Chapter 2: Basic Electromagnetic Analysis
Outline

Vector Potentials, Wave Equation

Far-field Radiation

Duality/Reciprocity

Transmission Lines
Antenna Theory Problems

Analysis Problem
Given an antenna structure or source current distribution, how does the antenna radiate?
Focus of this course (more mature and developed)

Synthesis problem
Given the desired operational characteristics (like the radiation pattern), find the antenna structure or source current that will generate this.
Challenging! Much less developed.
Topic for research.
Problem Statement

Given:

- Arbitrary volume $V$
- Filled with sources
  - $\overrightarrow{J}$ = electric currents (A/m$^2$)
  - $\overrightarrow{M}$ = magnetic currents (A/m$^2$)

Problem:

Compute fields $E$ and $H$ generated by currents

Solution:

Maxwell’s equations gives exact solution
Maxwell’s Equations

Equations (Differential Form)

\[ \nabla \cdot \mathbf{D} = \rho_v \]  Gauss’ Law (electric field)

\[ \nabla \cdot \mathbf{B} = 0 \]  Gauss’ Law (magnetic field)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M} \]  Faraday’s Law

\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \]  Ampere’s Law

Constitutive Relationships

\[ \mathbf{D} = \varepsilon \mathbf{E} \]
\[ \mathbf{B} = \mu \mathbf{H} \]
### Quantities

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>electric field</td>
<td>V/m</td>
</tr>
<tr>
<td>H</td>
<td>magnetic field</td>
<td>A/m</td>
</tr>
<tr>
<td>B</td>
<td>magnetic flux density</td>
<td>Tesla (T)</td>
</tr>
<tr>
<td>D</td>
<td>electric displacement</td>
<td>C/m²</td>
</tr>
<tr>
<td>J</td>
<td>electric current density</td>
<td>A/m²</td>
</tr>
<tr>
<td>M</td>
<td>magnetic current density</td>
<td>V/m²</td>
</tr>
<tr>
<td>ρ</td>
<td>electric charge density</td>
<td>C/m³</td>
</tr>
<tr>
<td>ε</td>
<td>permittivity</td>
<td>F/m</td>
</tr>
<tr>
<td>µ</td>
<td>permeability</td>
<td>H/m</td>
</tr>
<tr>
<td>η</td>
<td>Characteristic impedance</td>
<td>Ohm</td>
</tr>
</tbody>
</table>

\[
ε = ε₀ = 8.8542 \times 10^{-12} \text{ F/m} \quad µ = µ₀ = 4π \times 10^{-7} \text{ H/m (free space)}
\]

\[
ε = ε_r \cdot ε₀ \quad µ = µ_r \cdot µ₀
\]
Time Harmonic Fields

Assuming

Time-harmonic fields, or $\exp(j\omega t)$ variation

Linear, isotropic media

Maxwell’s equations become

$$\nabla \cdot \overrightarrow{D} = \rho_v$$
$$\nabla \cdot \overrightarrow{B} = 0$$
$$\nabla \times \overrightarrow{E} = -j\omega \overrightarrow{B} - \overrightarrow{M}$$
$$\nabla \times \overrightarrow{H} = j\omega \overrightarrow{D} + \overrightarrow{J}$$

where $\omega=2\pi f$ is the circular frequency (rad/s).
Boundary Conditions

\[
\hat{n} \times (\overline{E}_1 - \overline{E}_2) = \overline{M}_s \\
\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s \\
\hat{n} \cdot (\overline{D}_1 - \overline{D}_2) = \sigma_s \\
\hat{n} \cdot (\overline{B}_1 - \overline{B}_2) = \sigma^*_s
\]

Tangential E and H are

Continuous when no surface currents \((J, M)\) (e.g. dielectric media)
Differ according to the surface currents (conductive surface)

What if region 2 is perfect conductor?

PEC = perfect electrical conductor
PMC = perfect magnetic conductor
The Wave Equation

Maxwell famous for relating E and H fields by adding the $\partial D/\partial t$ term in Ampere’s law

Allows wave propagation to be predicted

Given source free-environment, take curl of Faraday’s law:

$$\nabla \times \nabla \times \mathbf{E} = -j \omega \mu \nabla \times \mathbf{H} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \frac{j}{\omega \mu} \left[ \nabla \times \nabla \times \mathbf{E} \right]$$

