* \cdot s).$$  \hspace{1cm} (15)

Using (1) we may compute the first term on the right side of (15) as

$$\varepsilon A \frac{dE}{dt} \cdot s^* = \int (\nabla \times H) \cdot s^* \, dA$$

$$= \int \alpha \frac{\partial H_z}{\partial y} \, dx \, dy - \int \beta \frac{\partial H_z}{\partial x} \, dx \, dy$$

$$= -\alpha \int H_z \, dx - \beta \int H_z \, dy$$  \hspace{1cm} (16)

where $s^* = (\alpha, \beta)$. The line integrals are performed around the two dual grid cells which share the dual side $s^*$, and $A$ is the area of these two dual grid cells. To compute $E \cdot s^*$ we note that because $s_*^*$ is orthogonal to the dual grid side $s^*$, a completely adequate approximation can be obtained from a simple finite difference of the magnetic field values at the barycenters which define the dual grid side $s^*$ (see Fig. 5(c)). We have

$$\varepsilon \frac{dE}{dt} \cdot s_*^* = \frac{(H_{z_2} - H_{z_1})}{d_{12}}$$  \hspace{1cm} (17)

where $d_{12}$ is the distance between the two barycenters.

To compute the magnetic field value at an element barycenter, we simply integrate or average the curl equation for $H$ over the element. This again is easily seen to be a line integral around the perimeter of the element and we have

$$\mu A \frac{dH_z}{dt} = -\int (\nabla \times E) \cdot \hat{\xi} \, dA$$

$$= -\oint E \cdot dl$$  \hspace{1cm} (18)

where $A$ is now the area of the element. Equations (15)-(18) constitute the Modified Finite-Volume (MFV) approximation method.

To numerically evaluate these line integrals, we assume that $H_z$ varies linearly between the barycenters on the dual grid, and that $E \cdot s$ is constant along the side of the regular grid which defines $s$. When this approximation is applied on an orthogonal grid, we note that $s_*^* = s$, $s^* \cdot s = 0$, and that this MFV approximation reduces to be identical to the canonical Yee finite-difference algorithm. The MFV approximation is thus a generalization of the usual finite-difference algorithm to nonorthogonal grids. The total number of unknowns is clearly equal to the number of sides plus the number of elements in the grid. The computation of a typical $E \cdot s$ will involve (in general) the use of six surrounding magnetic field values. The computation of any magnetic field value involves four electric field projected values. This algorithm is very general and can be applied to any convex polygonal grid. This increased generality results directly from the abandonment of the finite-element concept of piecewise polynomial representations of the field variables. For special grid discretizations involving only acute triangles, if the barycenter locations are replaced by the intersections of the side perpendicular bisectors, then the dual grid construction can be eliminated and a simple finite-difference algorithm for acute triangles results.

Exterior (perfect electric conductor) boundary conditions are very easily handled by simply setting $E_{tn} = E \cdot s = 0$ for those sides which form the boundary. Interior dielectric material interfaces are “naturally” treated due to the fact that only variables which represent the continuous tangent-
tial electric field components \((E \cdot s)\) occur on the interface.

8. Discrete differential forms

A general framework for the above discrete methods has been developed. It is based upon a discretization of the differential form version of Maxwell's equations. Because the differential form expressions are independent of the local coordinate systems, they represent a global formulation of the physics and naturally lead to the contour/area integrals appearing in the mixed-interpolant and modified finite-volume methods. In its pure form, the discretization of the differential form expressions leads to yet another algorithm that is restricted to Voronoi polygon constructions. Nevertheless, this approach has provided insight which helped to lead us to the present discrete approximations which have no mesh restrictions. The interested reader is referred to [9] for further information.

9. Numerical results

There are many different philosophical approaches for comparing numerical methods. It is neither our purpose nor our intent to exhaustively compare all of the different methods for solving Maxwell's equations in the time domain. Our intent is to demonstrate that accurate solutions can be obtained in the time domain through the use of irregular boundary conforming grids. We will not discuss run times or overall efficiencies because the results have been obtained on a variety of computers ranging from VAX stations to Cray-XMPs. Our computer software has not been carefully optimized to account for particular computer architectural features. Also, with most algorithms it is possible to trade memory storage requirements with running time. In summary, exhaustive numerical comparisons are far beyond the scope and intent of this paper.

