Chapter 4: Antenna Types
4.4 Aperture Antennas

High microwave frequencies
- Thin wires and dielectrics cause loss
- Coaxial lines: may have 10dB per meter
- Waveguides often used instead

Aperture antennas
- Transition from waveguide to free space
- Open end of a waveguide
- Or horn to allow smoother transition

Advantages
- Wideband, low loss
- Can be made directive
- Rigid, flush mounting (aerospace applications), easy integration
- Gain can be very accurately characterized
Radiated Fields from Apertures

1. Obtain approximation for fields at aperture opening

2. Transform to equivalent source problem (equivalence principle, images)

3. Compute fields radiated by the equivalent currents
Field Equivalence Principle

Uniqueness Theorem in EM

The fields in a (lossy) region are uniquely specified by
1. Sources in that region
2. Tangential components of electric or magnetic field over the complete boundary

Why useful?

Tells us what we need to preserve in a problem to have same solution
Allows us to develop simpler (but equivalent) problems
Field Equivalence Principle (2)

Problem (a)
Volume $V_1$ contains antenna
Has currents $J_1, M_1$
Gives rise to fields $E_1$ and $H_1$
Interested in radiated fields in $V_2$
Can replace with simpler problem

Problem (b)
Due to uniqueness, it is equivalent if
Tangential $E_1$ and $H_1$ same on surface
Created by fictitious currents $J_s, M_s$ on surface
Note: $E, H$ inside volume different!

(a) Original Problem

(b) Equivalent Problem
Ensuring proper tangential fields

Boundary conditions
\[ \hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_S \]
\[ -\hat{n} \times (\overline{E}_1 - \overline{E}_2) = \overline{M}_S \]

Ensure proper tangential fields with
\[ \overline{J}_S = \hat{n} \times (\overline{H}_1 - \overline{H}) \]
\[ \overline{M}_S = -\hat{n} \times (\overline{E}_1 - \overline{E}) \]

Can make $E$, $H$ whatever we like
So, usually make $E=0$ $H=0$
Radiated Fields

\[
\begin{align*}
\mathbf{J}_s &= \hat{n} \times (\mathbf{H}_1 - \mathbf{H}) \\
\mathbf{M}_s &= -\hat{n} \times (\mathbf{E}_1 - \mathbf{E}')
\end{align*}
\]

Substitute these currents into radiation integrals

\[
E_\theta(r, \theta, \phi) = -\frac{j k_0 e^{-j k_0 r}}{4 \pi r} \int_{S'} \left[ (-M_x \sin \phi + M_y \cos \phi) \\
+ \eta_0 (J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta) \right] e^{jk_0 \psi} dS'
\]

\[
E_\phi(r, \theta, \phi) = \frac{j k_0 e^{-j k_0 r}}{4 \pi r} \int_{S'} \left[ (M_x \cos \theta \cos \phi + M_y \cos \theta \sin \phi - M_z \sin \theta) \\
- \eta_0 (-J_x \sin \phi + J_y \cos \phi) \right] e^{jk_0 \psi} dS'
\]

\[
\psi = x' \sin \theta \sin \phi + y' \sin \theta \cos \phi + z' \cos \theta
\]
Image Technique

Idea

PEC and PMC surfaces can be replaced with images
Often simplifies problem

Images for (Tangential) Electric/Magnetic Currents

\[
\begin{align*}
\overline{J} & \rightarrow \quad \overline{J} \rightarrow \\
PEC & = \quad \overline{J} \leftarrow \\
\overline{M} & \rightarrow \quad \overline{M} \rightarrow \\
PEC & = \quad \overline{M} \leftarrow \\
\overline{J} & \rightarrow \quad \overline{J} \rightarrow \\
PMC & = \quad \overline{J} \leftarrow \\
\overline{M} & \rightarrow \quad \overline{M} \rightarrow \\
PMC & = \quad \overline{M} \leftarrow
\end{align*}
\]
Aperture Example: Uniform Field Over Infinite Ground Plane

Consider rectangular opening in infinite ground plane
Assume constant field in opening