Substituting into Ampere’s law

$$\frac{j}{\omega \mu} \left[ \nabla \times \nabla \times \mathbf{E} \right] = j \omega \varepsilon \mathbf{E}$$

$$\nabla \left[ \nabla \cdot \mathbf{E} \right] - \nabla^2 \mathbf{E}$$

$$-0$$

Wave equation

$$[\nabla^2 + k^2] \mathbf{E} = 0$$

$$k = \omega/c \quad \text{(wave number)}$$

$$c = 1/\sqrt{\mu \varepsilon} \quad \text{(wave speed)}$$
Basic Antenna Analysis Problem
Given currents \( \mathbf{J} \) and \( \mathbf{M} \), compute fields \( \mathbf{E} \) and \( \mathbf{H} \)
Option 1: Direct
Option 2: Use potential

Static Problems
Nice to use electric scalar potential \( V = \text{voltage} \) instead of \( \mathbf{E} \)
Why? Can solve scalar equations instead of vector equations

Dynamic Problems
Use the vector potential \( \mathbf{A} \) instead of \( \mathbf{B} \)
Can solve vector equations
instead of complicated dyadic (matrix) equations
Definition of \( \mathbf{A} \) chosen to make analysis as simple as possible
Exploit physical and vectorial properties
Vector Potential (2)

Properties we exploit in defining vector potential $A$

- Gauss’ Law: $\nabla \cdot \vec{B} = 0$
- Any vector: $\nabla \cdot (\nabla \times \vec{A}) = 0$

Therefore,

$$\overline{B}_A = \mu \overline{H}_A = \nabla \times \vec{A} \quad \text{(No loss of generality)}$$

Subscript A means “arising from $\vec{A}$” or from electric current $\vec{J}$

Substitute into Faraday’s law (with $\vec{M} = 0$)

$$\nabla \times \vec{E}_A = -j\omega \mu \overline{H}_A = -j\omega \nabla \times \vec{A}$$

$$\nabla \times (\vec{E}_A + j\omega \vec{A}) = 0$$

Scalar identity: $\nabla \times (\nabla \phi_e) = 0 \quad (\phi_e \text{ any scalar function})$

Finally:

$$\vec{E}_A + j\omega \vec{A} = -\nabla \phi_e$$

Note: when $\omega = 0$, have normal definition of scalar potential
Vector Potential (3)

Summarizing

We have chosen definition of $\vec{A}$ so vector properties ensure certain physical conditions are automatically satisfied.
Simplifies analysis!
Vector Potential (4)

Substituting \( \vec{E}_A + j\omega \vec{A} = -\nabla \phi_e \) into Ampere's Law

\[
\nabla \times \vec{H}_A = \frac{\vec{J}}{\mu_0} + j\omega \mu_0 \epsilon \vec{E}_A \\
\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J} + j\omega \mu_0 \epsilon \nabla \times \vec{E}_A \\
\n\nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \\
\n\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - j\omega \mu_0 \epsilon (\nabla \phi_e + j\omega \vec{A}) \\
\n\n= \mu_0 \vec{J} - j\omega \mu_0 \epsilon \nabla \phi_e + \frac{j\omega \mu_0 \epsilon}{\mu_0} \vec{A} \\
\n\n\nabla^2 \vec{A} + k^2 \vec{A} = -\mu_0 \vec{J} + \nabla (\nabla \cdot \vec{A}) + j\omega \mu_0 \epsilon \nabla \phi_e \\
\n\n= -\mu_0 \vec{J} + \nabla [\nabla \cdot \vec{A} + j\omega \mu_0 \epsilon \phi_e] \\
\n\]

Note: We specified \( \nabla \times \vec{A} \), but \( \vec{A} \) is not yet uniquely defined
So, choose \( \nabla \cdot \vec{A} = -j\omega \mu_0 \phi_e \) (Lorenz Condition)
Vector Potential (5)

Finally

\[ \nabla^2 \overline{A} + k^2 \overline{A} = -\mu \overline{J} \]

Relationship for \( \overline{A} \) that is just the non-homogeneous wave equation individually satisfied for each component (e.g. \( A_x, A_y, A_z \))

If we need fields,

\[ \overline{B}_A = \nabla \times \overline{A} \]
\[ \overline{E}_A = -\nabla \phi_e - j \omega \overline{A} = -j \omega \overline{A} - \frac{j}{\omega \mu \varepsilon} \nabla (\nabla \cdot \overline{A}) \]
Vector Potential (6)

Vector potential $\overline{F}$ for magnetic current $\overline{M}$

Almost identical procedure, but roles of $E$ and $H$ switch
See notes for derivation

$$\nabla^2 \overline{F} + k^2 \overline{F} = -\epsilon \overline{M}$$

$$\overline{H}_F = -j\omega \overline{F} - \frac{j}{\omega \mu \epsilon} \nabla (\nabla \cdot \overline{F})$$