We first consider a simple problem which is designed to provide some assessment of the wave propagation characteristics of each algorithm. We consider the propagation of a TEM pulse in a rectangular waveguide which is 0.4 meters in width and 2.0 meters in length. The pulse is launched by specifying the tangential field, \(E_y\), at the left boundary. The time function used to launch the pulse is

\[
f(t) = \begin{cases} 0.5[1 - \cos(10^9 \pi t)] & \text{if } 0 \leq t \leq 2 \times 10^{-9}, \\ 0 & \text{otherwise.} \end{cases}
\]

We assume that all of the waveguide walls are perfect electric conductors, i.e., \(E_\tan = 0\). The pulse is allowed to propagate until \(t = 5 \times 10^{-9}\) s, at which time we will examine the errors in the numerical solutions. The analytic solution is given by

\[
E_x(x, y, t) = 0, \\
E_y(x, y, t) = f(x - ct), \\
H_z(x, y, t) = \sqrt{\varepsilon/\mu} f(x - ct)
\]

where \(f\) is defined by (19) and \(c = 1/\sqrt{\varepsilon \mu}\) is the propagation speed. At the final time we measure the error in a maximum norm, i.e., we check the solution at all of the discrete variable locations and find the largest error.

We have used a variety of grids to numerically solve this problem. Clearly one can solve this problem using an orthogonal rectangular grid as shown in Fig. 6(a). However, to demonstrate the ability to use irregular grids, we deliberately skew the grid so that it is composed of quadrilaterals as shown in Fig. 6(b). To demonstrate the use of triangular discretizations, we have respectively derived the two triangular grids shown in Figs. 6(c) and (d) from the rectangular and skewed quadrilateral grids. Each of these four grids have the same exterior boundary and each is composed of 400 elements. With approximately 40 zones in the
propagation direction and a pulse-width of about 0.6 meters, we have about 12 zones per pulse-width on average. A time step size of $5 \times 10^{-11}$ s was used for all cases.

Table 1 summarizes the results obtained from the different algorithms using the various grids. The errors shown are for the electric field only. The errors for the magnetic field behave similarly.

<table>
<thead>
<tr>
<th>Method</th>
<th>Grid</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELHL</td>
<td>Rectangular</td>
<td>0.337</td>
</tr>
<tr>
<td>ELHC</td>
<td>Rectangular</td>
<td>0.093</td>
</tr>
<tr>
<td>ECHL</td>
<td>Rectangular</td>
<td>0.098</td>
</tr>
<tr>
<td>MFV</td>
<td>Rectangular</td>
<td>0.098</td>
</tr>
<tr>
<td>ELHL</td>
<td>Skewed Quadrilateral</td>
<td>0.399</td>
</tr>
<tr>
<td>ELHC</td>
<td>Skewed Quadrilateral</td>
<td>0.192</td>
</tr>
<tr>
<td>ECHL</td>
<td>Skewed Quadrilateral</td>
<td>0.199</td>
</tr>
<tr>
<td>MFV</td>
<td>Skewed Quadrilateral</td>
<td>0.042</td>
</tr>
<tr>
<td>MFV</td>
<td>Regular Triangular</td>
<td>0.047</td>
</tr>
<tr>
<td>MFV</td>
<td>Skewed Triangular</td>
<td>0.063</td>
</tr>
</tbody>
</table>

It is probable that one would expect the best results to be obtained using the MFV method on the rectangular grid (this being equivalent to the often used conventional finite-difference algorithm). However, we observe this not to be the case. The most accurate results are obtained using the MFV method with the skewed quadrilateral grid. Relatively accurate solutions are obtained by the MFV method with all of the grids. The ECHL and ELHC algorithms produce relatively accurate results when used with the rectangular grid but perform less well on the skewed irregular grid. The ELHL method behaves relatively poorly in general. Its propagation characteristics are poor when compared with the other methods. Though not presented here, we have observed a second-order convergence rate for all of the methods as the grid is successively refined. We expect that comparisons contained in Table 1 will remain valid even in the limit as the grid spacings converge to zero.