Solve by formulating an equivalent problem
Requires only knowledge of E field in opening!
Equivalent Problem Formulation

Step (i)

Place an imaginary surface over ground plane

Over ground $E_T = 0$

Means that $M_S = 0$ where ground plane was present

Original problem

\[ \overline{E_a} = E_0 \hat{y} \]
Step (ii)

Place a PEC sheet underneath surface

Let it approach the surface

Does not change the problem (not in the region of interest)
Equivalent Problem Formulation (3)

Step (iii)
Replace PEC with images

Original problem

\[
\hat{E}_a = E_0 \hat{y}
\]

\[
\begin{align*}
\mathcal{J}_s & \quad \mathcal{M}_s = -\hat{n} \times \mathcal{E}_s \\
-\mathcal{J}_s & \quad -\mathcal{J}_s
\end{align*}
\]
Equivalent Problem Formulation (4)

**Step (iv)**

- Electric currents cancel
- Magnetic currents double

**Notice**

Radiation only depends on E field in aperture!

\[
\begin{align*}
\mathcal{E}_a &= E_0 \hat{y} \\
\mathcal{J}_S &= 0 \quad \mathcal{M}_S &= 0 \\
\mathcal{M}_S &= -2\hat{n} \times \mathcal{E}_S \\
\mathcal{J}_S &= 0 \quad \mathcal{M}_S &= 0
\end{align*}
\]
Radiated Fields

Equivalent Currents

\[ \overline{M}_S = \begin{cases} -2\hat{n} \times \overline{E}_a = \hat{x}2E_0, & -a/2 \leq x' \leq a/2 \\ 0, & -b/2 \leq y' \leq b/2 \\ & \text{otherwise} \end{cases} \]

\[ \overline{J}_S = 0 \]

Substitution into Integral for \( E_\theta \)

\[
E_\theta(r, \theta, \phi) = -\frac{jk_0 e^{-jk_0 r}}{4\pi r} \int_{S'} \left[ (-M_x \sin \phi + M_y \cos \phi) \\
+ \eta_0 (J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta) \right] e^{jk_0 \rho} dS'
\]
Radiated Fields (2)

\[ E_{\theta}(r, \theta, \phi) = \frac{jk_0 e^{-jk_0r}}{4\pi r} \int_{y'=-b/2}^{b/2} \int_{x'=-a/2}^{a/2} M_x \sin \phi \ e^{jk_0 \sin \theta (x' \cos \phi + y' \sin \phi)} \, dx' \, dy' \]

\[ = \frac{jk_0 E_0 e^{-jk_0r} \sin \phi}{2\pi r} \left( \frac{2j \sin(k_0 a/2 \sin \theta \cos \phi)}{jk_0 \sin \theta \cos \phi} \right) \left( \frac{2j \sin(k_0 b/2 \sin \theta \sin \phi)}{jk_0 \sin \theta \sin \phi} \right) \]

\[ = \frac{jk_0 ab E_0 e^{-jk_0r}}{2\pi r} \sin \phi \frac{\sin X}{X} \frac{\sin Y}{Y} \]

\[ X = k_0 a/2 \sin \theta \cos \phi \]
\[ Y = k_0 b/2 \sin \theta \sin \phi \]

\[ E_r = 0 \]
\[ E_\theta = C \sin \phi \frac{\sin X}{X} \frac{\sin Y}{Y} \]
\[ E_\phi = C \cos \theta \cos \phi \frac{\sin X}{X} \frac{\sin Y}{Y} \]
\[ H_r = 0 \]
\[ H_\theta = -E_\phi/\eta_0 \]
\[ H_\phi = E_\theta/\eta_0 \]
Aperture Example 2: Uniform Field in Aperture
No Ground Plane

Now, imagine no ground plane present
Empirical approximation

\[
\begin{align*}
\overrightarrow{M}_S &= -\hat{n} \times \overrightarrow{E}_a \\
\overrightarrow{J}_S &= \hat{n} \times \overrightarrow{H}_a
\end{align*}
\]  \quad \begin{cases} 
-a/2 \leq x \leq a/2 \\
-b/2 \leq y \leq b/2
\end{cases}
\quad \overrightarrow{J}_S = 0 \quad \text{Elsewhere}
\quad \overrightarrow{M}_S = 0