Also, to get $E$

$$\overline{D}_F = \epsilon \overline{E}_F = -\nabla \times \overline{F}$$
Vector Potential Summary

Summarizing...
Transforms complicated wave equation involving $\vec{E}$ and $\vec{H}$
Simpler scalar wave equations for comps of $\vec{A}$ and $\vec{F}$

Approach
Given $\vec{J}$ and $\vec{M}$ solve for $\vec{A}$ and $\vec{F}$
When all finished, then find $\vec{E}$ and $\vec{H}$
Helps us manage the complexity of EM antenna problems
Finding $\overline{A}$ for Known $\overline{J}$

**Governing equation**

Need to solve

$$\nabla^2 \overline{A} + k^2 \overline{A} = -\mu \overline{J}$$

Direct solution wrt. some boundary conditions? Numerically...

**Alternative Approach**

Transform into integral equation

Green’s function technique

Advantage:
- Automatically incorporates boundary condition
- Single equation
Green’s Function Technique

Roadmap
1. Find fields radiated by a point source at origin

2. Translate solution to arbitrary source location

3. Solution for arbitrary current distribution given by superposition of point source solutions

Similar to signal analysis for LTI systems!

- Impulse response of system $h(t)$
- $\Rightarrow$ Response for arbitrary input is $y(t) = h(t) \ast x(t)$
1. Fields due to Point Source at Origin

Assume current distribution
\[ \mathbf{J} = \mathbf{\hat{z}} J_z \]

We need to solve
\[ \nabla^2 \mathbf{A} (\mathbf{r}) + k^2 \mathbf{A} (\mathbf{r}) = -\mu \mathbf{\hat{z}} J_z (\mathbf{r}) \]

Note:
\[ \mathbf{A} (\mathbf{r}) = \mathbf{\hat{x}} A_x (\mathbf{r}) + \mathbf{\hat{y}} A_y (\mathbf{r}) + \mathbf{\hat{z}} A_z (\mathbf{r}) \]
\[ \nabla^2 \mathbf{A} (\mathbf{r}) = \mathbf{\hat{x}} \nabla^2 A_x (\mathbf{r}) + \mathbf{\hat{y}} \nabla^2 A_y (\mathbf{r}) + \mathbf{\hat{z}} \nabla^2 A_z (\mathbf{r}) \]

Only have z-directed component on LHS, therefore also on RHS
\[ \nabla^2 A_z (\mathbf{r}) + k^2 A_z (\mathbf{r}) = -\mu J_z (\mathbf{r}) \]

Now, solve for point source (unit impulse) at origin: \[ J_z (\mathbf{r}) = \delta (\mathbf{r}) \]
\[ \text{call solution } A_z (\mathbf{r}) = g (\mathbf{r}) \]

Problem:
\[ \nabla^2 g (\mathbf{r}) + k^2 g (\mathbf{r}) = -\mu \delta (\mathbf{r}) \]
Case 1: Away from origin

Assume $\vec{r} \neq 0$

$$\nabla^2 g(\vec{r}) + k^2 g(\vec{r}) = 0$$

Simplify by invoking symmetry of problem
RHS is only a function of $r$ (point source), so $g(\vec{r})$ only should depend on distance from origin $r = |\vec{r}|$

$$\nabla^2 g = \frac{1}{r^2} \frac{3}{\partial r} \left( r^2 \frac{dg}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{dg}{\partial \theta} \right)$$
$$+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2}$$
$$\frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{dg(r)}{dr} \right] + k^2 g(r) = 0$$
Case 1: Away from origin (2)

\[
\frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{dg(r)}{dr} \right] + k^2 g(r) = 0
\]

Do a trick on the \( \frac{d}{dr} \) term

\[
\frac{d}{dr} \left[ r^2 \frac{dg(r)}{dr} \right] = r^2 \frac{d^2 g(r)}{dr^2} + 2r \frac{dg(r)}{dr}
\]

\[
= r \frac{d^2}{dr^2} \left[ rg(r) \right]
\]

So, need to solve

\[
\frac{1}{r} \frac{d^2}{dr^2} \left[ rg(r) \right] + k^2 g(r) = 0
\]

\[
\frac{d^2}{dr^2} \left[ \frac{rg(r)}{f(r)} \right] + k^2 \left[ rg(r) \right] = 0
\]
Case 1: Away from origin (4)