As a second comparison problem, we consider the scattering of a plane-wave pulse from a perfectly conducting cylinder with a radius of 0.1 meters. This fully legitimate two-dimensional problem again has a known analytic solution which can be expressed in series form [10]. Figure 7 shows the $15 \times 30$ grid together with the observation points that were used for solving this problem with the different methods. The computational domain extends to a radius of 1.1 meters. While this grid looks like a regular grid in cylindrical coordinates, we remind the reader that all of our computations are performed in cartesian $x$-$y$ geometry (where the grid is quite irregular). Again, we compute to $t = 5 \times 10^{-9}$ s using time steps of size $5 \times 10^{-11}$ s. The plane-wave pulse is defined by
Fig. 7. Quadrilateral grid for the cylinder scattering problem, and observation points.

Fig. 8. Computed versus analytic solutions for the cylinder scattering problem at point nearest to $\rho = 0.3$, $\phi = 130^\circ$: (a) MFV; (b) ECHL; (c) ELHL.
the Gaussian-like function
\[
f(t) = \begin{cases} 
\frac{\exp(-10[10^9 t - 1]^2) - \exp(-10)}{1 - \exp(-10)} & \text{if } 0 \leq t \leq 2 \times 10^{-9}, \\
0 & \text{otherwise}.
\end{cases}
\]

Rather than present the results in tabular form, we choose to graphically display time history plots of the computed solutions at two representative points (shown as dots in Fig. 7). Figures 8 and 9 compare the analytic solutions for the time histories at these two points to computed solutions obtained using the MFV, ECHL, and ELHL methods. (Results for the ELHC method would closely resemble those for ECHL and are not presented.) It is observed that the ECHL results are relatively good, but less accurate than those obtained from the MFV method. The ELHL method consistently produces much poorer results.

It was previously stated that the various methods could be used with either quadrilateral or triangular discretizations. Since the MFV method is a generalization of the conventional finite difference method, one can envision a method and code structured so that the majority of grid ele-
elements used are regular rectangles but which allow
the use of a few irregular quadrilateral or triangular
elements so as to more accurately match irregular
boundaries. The MFV method seems to be an
interesting candidate for this approach. To demon-
strate this idea we again solve the cylinder scatter-
ing problem. Figure 10(a) shows the grid used
along with the two observation points where the
time history solutions are compared with the
analytic solution. This grid consists of 1788 ele-
ments: 1774 of which are uniform squares, and 14
are irregular quadrilaterals or triangles. Figure
10(b) shows an enlargement of the cylinder region
illustrating how the use of 4 triangles and 10 quad-
rilaterals can very adequately approximate the
curved cylinder and yet match up with the under-
lying square grid. Figures 11(a) and (b) show the
computed solutions compared with the analytic
solutions at the two observation points. Very cred-
ible and accurate solutions are obtained in spite
of the rather few number of element sides which
approximate the cylinder. For comparisons of
ECHL results with “stair-stepped” finite-
difference solutions, we refer the reader to [9].

The numerical results presented here together
with other computational experience with all of
the presented methods lead us to believe that the
MFV method is one with significant advantages.
First, it is a direct generalization of the well-known
finite-difference method for Maxwell’s equations,
so its approximation properties on regular

![Fig. 10. Mixed polygonal discretization for the cylinder scattering problem: (a) entire grid; (b) enlargement of the mixed polygonal region.](image)

![Fig. 11. Computed mixed polygonal versus analytic solutions for the cylinder scattering problem at point nearest to: (a) ρ = 0.3,
ϕ = 135°; (b) ρ = 0.3, ϕ = 36°.](image)
orthogonal grids are quite well understood. Second, it is quite accurate even when used with irregularly structured grids. Third, it is very flexible and can be used with various combinations of different polygonal discretizations. The most significant disadvantage of the MFV method is the larger amount of information (regular grid structure plus dual grid structure) that is required. The generalizations of the MFV method to three dimensions will be reported in future papers.

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