Far-fields

\[
\begin{align*}
E_r &= H_r = 0 \\
E_\theta &= \frac{C'}{2} \sin \phi (1 + \cos \theta) \frac{\sin X}{X} \frac{\sin Y}{Y} \\
E_\phi &= \frac{C'}{2} \cos \phi (1 + \cos \theta) \frac{\sin X}{X} \frac{\sin Y}{Y}
\end{align*}
\]
Summary of Aperture Antennas

Table 12.1 EQUivalents, fields, beamwidths, side lobe levels, and directivities of rectangular apertures

<table>
<thead>
<tr>
<th>Aperture distribution of tangential components (analytical)</th>
<th>Uniform Distribution Aperture on Ground Plane</th>
<th>Uniform Distribution Aperture in Free-Space</th>
<th>TE_{10}-Mode Distribution Aperture on Ground Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aperture distribution of tangential components (graphical)</td>
<td><img src="image" alt="Graphical Representation" /></td>
<td><img src="image" alt="Graphical Representation" /></td>
<td><img src="image" alt="Graphical Representation" /></td>
</tr>
<tr>
<td>Equivalent ( M_x = \begin{cases} -2\hat{\alpha} \times E_\phi &amp; -a/2 \leq x' \leq a/2 \ 0 &amp; b/2 \leq y' \leq b/2 \end{cases} )</td>
<td>( M_x = \hat{\alpha} \times E_\phi )</td>
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</tr>
<tr>
<td>Equivalent ( J_x = 0 ) everywhere</td>
<td>( J_x = \hat{\alpha} \times E_\phi )</td>
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</tr>
<tr>
<td>Equivalent ( M_y = 0 ) everywhere</td>
<td>( M_y = 0 ) everywhere</td>
<td>( M_y = 0 ) everywhere</td>
<td>( M_y = 0 ) everywhere</td>
</tr>
<tr>
<td>Far-zone fields ( X = \frac{ka}{2} \sin \theta \cos \phi ) ( Y = \frac{k\beta}{2} \sin \theta \sin \phi ) ( C = \frac{j abk E_\phi e^{-j\phi}}{2\pi r} )</td>
<td>( E_r = H_r = 0 ) ( E_\phi = C \sin \phi \frac{\sin X \sin Y}{X \cdot Y} ) ( H_\theta = -E_\phi \eta ) ( H_\phi = E_\theta \eta )</td>
<td>( E_r = H_r = 0 ) ( E_\phi = \frac{C}{2} \sin \phi (1 + \cos \theta) \frac{\sin X \sin Y}{X \cdot Y} ) ( H_\theta = -E_\phi \eta ) ( H_\phi = E_\theta \eta )</td>
<td>( E_r = H_r = 0 ) ( E_\phi = -\frac{\pi}{2} C \sin \phi \frac{\cos X \sin Y}{(X^2 - \frac{\pi^2}{4}) \cdot Y} ) ( H_\theta = -E_\phi \eta ) ( H_\phi = E_\theta \eta )</td>
</tr>
</tbody>
</table>
### Summary of Aperture Antennas (2)

<table>
<thead>
<tr>
<th>Half-power beamwidth (degrees)</th>
<th>E-plane $b \gg \lambda$</th>
<th>$50.6 \over b/\lambda$</th>
<th>$50.6 \over b/\lambda$</th>
<th>$50.6 \over b/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$-plane $a \gg \lambda$</td>
<td>$50.6 \over a/\lambda$</td>
<td>$50.6 \over a/\lambda$</td>
<td>$68.8 \over a/\lambda$</td>
<td>$68.8 \over a/\lambda$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>First null beamwidth (degrees)</th>
<th>E-plane $b \gg \lambda$</th>
<th>$114.6 \over b/\lambda$</th>
<th>$114.6 \over b/\lambda$</th>
<th>$114.6 \over b/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$-plane $a \gg \lambda$</td>
<td>$114.6 \over a/\lambda$</td>
<td>$114.6 \over a/\lambda$</td>
<td>$171.9 \over a/\lambda$</td>
<td>$171.9 \over a/\lambda$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>First side lobe max. (to main max.) (dB)</th>
<th>E-plane $a \gg \lambda$</th>
<th>$-13.26$</th>
<th>$-13.26$</th>
<th>$-13.26$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$-plane $a \gg \lambda$</td>
<td>$-13.26$</td>
<td>$-13.26$</td>
<td>$-23$</td>
<td>$-23$</td>
</tr>
</tbody>
</table>