Only depends on single var r, so have famous ODE

\[ f''(r) + \kappa^2 f(r) = 0 \]

Solutions

\[ f(r) = r g(r) = c_1 e^{-j\kappa r} + c_2 e^{j\kappa r} \]

Simplification: \( c_2 = 0 \)
Case 2: Include contribution at origin

Have solution

\[ r_1(r) = c_1 e^{-jk_0r} \]

But how to find \( c_1 \)?
Need to include effect of source at origin

Consider original problem

\[ \nabla^2 g(\vec{r}) + k^2 g(\vec{r}) = -\mu \delta(\vec{r}) \]

Integrate both sides over a sphere
Let sphere shrink to simplify solution
Case 2: Include contribution at origin (2)

\[ \nabla^2 g(\vec{r}) + k^2 g(\vec{r}) = -\mu \delta(\vec{r}) \]

Note: \( \delta(\vec{r}) = \delta(x) \delta(y) \delta(z) = \frac{1}{4\pi r^2} \delta(r) \)

\[ dv = r^2 \sin \theta d\theta dr d\phi \]

\[
\lim_{\Delta \to 0} \left[ \int_0^{2\pi} \int_0^{\pi} \int_0^\Delta \nabla^2 g(r) \, dv + \int_0^{2\pi} \int_0^{\pi} \int_0^\Delta k^2 g(r) \, dv \right]
\]

\[
= -\mu \int_0^{2\pi} \int_0^{\pi} \int_0^\Delta \delta(r) \, dv
\]

\( \Rightarrow 0 \) as \( \Delta \to 0 \)

\( = 1 \)

Simplify first term with Divergence Theorem

Integrating divergence of something over volume

= Integration of outward normal component on surface

\[ \nabla^2 g = \nabla \cdot (\nabla g) \]
Case 2: Include contribution at origin (3)

\[
\lim_{\Delta \to 0} \int_0^{2\pi} \int_0^{\pi} \int_{\Delta} \nabla^2 g(r) \, dV = -\mu
\]

\[
\nabla^2 g = \nabla \cdot (\nabla g)
\]

Becomes

\[
\lim_{\Delta \to 0} \left. \int_0^{2\pi} \int_0^{\pi} \hat{r} \cdot \nabla g(r) \, ds \right|_{r=\Delta} = -\mu
\]

\[
ds = r^2 \sin \theta \, d\theta \, d\phi
\]

Just need to insert previous result and solve for \(c_1\)

\[
g(r) = c_1 \frac{e^{-jkr}}{r}
\]
Case 2: Include contribution at origin (4)

\[ g(r) = c_1 \frac{e^{-jkr}}{r} \]

\[ \frac{dg(r)}{dr} = c_1 \left[ -\frac{jkr e^{-jkr}}{r^2} - \frac{e^{-jkr}}{r^2} \right] = \]

\[ = -c_1 \frac{e^{-jkr}}{r^2} \left[ 1 + jkr \right] \]

\[ \lim_{\delta \to 0} \int_0^{2\pi} \int_0^{\pi} c_1 \frac{e^{-jkr}}{r^2} (1 + jkr) r^2 \sin \theta d\theta d\phi \bigg|_{r=\delta} = -\mu \]

\[ c_1 \int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\phi = \mu \]

\[ 4\pi c_1 = \mu \]

\[ c_1 = \frac{\mu}{4\pi} \]

\[ g(r) = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \]
2. Translate Solution

Have solution for point at origin

\[ g(r) = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \]

For arbitrary source point \( \vec{r}' \)

\[ r = |\vec{r} - \vec{r}'| \quad \vec{r}' = \hat{x}x' + \hat{y}y' + \hat{z}z' \]

\[ g(\vec{r}, \vec{r}') = \frac{\mu}{4\pi} \frac{e^{-jk|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \]
3. Arbitrary Current Distribution

Have solution for point source

\[ g(\vec{r}, \vec{r}') = \frac{\mu}{4\pi} \frac{e^{-jk|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \]

What about arbitrary source \( \vec{J}(\vec{r}') \)?

\[ \nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J} \]

Big Idea: Can solve this equation by substituting the point source solution into the proper integral equation.
Arbitrary Current Distribution (2)

Consider integral equation

\[
I = \iiint_{V} \left[ A_{x}(\vec{r}) \nabla^2 g(\vec{r}, \vec{r}') - g(\vec{r}, \vec{r}') \nabla^2 A_{x}(\vec{r}) \right] d\vec{r}
\]