| Directivity $D_0$ (dimensionless) | $\frac{4\pi}{\lambda^2} (\text{area}) = 4\pi \left( \frac{ab}{\lambda^2} \right)$ | $\frac{4\pi}{\lambda^2} (\text{area}) = 4\pi \left( \frac{ab}{\lambda^2} \right)$ | $\frac{8}{\pi^2} \left[ \frac{4\pi (ab)^2}{\lambda^2} \right] = 0.81 \left[ \frac{4\pi (ab)^2}{\lambda^2} \right]$ |

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**Table 12.1 (Continued)**

- **Antennas and Propagation**  
  *Slide 18*  
  *Chapter 4*
Reflector Antennas

Basic idea

Use a large reflective surface to direct / focus radiated energy
Increases effective area of element

Note

Whole books are dedicated to reflector antennas
We will only scratch the surface...
Corner Reflector

90° corner reflector

Feed produces far-field pattern

\[ f(\theta, \varphi) \]

How do we find solution with reflector?

Fields can be found using images
Corner Reflector: Image Method

(i) Original Problem

(ii) Replace Plate 2

(i) Replace Plate 1

Need to add contribution of images to find radiated fields
Far-fields of a Shifted Source

Assume we have current density $\overline{J}(\overline{r})$

What happens if we shift by $\Delta \overline{r}$?

$$f(\theta, \phi) = C \int \overline{J}(\overline{r}') e^{ik(x' \sin \theta \cos \phi + y' \sin \theta \sin \phi + z' \cos \theta)} d\overline{r}'$$

$$f'(\theta, \phi) = C \int \overline{J}(\overline{r}' - \Delta \overline{r}) e^{ik(x' \sin \theta \cos \phi + y' \sin \theta \sin \phi + z' \cos \theta)} d\overline{r}'$$

$$= C \int \overline{J}(\overline{r}') e^{ik[(x' + \Delta x) \sin \theta \cos \phi + (y' + \Delta y) \sin \theta \sin \phi + (z' + \Delta z) \cos \theta]} d\overline{r}'$$

$$= f(\theta, \phi) e^{ik(\Delta x \sin \theta \cos \phi + \Delta y \sin \theta \sin \phi + \Delta z \cos \theta)}$$

Just have a phase shift $\psi$
\[ \psi = jk(\Delta x \sin \theta \cos \phi + \Delta y \sin \theta \sin \phi + \Delta z \cos \theta) \]

\[ \psi_1 = ks \sin \theta \cos \phi \]
\[ \psi_2 = ks \sin \theta \sin \phi \]
\[ \psi_3 = -ks \sin \theta \cos \phi \]
\[ \psi_4 = -ks \sin \theta \sin \phi \]

\[ E(r, \theta, \phi) = \left[ e^{j\psi_1} - e^{j\psi_2} + e^{j\psi_3} - e^{j\psi_4} \right] f(\theta, \phi) \frac{e^{jkr}}{r} \]

\[ E(r, \theta, \phi) = 2[\cos(ks \sin \theta \cos \phi) - \cos(ks \sin \theta \sin \phi)] f(\theta, \phi) \frac{e^{-jkr}}{r} \]

Turns out that the corner reflector increases directivity of the dipole antenna significantly (6dBi instead of 1.8dBi)
Parabolic Reflector

Curved reflectors
- Paraboloid most common type (parabola rotated about vertex)
- Collimates rays (makes plane waves) from point source at feed
- By reciprocity, plane waves are focused to feed for reception
Curved Reflector: Analysis

Geometrical Optics

Provides approximate understanding
But, not exact...
1. Real feeds not a point source
2. Curvature causes diffraction
   (not flat relative to waves)

Two methods for analyzing

(i) Induced current density

   Current on the surface is found with
   If the surface is (approximately)
   an infinite plane

(ii) Aperture density method

   Approximate values in aperture are found
   Simplifies integration

\[
\mathbf{J}_S = \hat{n} \times \mathbf{H} = \hat{n} \times (\mathbf{H}_i + \mathbf{H}_r)
\]

\[
\hat{n} \times \mathbf{H}_i = \hat{n} \times \mathbf{H}_r
\]

\[
\mathbf{J}_S = 2\hat{n} \times \mathbf{H}_i
\]
Broadband / Frequency Independent Antennas

Very wide bandwidths are needed for some applications

- Television broadcasting / reception
- Spectrum monitoring
- Feeds for reflector antennas
- Ultra-wideband communications (UWB)
- Cognitive radio

Bandwidths $\geq 40:1$

- Realized with frequency independent antennas
- Antenna shape is independent of scale
- Completely specified by dimensions
Scale Model Measurements

Related to theory of frequency independent antennas

Consider

Want to model a very large 1 GHz antenna

We can reduce size by factor of 4

Test the structure with 4x frequency

Identical operation as long as material properties do not change!
Bi-conical Antenna

Classic example of frequency independent antenna

Problems:
- Heavy / bulky (if made solid)
- Ideally antenna must extend to infinity
- Careful truncation is necessary since currents do not decay to 0
Spiral Antennas

General conditions for frequency independence

Consider geometry in spherical coordinates
Input terminals infinitely close to origin at $\theta=0, \pi$
Antenna is described by the curve
$$r = F(\theta, \phi)$$
If want to scale frequency operation by $1/K$ need new surface
$$r = KF(\theta, \phi)$$

Relaxed criterion

Assuming only input properties are important (not pattern)
Tolerate rotation in $\phi$ (but not $\theta$ due to feeds)
$$KF(\theta, \phi) = F(\theta, \phi + C')$$
K and C interrelated $\Rightarrow$ radiation pattern can change with freq
Input properties identical
Spiral Antennas (2)

\[ KF(\theta, \phi) = F(\theta, \phi + C) \]

**Finding functional form of \( F(\theta, \phi) \)**

Differentiate (both sides) relation with respect to \( C \) and \( \phi \)

\[
\frac{d[KF(\theta, \phi)]}{dC} = \frac{dK}{dC} F(\theta, \phi) = \frac{\partial[F(\theta, \phi + C)]}{\partial C} = \frac{\partial[F(\theta, \phi + C)]}{\partial[\phi + C]}
\]

\[
\frac{\partial[KF(\theta, \phi)]}{\partial \phi} = K \frac{\partial[F(\theta, \phi)]}{\partial \phi} = \frac{\partial[F(\theta, \phi + C)]}{\partial \phi} = \frac{\partial F(\theta, \phi + C)}{\partial[\phi + C]}
\]

\[ \Rightarrow \frac{dK}{dC} F(\theta, \phi) = K \frac{\partial F(\theta, \phi)}{\partial \phi} \]

\[ \Rightarrow \frac{1}{K} \frac{dK}{dC} = \frac{1}{r} \frac{\partial r}{\partial \phi} \]

\[ r = F(\theta, \phi) \]
Spiral Antennas (3)

\[ \frac{1}{K} \frac{dK}{dC} = \frac{1}{r} \frac{\partial r}{\partial \phi} \]

Note: left-hand side is not a function of \( \theta \) or \( \phi \)

\[ \frac{1}{r} \frac{\partial r}{\partial \phi} = \frac{1}{K} \frac{dK}{dC} = \alpha \]

\[ \frac{1}{r} dr = \alpha \, d\phi \]

\[ \ln r = \alpha \phi + b(\theta) \]

\[ r = e^{b(\theta)} e^{\alpha \phi} \]

\[ \Rightarrow r = f(\theta) e^{\alpha \phi} \]

What is the shape of this function? Represents a spiral that unfolds exponentially with \( \phi \)
Planar Spiral