Use the vector identity: \( \nabla \nabla \cdot \vec{V} = \nabla \cdot (\nabla \vec{V}) - \nabla (\nabla \cdot \vec{V}) \)

\[
I = \iiint_{V} \left[ \nabla \cdot \left[ A_{x}(\vec{r}) \nabla g \right] - \nabla A_{x}(\vec{r}) \nabla g - \nabla \cdot \left[ g \nabla A_{x}(\vec{r}) \right] \right. \\
+ \left. \nabla g \cdot \nabla A_{x}(\vec{r}) \right] d\vec{r}
\]

\[
= \iiint_{V} \nabla \cdot \left\{ A_{x}(\vec{r}) \nabla g(\vec{r}, \vec{r}') - g(\vec{r}, \vec{r}') \nabla A_{x}(\vec{r}) \right\} d\vec{r}
\]
But, what is $I$ equal to?

Divergence Theorem

$$I = \oint_S \hat{n} \cdot \left( A_z(\vec{r}) \nabla g(\vec{r}, \vec{r}') - g(\vec{r}, \vec{r}') \nabla A_z(\vec{r}) \right) dS$$

Note: Surf. integral $\to 0$ as volume becomes large
Called the “radiation condition”
Intuitively: If we are far from all sources, $A_z$ and $g$ will shrink to 0.
Arbitrary Current Distribution (4)

\[ I = \iiint_V \left[ A_2(r) \nabla^2 g(r, r') - g(r, r') \nabla^2 A_2(r) \right] dV = 0 \]

Recall that we have

\[ \nabla^2 A_2(r) = -\mu \mathcal{J}_2(r-r') - k^2 A_2(r) \]
\[ \nabla^2 g(r, r') = -\mu \delta(r-r') - k^2 g(r, r') \]

Substitute into volume equation

\[ 0 = \iiint_V \left[ A_2(r) \left[ -\mu \mathcal{J}_2(r-r') - k^2 g(r, r') \right] - g(r, r') \left[ -\mu \mathcal{J}_2(r) - k^2 A_2(r) \right] \right] dV \]
Arbitrary Current Distribution (5)

\[ 0 = \iiint_{V} \left( A_{2}(\mathbf{r}) \left[ -\mu S(\mathbf{r} - \mathbf{r'}) - k^2 g(\mathbf{r}, \mathbf{r'}) \right] \right. \\
- g(\mathbf{r}, \mathbf{r'}) \left[ -\mu J_{2}(\mathbf{r}) - k^2 A_{2}(\mathbf{r}) \right] \left. \right) \mathbf{r} \, d\mathbf{r} \\
= -\mu A_{2}(\mathbf{r'}) + \iiint_{V} \left\{ k^2 A_{2}(\mathbf{r}) \sqrt{g(\mathbf{r}, \mathbf{r'}) - k^2 A_{2}(\mathbf{r})} \right\} \mathbf{r} \, d\mathbf{r}' \\
+ \mu g(\mathbf{r}, \mathbf{r'}) J_{2}(\mathbf{r}) \mathbf{r} \, d\mathbf{r}' \\
\]

\[ A_{2}(\mathbf{r}) = \iiint_{V} g(\mathbf{r}, \mathbf{r'}) J_{2}(\mathbf{r'}) \, d\mathbf{r}' \]

Note: We have switched \( r \) and \( r' \) (symmetry)

\[ g(\mathbf{r}', \mathbf{r}) = g(\mathbf{r}, \mathbf{r}') \]
Arbitrary Current Distribution (6)

\[
A_2(\vec{r}) = \frac{\mu}{4\pi} \iiint_V J_2(\vec{r}') \, e^{-jkr - k'r'} \, d\vec{r}'
\]

Interpretation?

Since we have the same equation for each Cart. component

\[
\overline{A}(\vec{r}) = \frac{\mu}{4\pi} \iiint_V \overline{J}(\vec{r}') \, e^{-jkr - k'r'} \, d\vec{r}'
\]

Similarly for magnetic currents

\[
\overline{F}(\vec{r}) = \frac{\varepsilon}{4\pi} \iiint_V \overline{M}(\vec{r}') \, e^{-jkr - k'r'} \, d\vec{r}'
\]
Summary of Green’s Function Analysis

Goal
Compute fields radiated by arbitrary current source (antenna).

Solution
Use vector potentials to make problem simpler
Simple inhomogeneous wave equation for A

Solve the equation for a point current source (= Green’s function)

Fields for arbitrary current given by integral equation
(Superposition or convolution)