Antenna confined to a plane

\[
\frac{df}{d\theta} = f'(\theta) = A \delta \left( \frac{\pi}{2} - \theta \right)
\]
(As change \( \theta \) there is an abrupt jump in \( r \))

\[
r \bigg|_{\theta = \pi/2} = \rho = \begin{cases} 
A e^{\alpha \phi}, & \theta = \pi/2, \\
0, & \text{elsewhere}
\end{cases}
\]

Let \( A = \rho_0 e^{-\alpha \phi_0} \)

\[
\rho = \begin{cases} 
\rho_0 e^{\alpha (\phi - \phi_0)}, & \theta = \pi/2, \\
0, & \text{elsewhere}
\end{cases}
\]
Planar Spiral (2)

Variants

- Infinitely thin wires for spiral (impractical)
- Instead, make curves *edges* of metallic surface

But what about infinite structure?
Freq. governed by dimensions \((\lambda/2)\)
Conical Spiral

Antenna does not have to be confined to plane

\[ \frac{df}{d\theta} = f'(\theta) = A\delta(\beta - \theta), \quad 0 \leq \beta \leq \pi \]

Represents spiral wrapped onto conical surface

**Advantage**

- Unidirectional pattern
- No need for a back cavity
Log-Periodic Antennas

Idea:
- Antenna shape varies periodically with frequency
- More flexibility than frequency-independent
- Antenna performance changes with frequency
- Not a problem when freq. coverage is main goal
Log-Periodic Antennas (2)

Demonstration of frequency scaling
Log-Periodic Antennas (3)

In spherical coordinates

Describe antenna with
\[ \theta = \text{periodic function of } [b \ln r] \]

For example
\[ \theta = \theta_0 \sin \left( b \ln \left( \frac{r}{r_0} \right) \right) \]

Since roughly \( \theta \propto \ln r, \quad r \propto e^\theta \)
similar to frequency independent antennas
Log-Periodic Antennas (4)

Define antenna using electrical size at $f_0$

$$\theta(r) = F[b \ln r]$$

As we scale frequency, dimensions change. Describe new antenna with subs

$$r \rightarrow \frac{f}{f_0}r, \quad \theta \rightarrow \theta, \quad \theta(r) = F(b \ln \frac{f}{f_0}r)$$

At what point does antenna look the same? I.e., when are scaled and original antenna described by same function?

$$b \ln \left( \frac{f}{f_0}r \right) = b \ln r + P, \quad b \ln \frac{f}{f_0} = P, \quad \ln \frac{f}{f_0} = \frac{P}{b}$$

This is reason for name log-periodic antenna. Performance changes periodically with log of frequency.
Examples of Log-Periodic Antennas
Examples of Log Periodic Antennas (2)
Electrically Small Antennas

Antenna Miniaturization

Antenna of fixed (electrical) size
fundamental limit on the minimum Q

Very high Q for antennas can be bad:
1. High ohmic losses
2. Very narrow bandwidth
3. Sensitivity in matching

Means that it is difficult or impossible to miniaturize antennas like we do with transistors
Chu Limit


Analyze radiating modes of a sphere of radius \( r \)

Radiated fields: sum of orthogonal modes

Each mode

- Can be driven independently within sphere
- Modeled with ladder network

For lowest order TM mode

\[
Q = \frac{1 + 2(kr)^2}{(kr)^3[1 + (kr)^2]} \approx \frac{1}{(kr)^3}
\]

\[
\frac{\Delta f}{f_0} = \frac{1}{Q}
\]

Radiation resistance of mode

\( kr \ll 1 \)
Electrically Small Antennas

Chu Limit

Plot shows
Minimum obtainable Q
(lower is better)

η = Efficiency of antenna

Also shown are measured practical antennas
Summary

Introduced you to many antennas

- Resonant Antennas
  - Dipole (Hertzian, finite length)
  - Patch
- Aperture antennas
  - Open waveguide
- Reflector antennas
  - Corner reflector

Frequency-independent / log-periodic

Electrically small antennas

Purpose

- Show salient features of different antenna types
- See analysis tools required for different structures